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ON THE RESIDUE CLASS DISTRIBUTION OF THE NUMBER OF PRIME DIVISORS OF AN INTEGER

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Abstract. Let $\Omega(n)$ denote the number of prime divisors of n counting multiplicity. One can show that for any positive integer m and all $j=0,1,\dots,m-1$, we have

$$\#\{n\le x:\Omega(n)\equiv j \pmod m\}=\frac{x}{m}+o(x^\alpha),$$

with $\alpha=1$. Building on work of Kubota and Yoshida, we show that for $m>2$ and any $j=0,1,\dots,m-1$, the error term is not $o(x^\alpha)$ for any $\alpha<1$.

§1. Introduction

The *Liouville function*, denoted $\lambda(n)$, is defined by $\lambda(n):=(-1)^{\Omega(n)}$, where $\Omega(n)$ is the number of prime divisors of n counting multiplicity. The Liouville function is closely connected to the Riemann zeta function and hence to many results and conjectures in prime number theory. Recall from [5, pp. 617−621] that for $\Re s>1$, we have

$$\sum_{n\ge 1}\frac{\lambda(n)}{n^s}=\frac{\zeta(2s)}{\zeta(s)},$$

so that $\zeta(s)\ne 0$ for $\Re s\ge\vartheta$, provided that $\sum_{n\le x}\lambda(n)=o(x^\vartheta)$. The prime number theorem allows the value $\vartheta=1$, so that for $j=0,1$, we have

$$\#\{n\le x:\Omega(n)\equiv j \pmod 2\}\sim\frac{x}{2}.$$

If the Riemann hypothesis holds, we even have, for $j=0,1$ and every $\alpha>1/2$,

$$\#\{n\le x:\Omega(n)\equiv j \pmod 2\}=\frac{x}{2}+o(x^\alpha).$$

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Kubota and Yoshida [4] investigated whether similar asymptotic properties could hold in general for the functions

$$N_{m,j}(x) := \#\big\{n \leq x : \Omega(n) \equiv j \pmod{m}\big\}, \quad m \in \mathbb{Z}_{>0},\, j = 0, 1, \ldots, m-1.$$

To this end, they introduced and studied generalizations of the Liouville function.

The question of whether for all $m \in \mathbb{Z}_{>0}$ and $j = 0, 1, \ldots, m-1$, we have

$$N_{m,j}(x) = \frac{x}{m} + o(x^{\alpha}) \tag{1}$$

with $\alpha = 1$ left open by Kubota and Yoshida [4], but it turns out that this follows from a result of Rivat, Sárközy, and Stewart [6]. In Section 2, we show that this also follows very quickly from a result of Hall [3] on the mean values of multiplicative functions.

As for the question of whether (1) can hold with $\alpha < 1$ if $m > 2$, Kubota and Yoshida obtained the following surprising result.

THEOREM 1 ([4, Theorem 4]). *Let $m \in \mathbb{Z}_{>2}$, and let $\alpha < 1$. Then for at least one $j = 0, 1, \ldots, m-1$, we have that (1) does not hold.*

This is in striking contrast to the expected result for $m = 2$. The result of Kubota and Yoshida still leaves open the possibility that, for some $m > 2$ and some $j = 0, 1, \ldots, m-1$, equation (1) holds with some $\alpha < 1$. Our main result is that this is impossible.

THEOREM 2. *Let $m \in \mathbb{Z}_{>2}$, and let $\alpha < 1$. Then for all $j = 0, 1, \ldots, m-1$, equation (1) does not hold.*

A proof, building on the work of Kubota and Yoshida [4], is given in Section 3.

§2. Generalizations of the Liouville function

Let $m \in \mathbb{Z}_{>0}$, and let $\zeta_m := e^{2\pi i/m}$ be a primitive mth root of unity. As a generalization of Liouville's function, define for $k = 0, 1, \ldots, m-1$ the function

$$\lambda_{m,k}(n) := \zeta_m^{k\Omega(n)}.$$

The functions $\lambda_{m,k}(n)$ were introduced by Kubota and Yoshida [4] to study the asymptotics of $N_{m,j}(x)$ for $m > 2$. To investigate the properties of

$N_{m,j}(x)$, it is natural to look at the partial sums

$$S_{m,k}(x) := \sum_{n \le x} \lambda_{m,k}(n).$$

First of all, there is a simple but very useful linear relationship between $S_{m,k}(x)$ and $N_{m,j}(x)$. For $k = 0, 1, \ldots, m-1$, we have

$$S_{m,k}(x) = \sum_{n \le x} \zeta_m^{k\Omega(n)} = \sum_{j=0}^{m-1} \sum_{\substack{n \le x \\ \Omega(n) \equiv j \pmod{m}}} \zeta_m^{k\Omega(n)} = \sum_{j=0}^{m-1} \zeta_m^{kj} N_{m,j}(x). \tag{2}$$

Conversely, for $j = 0, 1, \ldots, m-1$, we have

$$\begin{aligned} N_{m,j}(x) &= \sum_{\substack{n \le x \\ \Omega(n) \equiv j \pmod{m}}} 1 = \sum_{n \le x} \frac{1}{m} \sum_{k=0}^{m-1} \zeta_m^{k(\Omega(n)-j)} \\ &= \frac{1}{m} \sum_{k=0}^{m-1} \zeta_m^{-jk} S_{m,k}(x). \end{aligned} \tag{3}$$

Second, since $\lambda_{m,k}(n)$ is a multiplicative function with values in the unit disk, we can apply the following theorem of Hall [3] to give an asymptotic bound of $S_{m,k}(x)$.

THEOREM 3 (see [3]). *Let D be a convex subset of the closed unit disk in $\mathbb{C}$ containing zero with perimeter $L(D)$. If $f : \mathbb{Z}_{>0} \to \mathbb{C}$ is a multiplicative function with $|f(n)| \le 1$ for all $n \in \mathbb{Z}_{>0}$ and $f(p) \in D$ for all primes p, then*

$$\frac{1}{x} \Big| \sum_{n \le x} f(n) \Big| \ll \exp\left(-\frac{1}{2}\Big(1 - \frac{L(D)}{2\pi}\Big) \sum_{p \le x} \frac{1 - \Re f(p)}{p} \right). \tag{4}$$

LEMMA 4. *For every $m \in \mathbb{Z}_{>0}$ there exists an $A > 0$ such that for all $k = 1, 2, \ldots, m-1$, we have*

$$|S_{m,k}(x)| \ll \frac{x}{\log^A x}.$$

Proof. Set D equal to the convex hull of the mth roots of unity, and set $f(n) = \lambda_{m,k}(n)$. Because D is a convex subset strictly contained in the closed unit disk of $\mathbb{C}$, we have $L(D) < 2\pi$. This gives

$$c := \frac{1}{2}\Big(1 - \frac{L(D)}{2\pi}\Big) > 0.$$

Applying Theorem 3 yields

$$\frac{1}{x}\Big|\sum_{n\le x}\lambda_{m,k}(n)\Big| \ll \exp\Big(-c\sum_{p\le x}\frac{1-\Re\lambda_{m,k}(p)}{p}\Big) = \exp\Big(-c(1-\Re\zeta_m^k)\sum_{p\le x}\frac{1}{p}\Big).$$

Since $\sum_{p\le x} p^{-1} = \log\log x + O(1)$, this quantity is

$$\ll \exp\big(-c(1-\Re\zeta_m^k)\log\log x\big) = \Big(\frac{1}{\log x}\Big)^{c(1-\Re\zeta_m^k)}.$$

Noting that $0<k<m$, we have $c(1-\Re\zeta_m^k)>0$. Set $A := \min_{0<k<m}\{c(1-\Re\zeta_m^k)\}$. Then $A>0$, and we obtain

$$\Big|\sum_{n\le x}\lambda_{m,k}(n)\Big| \ll \frac{x}{\log^A x}. \qquad \square$$

As in the work of Rivat, Sárközy, and Stewart [6], this bound for the partial sums $S_{m,k}(x)$ immediately leads to an asymptotics result for the counting functions $N_{m,j}(x)$.

Corollary 5. *Let $m\in\mathbb{Z}_{>0}$. There exists an $A>0$ (depending on m) such that for all $j=0,1,\dots,m-1$, we have*

$$N_{m,j}(x) = \frac{x}{m} + O\Big(\frac{x}{\log^A x}\Big).$$

In particular, for all $j=0,1,\dots,m-1$, we have that (1) holds with $\alpha=1$.

Proof. From (3) we immediately get

$$N_{m,j}(x) = \frac{1}{m}S_{m,0}(x) + \frac{1}{m}\sum_{k=1}^{m-1}\zeta_m^{-jk}S_{m,k}(x). \tag{5}$$

The first term of the right-hand side of (5) is

$$\frac{1}{m}S_{m,0}(x) = \frac{1}{m}\sum_{n\le x}1 = \frac{x}{m} + O(1).$$

Applying the triangle inequality and Lemma 4, we get that the absolute value of the second term of the right-hand side of (5) is

$$\Big|\frac{1}{m}\sum_{k=1}^{m-1}\zeta_m^{-jk}S_{m,k}(x)\Big| \le \frac{1}{m}\sum_{k=1}^{m-1}|S_{m,k}(x)| \ll \frac{x}{\log^A x}$$

for some $A>0$. This gives us our desired result. $\square$

The constant A in Corollary 5 can easily be made explicit, but it is not the purpose of this paper to determine a good value for A. Readers interested in the constant A may wish to consult [6].

§3. Lower bounds for the error terms

Let $m \in \mathbb{Z}_{>0}$, and let $j = 0, 1, \ldots, m-1$. We introduce the error term

$$R_{m,j}(x) := N_{m,j}(x) - \frac{x}{m}.$$

Our main result, Theorem 2, obviously translates as follows.

THEOREM 6. *Let $m \in \mathbb{Z}_{>2}$, and let $\alpha < 1$. None of $R_{m,0}, R_{m,1}, \ldots, R_{m,m-1}$ are $o(x^\alpha)$.*

To prove Theorem 6, keeping with [4], we use the following results.

LEMMA 7. *Let $\{a_n\}_{n\in\mathbb{Z}_{>0}}$ be a sequence of complex numbers, and let $\alpha > 0$. If the partial sums satisfy $\sum_{n\le x} a_n = o(x^\alpha)$, then the Dirichlet series $\sum_{n\ge1} a_n n^{-s}$ converges for $\Re s > \alpha$ to a holomorphic (single-valued) function.*

Proof. This follows directly from Perron's formula (see [1, p. 243, Lemma 4]). □

For $\Re s > 1$, denote

$$L_{m,k}(s) := \sum_{n\ge1} \frac{\lambda_{m,k}(n)}{n^s}.$$

Kubota and Yoshida [4] introduced the function $L_{m,k}(s)$ and gave a multivalued analytic continuation of $L_{m,1}(s)$ to the region $\Re s > 1/2$; their proof easily generalizes to give the result for all $k = 1, 2, \ldots, m-1$; thus, we attribute to them the generalization as well.

THEOREM 8 (see [4]). *Let $m \in \mathbb{Z}_{>2}$, and let $k = 1, 2, \ldots, m-1$. The Dirichlet series $L_{m,k}(s)$ can be analytically continued to a multivalued function on $\Re s > 1/2$ given by the product $\zeta(s)^{\zeta_m^k} G_{m,k}(s)$, where $G_{m,k}(s)$ is a holomorphic function for $\Re s > 1/2$. In particular, if $k \neq m/2$, then for any $\alpha < 1$, the Dirichlet series $L_{m,k}(s)$ does not converge for all s with $\Re s > \alpha$.*

Proof. The first part follows from (the proof of) [4, Theorem 1]. Note that ζ_m^k is not rational for $k \neq m/2$. Since $\zeta(s)$ has a pole at $s=1$, this means that no branch of $\zeta(s)^{\zeta_m^k}$ is holomorphic in a neighborhood of $s=1$. ∎

Remark 9. Using these results, we can quickly obtain that if $m>2$, at least two of the error terms are not $o(x^\alpha)$ for any $\alpha<1$. For $k=1,2,\ldots,m-1$, using (2), we have

$$S_{m,k}(x)=\sum_{j=0}^{m-1}\zeta_m^{jk}R_{m,j}(x).$$

By Lemma 7 and Theorem 8, $S_{m,1}(x)$ is not $o(x^\alpha)$ for any $\alpha<1$, so that at least one of the error terms $R_{m,j}(x)$ is not $o(x^\alpha)$, which is the result of Kubota and Yoshida [4, Theorem 1]. From (2) with $k=0$, we obtain

$$\sum_{j=0}^{m-1}R_{m,j}(x)=S_{m,0}(x)-x=-\{x\},$$

where $\{x\}$ denotes the fractional part of x. This shows that it is impossible that all but one of the error terms $R_{m,j}(x)$ are $o(x^\alpha)$ for an $\alpha<1$.

Let $m>2$, and let $j=0,1,\ldots,m-1$. From (3) we get

$$R_{m,j}(x)=\frac{1}{m}\sum_{k=1}^{m-1}\zeta_m^{-jk}S_{m,k}(x)-\frac{\{x\}}{m}.$$

In light of Lemma 7, to obtain that $R_{m,j}(x)$ is not $o(x^\alpha)$ for any $\alpha<1$, it suffices to show that the generating function of $R_{m,j}(x)+\{x\}/m$, which is

$$\frac{1}{m}\sum_{k=1}^{m-1}\zeta_m^{-jk}L_{m,k}(s),$$

cannot be analytically continued to a holomorphic (single-valued) function in the half-plane $\Re s>\alpha$.

We now proceed with the proof of Theorem 6.

Proof of Theorem 6. Let $1/2<\alpha<1$, and let $c_1,c_2,\ldots,c_{m-1}\in\mathbb{C}^*$. We will prove that the linear combination

$$f(s):=\sum_{k=1}^{m-1}c_kL_{m,k}(s)$$

cannot be analytically continued to a holomorphic (single-valued) function in the half-plane $\Re s > \alpha$. Suppose, to the contrary, that it can, and assume for now that $L_{m,1}(s), L_{m,2}(s), \ldots, L_{m,m-1}(s)$ are linearly independent over $\mathbb{C}$, which will be shown later. Let C denote a smooth path in the half-plane $\Re s > \alpha$, starting and ending in an s_0 with $\Re s_0 > 1$, winding around $s = 1$ once in the positive direction and not winding around (and not passing) any zeros of $\zeta(s)$. (One way to obtain rigorous statements below is to consider all linear combinations of $L_{m,k}(s)$ and analytic continuations along C thereof as single-valued holomorphic functions in the half-plane $\Re s > 1$.) By Theorem 8, as pointed out in [4, Remark 1], the analytic continuation of $L_{m,k}(s)$ along C gives us $\exp(-2\pi i \zeta_m^k) L_{m,k}(s)$. From the holomorphicity assumption on $f(s)$, it follows that the analytic continuation of $f(s)$ along C is $f(s)$ itself. So we have

$$\sum_{k=1}^{m-1} c_k L_{m,k}(s) = \sum_{k=1}^{m-1} c_k \exp(-2\pi i \zeta_m^k) L_{m,k}(s),$$

and from the linear independence over $\mathbb{C}$ of the functions $L_{m,k}(s)$, we obtain that $\exp(-2\pi i \zeta_m^k) = 1$ for $k = 1, 2, \ldots, m-1$. This means that $\zeta_m^k \in \mathbb{Z}$ for $k = 1, 2, \ldots, m-1$, a contradiction if $m > 2$.

We are left with proving that $L_{m,1}(s), L_{m,2}(s), \ldots, L_{m,m-1}(s)$ are linearly independent over $\mathbb{C}$. By the uniqueness of Dirichlet series (see, e.g., [1, Theorem 11.3]), this would follow from the linear independence over $\mathbb{C}$ of the functions $\lambda_{m,k}(n) = \zeta_m^{k\Omega(n)}$ for $k = 1, 2, \ldots, m-1$. To prove the latter, suppose that for some $d_1, d_2, \ldots, d_{m-1} \in \mathbb{C}$, we have that $\sum_{k=1}^{m-1} d_k \zeta_m^{k\Omega(n)} = 0$ for all $n \in \mathbb{Z}_{>0}$. Then, in particular, $\sum_{k=1}^{m-1} d_k (\zeta_m^k)^i = 0$ for $i = 0, 1, \ldots, m-2$. This defines a system of linear equations in the d_k with matrix M of Vandermonde type. The values ζ_m^k for $k = 1, 2, \ldots, m-1$ are all distinct, so $\det M \neq 0$. Therefore, $d_1, d_2, \ldots, d_{m-1}$ must all be zero; that is, $\lambda_{m,1}(n), \lambda_{m,2}(n), \ldots, \lambda_{m,m-1}(n)$ are linearly independent over $\mathbb{C}$. This completes the proof. ∎

REMARK 10. In the spirit of prime number races, it seems fitting that further study should be taken to investigate the sign changes of $N_{m,j}(x) - N_{m,j'}(x)$ for $j \neq j'$. For the case $m = 2$, some such investigations have been undertaken (see [2] and the references therein).

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