Well-posedness for nonlinear Dirac equations in one dimension

Shuji Machihara, Kenji Nakanishi, and Kotaro Tsugawa

Abstract We completely determine the range of Sobolev regularity for the Dirac-Klein-Gordon system, the quadratic nonlinear Dirac equations, and the wave-map equation to be well posed locally in time on the real line. For the Dirac-Klein-Gordon system, we can continue those local solutions in nonnegative Sobolev spaces by the charge conservation. In particular, we obtain global well-posedness in the space where both the spinor and scalar fields are only in $L^2(\mathbb{R})$. Outside the range for well-posedness, we show either that some solutions exit the Sobolev space instantly or that the solution map is not twice differentiable at zero.

Contents

1. Introduction

Our primary purpose in this article is to study the Cauchy problem of the Dirac-Klein-Gordon system (DKG) in one spatial dimension:

Kyoto Journal of Mathematics, Vol. 50, No. 2 (2010), 403–451
DOI 10.1215/0023608X-2009-018,
Received July 13, 2009. Accepted December 3, 2009.
Mathematics Subject Classification: Primary 35L70; Secondary 35L71, 35B44, 35B65, 35Q41.

$$\text{(1.1)} \qquad \text{DKG} \quad \begin{cases} (i\gamma_0\partial_t + \gamma_1\partial_x)\psi + m\psi = \phi\psi, \\ (\partial_t^2 - \partial_x^2 + M^2)\phi = \psi^*\gamma_0\psi, \end{cases}$$

where $\psi(t,x) : \mathbb{R}^{1+1} \to \mathbb{C}^2$ is a 2-spinor field, $\phi(t,x) : \mathbb{R}^{1+1} \to \mathbb{R}$ is a scalar field, m and M are nonnegative mass constants, and $*$ denotes the adjoint (transposed complex conjugate); γ_0 and γ_1 are fixed 2×2 Hermitian matrices satisfying

$$\text{(1.2)} \qquad \gamma_j\gamma_k + \gamma_k\gamma_j = 2\delta_{j,k}I_2$$

for $j,k \in \{0,1\}$, so that we have $(i\gamma_0\partial_t + \gamma_1\partial_x)^2 = (-\partial_t^2 + \partial_x^2)I_2$. For example, we can choose γ_0 and γ_1 from the Pauli matrices:

$$\text{(1.3)} \qquad \sigma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We investigate time-local well-posedness of the Cauchy problem for the above system with the initial data

$$\text{(1.4)} \qquad \psi(0,x) \in H^a, \qquad \phi(0,x) \in H^s, \qquad \partial_t\phi(0,x) \in H^{s-1},$$

for all possible choices of $(a,s) \in \mathbb{R}^2$, where $H^s = H^s(\mathbb{R})$ denotes the usual L^2 Sobolev space on $\mathbb{R}$. There have been many results on this problem (even restricted to the one-dimensional case; see [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [15], [19], [20], [21], [22], [23], [25], [26]). Except for some earlier works, they all exploit the null structure of nonlinearity, estimating the solutions in the Fourier restriction norms, which allow them to work with rough regularity. However, the following cases have been left unsolved:

(1) low regularity: $a \le -1/4$,
(2) endpoint: $(a,s) = (0,0)$.

They are concerned, respectively, with the trace to a fixed time slice and the Sobolev embedding into L^∞, both of which are due to the L^2 nature of the space-time norms of the Fourier restriction. We resolve these problems by quite simple ideas, and thereby completely determine the region of (a,s) where DKG is locally well posed. By using the L^2 conservation of the spinor field, we also obtain global well-posedness.

THEOREM 1.1
Let $(a,s) \in \mathbb{R}^2$ satisfy $a > -1/2, |a| \le s \le a+1$. Then DKG is time-locally well posed in the space $(\psi,\phi,\partial_t\phi) \in H^a \times H^s \times H^{s-1}$. If in addition $a \ge 0$, then it is globally well posed.

In particular, DKG is globally well posed in $L^2 \times L^2 \times H^{-1}$. The following two theorems show that the above local well-posedness is optimal: in the other region of exponents, it is either ill posed or the solution map (if it exists) is not regular. On the other hand, the global well-posedness part is not optimal. It is already proved in [25] that one can go slightly below L^2 for ψ (for $s \ge |a| > 0$). With the improved local well-posedness, we can probably improve the global result as well, but we do not pursue it here.

THEOREM 1.2

DKG is ill posed if either $a > \max(0, s)$ *or* $s > \max(a+1, 1/2)$. *More precisely, there is a local solution* (ψ, ϕ), *given by Theorem 1.1 with some different regularity exponent* (a', s') *in the well-posedness region, which satisfies* $(\psi(0), \phi(0), \partial_t\phi(0)) \in H^a \times H^s \times H^{s-1}$ *but does not stay there for any small* $t \neq 0$.

If $a > s \geq 0$ or $s > a+1 > 1/2$, then we can choose $a' \leq a$ and $s' \leq s$, so that we have ill-posedness at (a, s) by nonexistence. Otherwise, we can choose a sequence of smooth initial data which converge (in both spaces) to that in the theorem, and thus we deduce ill-posedness at least by discontinuity of the solution map (from the initial data at (a, s) to the solution, even in the space-time distributions). In the remaining region we have the following.

THEOREM 1.3

Let $a + s < 0$, *or let* $(a, s) = (-1/2, 1/2)$. *Then for any small* $T > 0$, *the flow map of* $DKG : (\psi(0), \phi(0), \partial_t\phi(0)) \to (\psi, \phi, \partial_t\phi)$ *cannot be twice differentiable (at zero) from* $H^a \times H^s \times H^{s-1}$ *to* $C([0, T]; H^a \times H^s \times H^{s-1})$.

Here we consider the second derivative in the sense of Fréchet.

DEFINITION 1.4

Let X, Y be normed spaces. We say that N is *twice differentiable at zero from* X *to* Y if N is a map from a zero neighborhood of X to Y and there exist $N_0' : X \to Y$ bounded linear and $N_0'' : X^2 \to Y$ bounded symmetric bilinear such that

$$\left\| N(u) - N(0) - N_0'(u) - \frac{1}{2} N_0''(u, u) \right\|_Y = o(\|u\|_X^2) \tag{1.5}$$

as $u \to 0$ in X. (It is clear that N_0' and N_0'' are unique.)

Theorem 1.3 does not really imply ill-posedness but precludes proofs of well-posedness by the simple iteration argument. We mention that Holmer [16] had obtained similar ill-posedness results for the 1-dimensional Zakharov system.

To prove the well-posedness result Theorem 1.1, we give two types of approach. Both cases are based on the Sobolev spaces on the null coordinates:

$$H_\alpha^{s_1} H_\beta^{s_2}, \quad (\alpha, \beta) := (t + x, t - x). \tag{1.6}$$

It turns out that the low regularity problem for $a \leq -1/4$ consists in transfer from the space-time Sobolev spaces on the null coordinates to the Sobolev spaces on each time slice, especially to the initial time, namely, in the trace operator. Let us explain more details about it.

Solving the Klein-Gordon equation with a given source term is used essentially to integrate in both the two null directions α and β, which adds one regularity to each direction in the Sobolev norms. However, since we are dealing with the initial data problem, we have to impose some condition at $t=0$, which requires us to take the trace after each integration. Hence if we start with the Sobolev space with the critical regularity $-1/4$ in both directions, then we need the trace in the Sobolev space $H^{-1/4}H^{3/4}$ or $H^{3/4}H^{-1/4}$. But the trace operator to $t=0$ requires that $s_1+s_2>1/2$, and so we need $a>-1/4$ in this way. Thus the linear estimate fails at $a=-1/4$ in the Sobolev spaces $H^{s_1}_\alpha H^{s_2}_\beta$, even though the product estimate in these spaces do not encounter any difficulty at this regularity. Note that the Sobolev space with a slightly different weight,

$$\|\langle\xi\rangle^{s_0}\langle|\tau|+|\xi|\rangle^{s_1}\langle|\tau|-|\xi|\rangle^{s_2}\mathcal{F}_{t,x}u(\tau,\xi)\|_{L^2_{\tau,\xi}}, \tag{1.7}$$

suffers essentially the same problem. More precisely, the above trace problem forces one to choose $s_1+s_2>3/2$ while reducing s_0. Since in the bilinear estimate the high-high interaction of the Dirac fields getting into low $|\xi|$- and high $|\tau|$-frequencies has the order $|\tau|^{-2a-2}L^2_{\tau,\xi}$, one needs $2a+2\geq s_1+s_2>3/2$, namely, $a>-1/4$ (see [26] for a counterexample of the bilinear estimate in the above norm for $a\leq -1/4$).

In our first approach to resolving the above problem, we divide the solution into the free part and the nonlinear part, and we estimate their contributions separately in the nonlinear terms. Then we take advantage of the fact that the null structure works more effectively on the free part while the nonlinear part has more regularity. Thus the remaining task is reduced to the standard product estimate and the trace estimate with sufficient regularity. At the endpoint $(a,s)=(0,0)$, we replace the Sobolev spaces with mixed L^p spaces, but the proof remains quite elementary. A similar approach was used for the 3D cubic wave equation in [24].

The second approach is to recover the trace estimate in the lower regularity by adding to the norm another component, which is L^1 in the time Fourier variable. This approach works except for the endpoint $(a,s)=(0,0)$. It requires more work to prove the bilinear estimates but not so much regularity of the nonlinear terms. In particular, it can be applied to other similar equations, for example, nonlinear Dirac equations with quadratic terms (QD):

$$(i\gamma_0\partial_t+\gamma_1\partial_x)\psi+m\psi=C(\psi^*\gamma_0\psi), \tag{1.8}$$

where $\psi:\mathbb{R}^{1+1}\to\mathbb{C}^2$, and C is a constant 2×2 complex matrix, or the wave map equation (WM):

$$(\partial_t^2-\partial_x^2)\phi_j=\sum_{k,l}g(\phi)_j^{k,l}(\partial_t\phi_k\partial_t\phi_l-\partial_x\phi_k\partial_x\phi_l), \tag{1.9}$$

where $\phi:\mathbb{R}^{1+1}\to\mathbb{R}^N$, $N\in\mathbb{N}$, and $g:\mathbb{R}^N\to\mathbb{R}^{N^3}$ is a fixed smooth function.* Now we state our results for QD and WM.

*For regularity of g, it suffices to have $g\in C^{2r+1}$ for some $\mathbb{N}\ni r\geq s$ (cf. Theorem 4.5.)

THEOREM 1.5

QD is time-locally well posed for any C in the space $\psi \in H^a$ if and only if $a > -1/2$. For the ill-posedness, there are some C and a local solution ψ which satisfies $\psi(0) \in H^{-1/2}$, but $\psi(t) \notin H^{-1/2}$ for any small $t \neq 0$.

THEOREM 1.6

WM is time-locally well posed for any N and any g in the space $(\phi, \partial_t \phi) \in H^s \times H^{s-1}$ if and only if $s > 1/2$. For the illposedness, there are some g and a local solution ϕ which satisfies $(\phi(0), \partial_t \phi(0)) \in H^{1/2} \times H^{-1/2}$, but $\phi(t) \notin H^{1/2}$ for any small $t \neq 0$.

The ill-posedness part is essentially known. There have been a few well-posedness results on QD (see [3], [19], [20]), with the same regularity restriction $a > -1/4$. Here we should remark that Keel and Tao in [17] claimed the local well-posedness for WM in the space $(\phi, \partial_t \phi) \in H^s \times H^{s-1}$ for $s > 1/2$, but their proof has a gap for $s \leq 3/4$ by the same problem as explained above (see Remark 2.6 for more details).

We conclude the introduction with some notation used throughout the article. We denote the null coordinate and its dual (the Fourier variable) by

$$(1.10) \qquad (\alpha, \beta) = (t + x, t - x), \qquad (\mu, \nu) = \frac{(\tau + \xi, \tau - \xi)}{2},$$

where (τ, ξ) denotes the Fourier variable for (t, x). Hence we have

$$(1.11) \qquad (\partial_\alpha, \partial_\beta) = \frac{(\partial_t + \partial_x, \partial_t - \partial_x)}{2}, \qquad (\partial_\mu, \partial_\nu) = (\partial_\tau + \partial_\xi, \partial_\tau - \partial_\xi).$$

To switch the coordinates, we use the following convention:

$$(1.12) \qquad \begin{aligned} f(\alpha, \beta)^\times &:= f\Big(\frac{(\alpha + \beta)}{2}, \frac{(\alpha - \beta)}{2}\Big) \ (= f(t, x)), \\ g(\mu, \nu)_\times &:= g(\mu + \nu, \mu - \nu) \ (= g(\tau, \xi)). \end{aligned}$$

Using (α, β), we can rewrite the system DKG in a simpler form. Choosing $(\gamma_0, \gamma_1) = (\sigma_0, \sigma_1)$ in (1.3) and putting $\psi = (u, v)$, we get*

$$(1.13) \qquad \begin{aligned} 2\partial_\alpha u = i(m - \phi)v, \qquad 2\partial_\beta v = i(m - \phi)u, \\ 4\partial_\alpha \partial_\beta \phi = -M^2 \phi + 2\Re(u\overline{v}). \end{aligned}$$

We denote the Fourier transform in one and two variables, respectively, by

$$(1.14) \qquad \widehat{\varphi}(\xi) = \int_{\mathbb{R}} \varphi(x) e^{-ix\xi}\, dx, \quad \widetilde{f}(\xi, \eta) = \iint_{\mathbb{R}^2} f(x, y) e^{-ix\xi - iy\eta}\, dx\, dy.$$

*For any pair of Hermitian matrices (γ_0, γ_1) satisfying (1.2), there is a unitary matrix U such that $U^* \gamma_j U = \sigma_j$ $(j = 0, 1)$. Hence $(U\psi, \phi)$ satisfies DKG with the above choice $(\gamma_0, \gamma_1) = (\sigma_0, \sigma_1)$, and so it suffices to treat this special case.

Then we have in the null coordinate

$$\widetilde{f}(\mu,\nu)_\times = \frac{1}{2}\iint_{\mathbb{R}^2} f(\alpha,\beta)^\times e^{-i\alpha\mu - i\beta\nu}\, d\alpha\, d\beta. \tag{1.15}$$

The constant multiple $1/2$ plays no role. The Sobolev spaces on these coordinate systems are defined by the norms

$$\|\varphi\|_{H^s_x} = \|\langle\xi\rangle\widehat{\varphi}(\xi)\|_{L^2_\xi(\mathbb{R})}, \qquad \|f\|_{H^{s_1}_\alpha H^{s_2}_\beta} = \|\langle\mu\rangle^{s_1}\langle\nu\rangle^{s_2}\widetilde{f}(\mu,\nu)_\times\|_{L^2_{\mu,\nu}(\mathbb{R}^2)}, \tag{1.16}$$

where $\langle x\rangle = \sqrt{1+|x|^2}$. We often abbreviate

$$\|u\|_{s_1,s_2} := \|u\|_{H^{s_1}_\alpha H^{s_2}_\beta}. \tag{1.17}$$

We denote the solution for DKG by

$$\mathfrak{u}(t) := \big(u(t), v(t), \phi(t), \partial_t\phi(t)\big) \tag{1.18}$$

and its space by

$$\mathcal{H}^{a,s} := H^a \times H^a \times H^s \times H^{s-1}. \tag{1.19}$$

For the product and trace estimates, we define the following relation between any three real numbers a, b, c:

$$c \prec \{a,b\} \tag{1.20}$$

holds true if and only if

$$a+b \geq 0, \qquad c \leq a, \qquad c \leq b, \qquad c \leq a+b-\frac{1}{2}, \tag{1.21}$$

and

$$c = a+b-\frac{1}{2} \implies a+b>0, \quad c<a,\ c<b. \tag{1.22}$$

The above relation gives the necessary and sufficient condition for the product estimate in the Sobolev space (see Lemma 2.2).

2. Local well-posedness of DKG: First proof

In this section, we prove the local well-posedness part of Theorem 1.1 separately in the three cases $s > -a$, $s = -a > 0$, and $s = -a = 0$ by the first approach, decomposing the solution into the free and the nonlinear parts. We recall that the second proof does not work at the endpoint $s = a = 0$.

2.1. Integral equations

First, note that we can make the initial norm and the mass constants m, M as small as we want by the rescaling

$$\begin{aligned} &\psi(t,x) \mapsto \lambda^{3/2}\psi(\lambda t, \lambda x), \qquad \phi(t,x) \mapsto \lambda\phi(\lambda t,\lambda x), \\ &m \mapsto \lambda m, \qquad M \mapsto \lambda M \end{aligned} \tag{2.1}$$

with $\lambda \to +0$. The critical exponent is $(a,s) = (-1,-1/2)$, which is quite below the well-posedness region.

Next, we recall that we may localize the problem in space-time thanks to the finite propagation property. More precisely, let $\chi(x) \in C^\infty(\mathbb{R})$ be a cutoff function satisfying

$$\chi(-x) = \chi(x), \quad \chi(x) = \begin{cases} 1, & (|x| \le 1), \\ 0, & (|x| \ge 2), \end{cases} \tag{2.2}$$

and $\chi_T(x) = \chi(x/T)$ for any $T > 0$.

For any $T > 0$ and $k \in \mathbb{Z}$, let $\mathfrak{u}_k(t)$ be a solution for $|t| < 2T$ satisfying

$$\mathfrak{u}_k(0) = \chi_{2T}(x - kT)\mathfrak{u}(0). \tag{2.3}$$

If we can construct $\mathfrak{u}_k(t)$ by the iteration or the fixed point argument, then the finite propagation property is inherited from the linear Dirac and Klein-Gordon equations; hence we have

$$|t| \le T, \quad |x - kT| \le T \implies \mathfrak{u}_k(t,x) = \mathfrak{u}(t,x). \tag{2.4}$$

At any $(a,s) \in \mathbb{R}^2$, (2.3) and (2.4), respectively, imply that

$$\sum_{k\in\mathbb{Z}} \|\mathfrak{u}_k(0)\|^2_{\mathcal{H}^{a,s}} \sim \|\mathfrak{u}(0)\|^2_{\mathcal{H}^{a,s}}, \qquad \|\mathfrak{u}(t)\|^2_{\mathcal{H}^{a,s}} \lesssim \sum_{k\in\mathbb{Z}} \|\mathfrak{u}_k(t)\|^2_{\mathcal{H}^{a,s}}, \tag{2.5}$$

for $|t| \le T$ uniformly in $T > 0$. Hence, for the local well-posedness, it suffices to solve $\mathfrak{u}_k$ by the iteration argument, and so, after translation in space-time, we may assume that the initial data is compactly supported around zero.

Next, we rewrite the equations by using the following integral and trace operators:

$$\begin{aligned} &J_\alpha f(\alpha,\beta)^\times := \int_0^\alpha f(\gamma,\beta)^\times \, d\gamma, \qquad J_\beta f(\alpha,\beta)^\times := \int_0^\beta f(\alpha,\delta)^\times \, d\delta, \\ &R_\alpha f(\alpha,\beta)^\times := f(0,\alpha), \qquad R_\beta f(\alpha,\beta)^\times := f(0,-\beta). \end{aligned} \tag{2.6}$$

Let u_F, v_F, ϕ_F denote the free parts of the solution given by

$$\begin{aligned} &u_F = R_\beta u, \qquad v_F = R_\alpha v, \qquad \phi_F(\alpha,\beta)^\times = \phi_+(\alpha) + \phi_-(\beta), \\ &\phi_\pm(x) = \phi(0,\pm x) \pm \int_0^{\pm x} \partial_t \phi(0,y)\, dy. \end{aligned} \tag{2.7}$$

Then the system (1.13) is equivalent to

$$\begin{aligned} u(\alpha,\beta)^\times &= u_F + (1 - R_\beta) J_\alpha (c_1 v + c_2 \phi v), \\ v(\alpha,\beta)^\times &= v_F + (1 - R_\alpha) J_\beta (c_3 u + c_4 \phi u), \\ \phi(\alpha,\beta)^\times &= \phi_F + (J_\alpha J_\beta - J_\alpha R_\alpha J_\beta - J_\beta R_\beta J_\alpha)(c_5 \phi + c_6 uv), \end{aligned} \tag{2.8}$$

with some complex constants c_1–c_6, satisfying $|c_1| + |c_3| \lesssim m$, $|c_5| \lesssim M^2$. It suffices to solve its localized version. We consider the system

$$\begin{aligned} u &= \chi_T(\alpha)[u_F + \chi_T(\beta) I_\alpha (c_1 v + c_2 \phi v)], \\ v &= \chi_T(\beta)[v_F + \chi_T(\alpha) I_\beta (c_3 u + c_4 \phi u)], \end{aligned} \tag{2.9}$$

$$\phi = \chi_T(\alpha,\beta)[\phi_F + I_{\alpha,\beta}(c_5\phi + c_6 uv)],$$

where $\chi_T(\alpha,\beta) := \chi_T(\alpha)\chi_T(\beta)$ and the operators $I_\alpha, I_\beta, I_{\alpha,\beta}$ are defined by

$$\begin{aligned} I_\alpha &= (1-R_\beta)J_\alpha, \qquad I_\beta = (1-R_\alpha)J_\beta, \\ I_{\alpha,\beta} &= I_\alpha I_\beta = I_\beta I_\alpha = J_\alpha J_\beta - J_\alpha R_\alpha J_\beta - J_\beta R_\beta J_\alpha. \end{aligned} \tag{2.10}$$

For the term with R_α, we use the identities

$$\chi_T(\alpha)J_\alpha = \chi_T(\alpha)J_\alpha\chi_{2T}(\alpha), \qquad \chi_T(\alpha)R_\alpha = R_\alpha\chi_T(\beta). \tag{2.11}$$

The following inequality is convenient to dispose of a cutoff in the Fourier space: for any $s,x,y \in \mathbb{R}$, we have

$$\langle x+y\rangle^s \lesssim \langle x\rangle^{|s|}\langle y\rangle^s. \tag{2.12}$$

2.2. Basic estimates

To solve the above integral equation, we need only estimates for the localized integrals, the products, and the restrictions in the Sobolev spaces. We state the first two estimates without proof for they are quite well known.

LEMMA 2.1

Let $s > 1/2$. Then

$$\Big\|\chi(x)\int_0^x f(t)\,dt\Big\|_{H^s} \lesssim \|f\|_{H^{s-1}}. \tag{2.13}$$

LEMMA 2.2

Let $(a,b,c) \in \mathbb{R}^3$. Then we have the bilinear estimate

$$\|fg\|_{H^c} \lesssim \|f\|_{H^a}\|g\|_{H^b} \tag{2.14}$$

if and only if $c \prec \{a,b\}$.

For a product with smooth functions, we have the following.

LEMMA 2.3

For any $a,b \in \mathbb{R}$ and any $\lambda(t,x)$, we have

$$\|\lambda u\|_{H^a_\alpha H^b_\beta} \lesssim \|\langle\tau\rangle^N\langle\xi\rangle^N\widetilde{\lambda}(\tau,\xi)\|_{\mathcal{M}(\mathbb{R}^2)}\|u\|_{H^a_\alpha H^b_\beta} \tag{2.15}$$

for any $N \ge |a|+|b|$, where $\mathcal{M}(\mathbb{R}^2)$ denotes the Banach space of complex Radon measures on $\mathbb{R}^2$ normed by the total variation.

In particular, this allows multiplication with arbitrary C_0^∞ functions of x, t or (t,x) in $H^a_\alpha H^b_\beta$.

Proof of Lemma 2.3

The Fourier transform is

$$\widetilde{\lambda u} = \iint \widetilde{\lambda}(\tau-\sigma,\xi-\eta)\widetilde{u}(\sigma,\eta)\,d\sigma\,d\eta, \tag{2.16}$$

and by (2.12), we have

$$\langle \tau \pm \xi \rangle^s \lesssim \langle \tau - \sigma \rangle^N \langle \xi - \eta \rangle^N \langle \sigma \pm \eta \rangle^s \tag{2.17}$$

if $N \geq |s|$. Hence we get the desired estimate by Minkowski. □

The following trace estimate is also quite elementary.

LEMMA 2.4

Let $a, b, c \in \mathbb{R}^3$. Then we have the linear estimate

$$\|f(x,x)\|_{H^c_x} \lesssim \|f(x,y)\|_{H^a_x H^b_y} \tag{2.18}$$

if and only if $c \prec \{a, b\}$ and $a + b > 1/2$.

Proof

The necessity is easily seen by applying the estimate to functions of the forms $f(x,y) = g(x)h(y)$ and $f(x,y) = \chi_T(x)g(x-y)$ with $T \gg 1$, respectively, for $c \prec \{a, b\}$ and $a + b > 1/2$. More precisely, it is reduced to the necessity for the above product estimate and that for the Sobolev embedding $H^{a+b}(\mathbb{R}) \subset L^\infty_{\mathrm{loc}}$, respectively.

For sufficiency, we use the Fourier transform

$$\begin{aligned} &\int_{\mathbb{R}} e^{-ix\xi} f(x,x)\, dx \\ &= \int_{\mathbb{R}^2} e^{-ix\xi} f(x,y)\delta(x-y)\, dy\, dx \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^3} e^{-ix\xi + i\eta(x-y)} f(x,y)\, d\eta\, dy\, dx \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \widetilde{f}(\xi - \eta, \eta)\, d\eta. \end{aligned} \tag{2.19}$$

By setting $F(\xi,\eta) = \langle \xi \rangle^a \langle \eta \rangle^b \widetilde{f}(\xi,\eta)$, the boundedness is equivalent to the estimate

$$\left\| \langle \xi \rangle^c \int \langle \xi - \eta \rangle^{-a} \langle \eta \rangle^{-b} F(\xi - \eta, \eta)\, d\eta \right\|_{L^2_\xi} \lesssim \|F\|_{L^2_{\xi,\eta}}. \tag{2.20}$$

Applying Schwarz to the integral in η, the left-hand side is bounded by

$$\int \int |F(\xi - \eta, \eta)|^2\, d\eta \Big[\int \langle \xi \rangle^{2c} \langle \xi - \eta \rangle^{-2a} \langle \eta \rangle^{-2b}\, d\eta \Big]\, d\xi. \tag{2.21}$$

This is bounded by $\|F\|^2_{L^2}$, provided that the second integral in η is uniformly bounded in ξ, which is true if $a + b > 1/2$ and $c \prec \{a, b\}$. □

Combining the above estimates, we obtain the following estimates on the Dirac and the Klein-Gordon equations.

LEMMA 2.5

Let $s_1, s_2 \in \mathbb{R}$ and $T > 0$. Then we have the linear estimates

$$\begin{aligned} &\|\chi_T(\alpha,\beta) I_\alpha f\|_{H_\alpha^{s_1} H_\beta^{s_2}} \lesssim \|f\|_{H_\alpha^{s_1-1} H_\beta^{s_2}}, \\ &\|\chi_T(\alpha,\beta) I_\beta f\|_{H_\alpha^{s_2} H_\beta^{s_1}} \lesssim \|f\|_{H_\alpha^{s_2} H_\beta^{s_1-1}} \end{aligned} \tag{2.22}$$

if and only if

$$s_1 > \frac{1}{2}, \quad s_1 \ge s_2, \quad s_1 + s_2 > \frac{1}{2}. \tag{2.23}$$

We have the linear estimate

$$\|\chi_T(\alpha,\beta) I_{\alpha,\beta} f\|_{H_\alpha^{s_1} H_\beta^{s_2}} \lesssim \|f\|_{H_\alpha^{s_1-1} H_\beta^{s_2-1}} \tag{2.24}$$

if and only if

$$s_1 > \frac{1}{2}, \qquad s_2 > \frac{1}{2}, \quad |s_1 - s_2| \le 1,\ s_1 + s_2 > \frac{3}{2}. \tag{2.25}$$

These estimates are essentially known in previous works. Here we are more concerned with the necessary conditions. (In fact, we do not use (2.24) in our proof.)

Proof

For the estimate on I_α, we have, by using (2.11) and (2.15),

$$\begin{aligned} &\|\chi_T(\alpha,\beta)(1-R_\beta) J_\alpha f\|_{H_\alpha^{s_1} H_\beta^{s_2}} \\ &\quad \lesssim \|\chi_T(\beta)\chi_T(\alpha) J_\alpha f\|_{H_\alpha^{s_1} H_\beta^{s_2}} + \|\chi_T(\alpha) R_\beta \chi_T(\alpha) J_\alpha f\|_{H_\alpha^{s_1} H_\beta^{s_2}} \\ &\quad \lesssim \|\chi_T(\alpha) J_\alpha f\|_{H_\alpha^{s_1} H_\beta^{s_2}} + \|R_\beta \chi_T(\alpha) J_\alpha f\|_{H_\beta^{s_2}} \end{aligned} \tag{2.26}$$

for any s_1, s_2. Then it is bounded by $\|f\|_{H_\alpha^{s_1-1} H_\beta^{s_2}}$, by Lemmas 2.1 and 2.4, if

$$s_1 > \frac{1}{2}, \qquad s_2 \prec \{s_1, s_2\}, \qquad s_1 + s_2 > \frac{1}{2}, \tag{2.27}$$

which is equivalent to (2.23). The estimate on I_β is obtained in the same way. Similarly, the estimate on $I_{\alpha,\beta}$ is obtained as follows. We have, by (2.11),

$$\begin{aligned} \chi_T(\alpha,\beta) I_{\alpha,\beta} f = \chi_T(\alpha) J_\alpha \chi_T(\beta) J_\beta f& \\ - \chi_T(\beta)\chi_T(\alpha) J_\alpha R_\alpha \chi_{2T}(\beta) J_\beta f& \\ - \chi_T(\alpha)\chi_T(\beta) J_\beta R_\beta \chi_{2T}(\alpha) J_\alpha f.& \end{aligned} \tag{2.28}$$

Hence using Lemmas 2.1, 2.4, and 2.3, we obtain (2.24) if $s_1 > 1/2$, $s_2 > 1/2$, and

$$s_1 - 1 \prec \{s_1 - 1, s_2\}, \quad s_2 - 1 \prec \{s_1, s_2 - 1\}, \quad s_1 + s_2 - 1 > \frac{1}{2}, \tag{2.29}$$

which are satisfied under the conditions (2.25).

It remains to check the necessity. For the first estimate, let $f = g'(\alpha)h(\beta)$ with $g \in H^{s_1}$ and $h \in H^{s_2}$. Then $f \in H_\alpha^{s_1-1}H_\beta^{s_2}$ and

$$\chi_T(\alpha,\beta)I_\alpha f = \chi_T(\alpha,\beta)g(\alpha)h(\beta) - \chi_T(\alpha,\beta)g(-\beta)h(\beta), \tag{2.30}$$

where the first term on the right-hand side is in $H_\alpha^{s_1}H_\beta^{s_2}$. To have the last term in $H_\alpha^{s_1}H_\beta^{s_2}$ for all g and h, we need $g(-\beta)h(\beta) \in H^{s_2}$, for which, by Lemma 2.2, we need $s_2 \prec \{s_1, s_2\}$. This requires $s_1 > 1/2$ and $s_1 \geq s_2$.

To see the remaining condition, let $f = g'(t)\chi_{2T}(x)$ with $g \in H^{s_1+s_2}$. Then

$$\begin{aligned} \|f\|_{H_\alpha^{s_1-1}H_\beta^{s_2}} &\sim \|\langle\tau+\xi\rangle^{s_1-1}\langle\tau-\xi\rangle^{s_2}\widehat{g'}(\tau)\widehat{\chi_{2T}}(\xi)\|_{L^2_{\tau,\xi}} \\ &\lesssim \|\langle\tau\rangle^{s_1+s_2-1}\widehat{g'}(\tau)\|_{L^2_\tau}\|\langle\xi\rangle^N\widehat{\chi_{2T}}(\xi)\|_{L^2_\xi} \lesssim \|g'\|_{H^{s_1+s_2-1}} < \infty, \end{aligned} \tag{2.31}$$

where $N \geq |s_1 - 1| + |s_2|$ and we used (2.12). On the other hand, we have

$$\chi_T(\alpha,\beta)I_\alpha f = \chi_T(\alpha,\beta)g(t) - \chi_T(\alpha,\beta)g(0), \tag{2.32}$$

where the first term on the right-hand side belongs to $H_\alpha^{s_1}H_\beta^{s_2}$ by the same computation as above, while the last term is bounded for $g \in H^{s_1+s_2}$ only if $s_1 + s_2 > 1/2$. The necessity for I_β is seen by the symmetry.

Next, we check the necessity of (2.25). Let

$$f = \chi_{2T}(x)g''(t), \quad g \in H^{s_1+s_2}(\mathbb{R}). \tag{2.33}$$

Then we have $\|f\|_{H_\alpha^{s_1-1}H_\beta^{s_2-1}} \lesssim \|g''\|_{H^{s_1+s_2-2}} < \infty$ as in (2.31), and

$$\begin{aligned} \chi_T(\alpha,\beta)I_{\alpha,\beta}f &= \chi_T(\alpha,\beta)\int_0^t (t-s)g''(s)\,ds \\ &= \chi_T(\alpha,\beta)[g(t) - g(0) - tg'(0)], \end{aligned} \tag{2.34}$$

which is bounded for $g \in H^{s_1+s_2}$ only if $s_1 + s_2 - 1 > 1/2$. To see the remaining conditions, let $f = g'(\alpha)h'(\beta)$ with $g \in H^{s_1}$, $h \in H^{s_2}$, assuming that

$$s_1 + s_2 > \frac{3}{2}, \quad s_1 \leq s_2. \tag{2.35}$$

Then we have $f \in H_\alpha^{s_1-1}H_\beta^{s_2-1}$ and

$$I_{\alpha,\beta}f = g(\alpha)h(\beta) - g(\alpha)h(-\alpha) - \int_{-\alpha}^{\beta} g(-\gamma)h'(\gamma)\,d\gamma, \tag{2.36}$$

where the first term on the right-hand side is bounded in $H_\alpha^{s_1}H_\beta^{s_2}$, so is the second term after the cutoff by $\chi_T(\beta)$ since (2.35) implies $s_1 \prec \{s_1, s_2\}$. Let $H(x) = \int_0^x g(-y)h'(y)\,dy$. Then the last term equals $H(\beta) - H(-\alpha)$; hence for the bound (2.24) we need

$$\|f\|_{H_\alpha^{s_1-1}H_\beta^{s_2-1}} \gtrsim \|\chi_T(\alpha)H(\beta)\|_{H_\alpha^{s_1}H_\beta^{s_2}} \gtrsim \|H\|_{H^{s_2}} \gtrsim \|g(-x)h'(x)\|_{H^{s_2-1}}, \tag{2.37}$$

which requires $s_2 - 1 \prec \{s_1, s_2 - 1\}$, so we need $s_1 > 1/2$ and $s_1 \geq s_2 - 1$. By symmetry, we also need $s_2 > 1/2$ and $s_2 \geq s_1 - 1$ and thus all of (2.25). □

REMARK 2.6

The conditions $s_1+s_2>1/2$ in (2.23) and $s_1+s_2>3/2$ in (2.25) were the source for the lower bounds on the regularity in the previous works, $a>-1/4$ for DKG and QD, $s>3/4$ for WM. Here we explain the problem for WM in the context of [17]. Neglecting some unnecessary computations, we can extract the following estimate as the essence of their proof (see, e.g., [17, p. 1131])

$$\begin{aligned}\|\chi_T(t)I_{\alpha,\beta}(\partial_\alpha\phi\partial_\beta\phi)\|_{H^s_\alpha H^s_\beta} &\lesssim \|\partial_\alpha\phi\partial_\beta\phi\|_{H^{s-1}_\alpha H^{s-1}_\beta} \\ &\lesssim \|\partial_\alpha\phi\|_{H^{s-1}_\alpha H^s_\beta}\|\partial_\beta\phi\|_{H^s_\alpha H^{s-1}_\beta} \lesssim \|\phi\|^2_{H^s_\alpha H^s_\beta}.\end{aligned} \tag{2.38}$$

The second inequality follows from (2.14) for $s>1/2$, while the last one is trivial. However, the first inequality requires $2s>3/2$ by (2.25). The example in (2.33) is sufficient to see this condition. The authors in [17] claimed the above estimates for all $s>1/2$ and claimed more explicitly in [17, Lemma 3.5] that they had

$$\|\chi_T(t)I_{\alpha,\beta}\phi\|_{H^{s_1}_\alpha H^{s_2}_\beta} \lesssim \|\phi\|_{H^{s_1-1}_\alpha H^{s_2-1}_\beta} \tag{2.39}$$

for all $s_1,s_2\ge 1/2$, which is far from the correct condition (2.25). The error is in the second-to-last step of their proof, where they overlooked the region $|\xi|\ll|\tau|$ for ϕ. Hence their proof of well-posedness works only for $s>3/4$.

2.3. DKG for $s+a>0$

First, we prove the local well-posedness of DKG in the subcritical case $s+a>0$.

THEOREM 2.7

Let $(a,s)\in\mathbb{R}^2$ satisfy

$$a\le s\le a+1,\quad s+a>0,\ a>-\frac{1}{2}. \tag{2.40}$$

Then for any initial data $\mathfrak{u}(0)\in\mathcal{H}^{a,s}$, there exists $T=T(\|\mathfrak{u}(0)\|_{\mathcal{H}^{a,s}})>0$ such that (2.9) have a unique solution (u,v,ϕ) satisfying

$$\begin{aligned}&u,v\in C(H^a),\qquad \phi\in C(H^s),\qquad \partial_t\phi\in C(H^{s-1}),\\ &u\in H^n_\alpha H^a_\beta+H^b_\alpha H^s_\beta,\qquad v\in H^a_\alpha H^n_\beta+H^s_\alpha H^b_\beta,\\ &\phi\in H^n_\alpha H^s_\beta+H^s_\alpha H^n_\beta+H^{a+1}_\alpha H^{a+1}_\beta\end{aligned} \tag{2.41}$$

for any n and any b such that $b-1\prec\{a,s\}$. The solution is unique if it is in those spaces for some n and b satisfying (2.42).

REMARK 2.8

We solve the Cauchy problem locally in space-time by the fixed point argument. Strictly speaking, we thus obtain local bounds on the space-time norms in the statement (2.41). However, the fixed point argument implies that the space-time norms are bounded in each localized region by the corresponding localized initial data. Then taking the ℓ^2-sum over the decomposition by the same argument as in (2.5), we can recover the spatially global norms.

Proof
We may choose n and b (by increasing it if necessary) such that

$$n > \max(|a|, b) + 2, \quad b > \frac{1}{2}, \quad b \geq s(= |s|), \quad b - 1 \prec \{a, s\}, \tag{2.42}$$

thanks to (2.40). Note that $b > 1/2$ is impossible in the critical case $a + s = 0$, which is treated separately.

We estimate the iteration map $(u, v, \phi) \to (u^\sharp, v^\sharp, \phi^\sharp)$ for (2.9) defined by

$$\begin{aligned} u^\sharp &:= u_0 + u_1^\sharp, \quad u_0 := \chi_T(\alpha) u_F, u_1^\sharp := \chi_T(\alpha, \beta) I_\alpha (c_1 v + c_2 \phi v), \\ v^\sharp &:= v_0 + v_1^\sharp, \quad v_0 := \chi_T(\beta) v_F, v_1^\sharp := \chi_T(\alpha, \beta) I_\beta (c_3 u + c_4 \phi u), \\ \phi^\sharp &:= \phi_0 + \phi_1^\sharp, \quad \phi_0 := \chi_T(\alpha, \beta) \phi_F, \phi_1^\sharp := \chi_T(\alpha, \beta) I_{\alpha,\beta} (c_5 \phi + c_6 uv), \end{aligned} \tag{2.43}$$

in the following function spaces:

$$\begin{aligned} u &= u_0 + u_1 \in H_\alpha^b H_\beta^a, \quad u_0 \in H_\alpha^n H_\beta^a, u_1 \in H_\alpha^b H_\beta^s, \\ v &= v_0 + v_1 \in H_\alpha^a H_\beta^b, \quad v_0 \in H_\alpha^a H_\beta^n, v_1 \in H_\alpha^s H_\beta^b, \\ \phi &= \phi_0 + \phi_1 \in H_\alpha^s H_\beta^s, \quad \phi_0 \in H_\alpha^s H_\beta^s, \phi_1 \in H_\alpha^{a+1} H_\beta^{a+1}. \end{aligned} \tag{2.44}$$

Since the estimates on the free parts u_0, v_0, ϕ_0 are trivial, it suffices to estimate $u_1^\sharp$, $v_1^\sharp$, and $\phi_1^\sharp$. The estimate on $u_1^\sharp$ is immediate from Lemmas 2.5 and 2.2:

$$\|\chi_T(\alpha, \beta) I_\alpha(\phi v)\|_{b,s} \lesssim \|\phi v\|_{b-1,s} \lesssim \|\phi\|_{s,s} \|v\|_{a,b}, \tag{2.45}$$

where we used

$$b > \frac{1}{2}, \qquad b + s > \frac{1}{2}, \qquad s \prec \{b, s\}, \qquad b - 1 \prec \{a, s\}, \tag{2.46}$$

which follows from the assumptions together with (2.42). We obtain the estimate on $v_1^\sharp$ just by exchanging α and β.

For the estimate on $\phi_1^\sharp$, the above argument does not work for lower regularity. Instead, we expand $I_{\alpha,\beta}$ by (2.10) and also u and v, depending on the direction of integrations. The term without restriction is estimated simply by

$$\|\chi_T(\alpha, \beta) J_\alpha J_\beta uv\|_{a+1,a+1} \lesssim \|uv\|_{a,a} \lesssim \|u\|_{b,a} \|v\|_{a,b} \tag{2.47}$$

since $a > -1/2$ and $a \prec \{a, b\}$. For the term with R_α, we expand $v = v_0 + v_1$ and use the identities in (2.11) and

$$R_\alpha \chi_{2T}(\beta) J_\beta(uv) = v_F R_\alpha \chi_{2T}(\beta) J_\beta\big(u \chi_T(\beta)\big) + R_\alpha \chi_{2T}(\beta) J_\beta(uv_1), \tag{2.48}$$

where we used the fact that v_F is independent of β. Then we get

$$\begin{aligned} &\|\chi_T(\alpha, \beta) J_\alpha R_\alpha J_\beta uv\|_{a+1,a+1} \\ &\quad \lesssim \|\chi_{2T}(\alpha) R_\alpha J_\beta uv\|_{H_\alpha^a} \\ &\quad \lesssim \big\|v_F R_\alpha \chi_{2T}(\beta) J_\beta\big(u \chi_T(\beta)\big)\big\|_{H_\alpha^a} \\ &\qquad + \|R_\alpha \chi_{2T}(\beta) J_\beta(uv_1)\|_{H_\alpha^a}. \end{aligned} \tag{2.49}$$

The second last term is bounded by using Lemmas 2.2, 2.4, and 2.1:

$$\begin{aligned}
&\big\|v_F R_\alpha \chi_{2T}(\beta) J_\beta\big(u\chi_T(\beta)\big)\big\|_{H^a_\alpha} \\
&\lesssim \|v_F\|_{H^a_\alpha} \big\|R_\alpha \chi_{2T}(\beta) J_\beta\big(u\chi_T(\beta)\big)\big\|_{H^b_\alpha} \\
&\lesssim \|v(0)\|_{H^a} \big\|\chi_{2T}(\beta) J_\beta\big(u\chi_T(\beta)\big)\big\|_{H^{b,a+1}} \\
&\lesssim \|v(0)\|_{H^a} \|u\chi_T(\beta)\|_{H^{b,a}} \lesssim \|v(0)\|_{H^a} \|u\|_{H^{b,a}},
\end{aligned} \tag{2.50}$$

where we used

$$a \prec \{a,b\}, \quad b \prec \{b, a+1\}, \quad b+a > -\frac{1}{2}, \quad a > -\frac{1}{2}. \tag{2.51}$$

The last term in (2.49) is estimated by

$$\begin{aligned}
\|R_\alpha \chi_{2T}(\beta) J_\beta(uv_1)\|_{H^a_\alpha} &\lesssim \|\chi_{2T}(\beta) J_\beta(uv_1)\|_{s,a+1} \lesssim \|uv_1\|_{s,a} \\
&\lesssim \|u\|_{b,a} \|v_1\|_{s,b},
\end{aligned} \tag{2.52}$$

where we used

$$a \prec \{s, a+1\}, \quad s+a > -\frac{1}{2}, \quad a > -\frac{1}{2}, \quad s \prec \{s,b\}, \quad a \prec \{a,b\}. \tag{2.53}$$

The estimate for the term with R_β is obtained in the same way, and thus we get the desired estimate for $\phi^\sharp$.

Therefore we get a unique solution of (2.9) satisfying (2.44) by the iteration argument. It remains to check the time continuity in (2.41). For u, it suffices to estimate u_1 since $u_0 \in C(H^a)$ is obvious. We define an operator P_t for any fixed $t \in \mathbb{R}$ by the time translation

$$(P_t f)(\alpha, \beta)^\times := f(\alpha - t, \beta - t)^\times. \tag{2.54}$$

By Lemma 2.4, we have a uniform bound for all t,

$$\|u_1(t,x)\|_{H^a_x} \sim \|R_\alpha P_t u_1\|_{H^a_\alpha} \lesssim \|P_t u_1\|_{H^b_\alpha H^s_\beta} = \|u_1\|_{H^b_\alpha H^s_\beta} < \infty, \tag{2.55}$$

and the continuity follows from the fact that $P_t \to I$ strongly as $t \to 0$. Thus we get $u \in C(H^a)$. The continuity of v and ϕ is proved in the same way. For that of ϕ_t, it suffices to show

$$\partial_\alpha \phi, \partial_\beta \phi \in C_t(H^{s-1}_x), \tag{2.56}$$

because $\partial_t \phi = \partial_\alpha \phi + \partial_\beta \phi$. Again we treat only ϕ_1 since ϕ_0 is easy. After expanding $I_{\alpha,\beta}$ in $\partial_\alpha \phi_1$ by (2.10), the term where ∂_α hits $\chi_T(\alpha)$ is treated by the same estimates as for $\phi^\sharp$. The other part is given by

$$\begin{aligned}
R_\alpha P_t \chi_T(\alpha,\beta) \partial_\alpha I_{\alpha,\beta}(uv) &= R_\alpha P_t \chi_T(\alpha,\beta) I_\beta(uv) \\
&= R_\alpha P_t \chi_T(\alpha,\beta) J_\beta(uv) \\
&\quad - [R_\alpha P_t \chi_T(\alpha,\beta)][R_\alpha J_\beta(uv)],
\end{aligned} \tag{2.57}$$

which is estimated in H^{s-1}_α by the same argument as in (2.50)–(2.52) since $s-1 \le a$. Thus we have the desired estimate for $\partial_\alpha \phi$ as well as the continuity. Since the estimate for $\partial_\beta \phi$ is the same, we obtain the property of ϕ_t in (2.41). $\square$

2.4. DKG for $s+a=0$ and $s>0$

Next, we prove local well-posedness in the critical case where $s+a=0$ but $s>0$.

THEOREM 2.9

Let $(a,s)\in\mathbb{R}^2$ satisfy

$$s+a=0, \quad 0<s<\frac{1}{2}. \tag{2.58}$$

Then for any initial data $\mathfrak{u}(0)\in\mathcal{H}^{a,s}$, there exists $T=T(\|\mathfrak{u}(0)\|_{\mathcal{H}^{a,s}})>0$ such that (2.9) have a unique solution (u,v,ϕ) satisfying

$$\begin{aligned} &u,v\in C(H^a), \qquad \phi\in C(H^s), \qquad \phi_t\in C(H^{s-1}),\\ &u\in H^n_\alpha H^a_\beta+H^{1-s}_\alpha H^r_\beta+H^{b'}_\alpha H^b_\beta, \qquad v\in H^a_\alpha H^n_\beta+H^r_\alpha H^{1-s}_\beta+H^b_\alpha H^{b'}_\beta,\\ &\phi\in H^n_\alpha H^s_\beta+H^s_\alpha H^n_\beta+H^b_\alpha H^b_\beta, \end{aligned} \tag{2.59}$$

for any n, $b<1-s$, $b'<1/2$, and $0<r<s$. The uniqueness holds for the solution in those spaces for some n,b,b',r satisfying (2.60).

Proof

Since $s<1/2$ and $r<s$, we may assume that

$$b'-s>b-\frac{1}{2}>0, \quad b'+b+r>1, \quad b'+s-r\geq\frac{1}{2} \tag{2.60}$$

by increasing b, b', and r if necessary (e.g., let $b'=1/2-\varepsilon$, $b=1-s-2\varepsilon$, and $r=s-\varepsilon$ with $\varepsilon>0$ sufficiently small).

As in Theorem 2.7, we solve (2.9) by iteration for (2.43) in the following function spaces:

$$\begin{aligned} &u=u_0+u_1\in H^{b'}_\alpha H^a_\beta, \quad u_1\in H^{1-s}_\alpha H^r_\beta+H^{b'}_\alpha H^b_\beta,\\ &v=v_0+v_1\in H^a_\alpha H^{b'}_\beta, \quad v_1\in H^r_\alpha H^{1-s}_\beta+H^b_\alpha H^{b'}_\beta,\\ &\phi=\phi_0+\phi_1\in H^s_\alpha H^s_\beta, \quad \phi_1\in H^b_\alpha H^b_\beta. \end{aligned} \tag{2.61}$$

For the estimate on $u_1^\sharp$, we decompose ϕ and v differently as follows. Let

$$\phi=\phi_2+\phi_3, \quad \begin{aligned} &\phi_2:=\chi_T(\alpha,\beta)\phi_-(\beta)+\phi_1\in H^b_\alpha H^s_\beta,\\ &\phi_3:=\chi_T(\alpha,\beta)\phi_+(\alpha)\in H^s_\alpha H^b_\beta, \end{aligned} \tag{2.62}$$

where $\phi_\pm$ are given in (2.7), and

$$\begin{aligned} &v=v_0+v_1=v_2+v_3, \qquad v_1=v_1'+v_3, \qquad v_2=v_0+v_1',\\ &v_1'\in H^r_\alpha H^{1-s}_\beta,\ v_2\in H^a_\alpha H^b_\beta,\ v_3\in H^b_\alpha H^{b'}_\beta. \end{aligned} \tag{2.63}$$

Lemma 2.1 is not applicable to the contribution of $\phi_3 v_0$, so we replace J_α with $\widetilde{J}_\alpha$ defined by

$$\widetilde{J}_\alpha f=\chi_{2T}(\alpha)\int_{-\infty}^{\alpha}\chi_{2T}(\gamma)f(\gamma,\beta)^\times\,d\gamma, \tag{2.64}$$

which is bounded by $H^a_\alpha H^b_\beta \to H^{a+1}_\alpha H^b_\beta$ for any $a, b \in \mathbb{R}$. (It is easy for $a \in \mathbb{N}$ and is extended to $a \in \mathbb{R}$ by duality and interpolation.) We have

$$\chi_T(\alpha,\beta) I_\alpha = \chi_T(\alpha,\beta)(1-R_\beta)\widetilde{J}_\alpha, \tag{2.65}$$

so we can expand $u_1^\sharp$ as

$$\begin{aligned} &\|\chi_T(\alpha,\beta) I_\alpha(\phi v)\|_{H^{1-s}_\alpha H^r_\beta + H^{b'}_\alpha H^b_\beta} \\ &\quad \lesssim \|\chi_T(\alpha)\widetilde{J}_\alpha \phi v\|_{H^{1-s}_\alpha H^r_\beta + H^{b'}_\alpha H^b_\beta} + \|R_\beta \chi_T(\alpha)\widetilde{J}_\alpha \phi v\|_{H^r_\beta}. \end{aligned} \tag{2.66}$$

Lemma 2.4 implies that the last term is absorbed by the preceding one since

$$r \prec \{1-s, r\}, \qquad r \prec \{b', b\}, \quad 1-s+r > \frac{1}{2},\ b'+b > \frac{1}{2}. \tag{2.67}$$

We estimate the main term by using the decompositions (2.62) and (2.63), Lemma 2.2, and the regularity gain by $\widetilde{J}_\alpha$,

$$\begin{aligned} &\|\chi_T(\alpha)\widetilde{J}_\alpha \phi_3 v_2\|_{b',b} \lesssim \|\phi_3\|_{s,b}\|v_2\|_{-s,b}, \\ &\|\chi_T(\alpha)\widetilde{J}_\alpha \phi_3 v_3\|_{1+s,b'} \lesssim \|\phi_3\|_{s,b}\|v_3\|_{b,b'}, \\ &\|\chi_T(\alpha)\widetilde{J}_\alpha \phi_2 v_2\|_{1-s,s} \lesssim \|\phi_2\|_{b,s}\|v_2\|_{-s,b}, \\ &\|\chi_T(\alpha)\widetilde{J}_\alpha \phi_2 v_3\|_{1+b,r} \lesssim \|\phi_2\|_{b,s}\|v_3\|_{b,b'}, \end{aligned} \tag{2.68}$$

where we used

$$b > \frac{1}{2} > b' > s > 0, \quad b'-1 \prec \{s,-s\},\ r \prec \{s, b'\},\ c \prec \{c, b\}, \tag{2.69}$$

for $c = b, s, b'$. Thus we obtain

$$\|\chi(\alpha)\widetilde{J}_\alpha \phi v\|_{H^{1-s}_\alpha H^r_\beta + H^{b'}_\alpha H^b_\beta} \lesssim \|\phi\|_{H^b_\alpha H^s_\beta + H^s_\alpha H^b_\beta}\|v\|_{H^{-s}_\alpha H^b_\beta + H^b_\alpha H^{b'}_\beta} \tag{2.70}$$

and the desired estimate for $u^\sharp$. The estimate for $v^\sharp$ is the same by the symmetry.

Next, we estimate $\phi_1^\sharp$, expanding $I_{\alpha,\beta}$ by (2.10). For the part without R_*, we have, by Lemma 2.1,

$$\begin{aligned} \|\chi_T(\alpha,\beta) J_\alpha J_\beta(uv)\|_{b,b} &\lesssim \|uv\|_{b-1,b-1} \\ &\lesssim \|u\|_{H^b_\alpha H^{-s}_\beta + H^{b'}_\alpha H^b_\beta}\|v\|_{H^{-s}_\alpha H^b_\beta + H^b_\alpha H^{b'}_\beta}, \end{aligned} \tag{2.71}$$

where we used

$$b > b' > s > 0, \quad b-1 \prec \{b', -s\}. \tag{2.72}$$

For the term with R_α, we use (2.11) and (2.48). Then we estimate as in (2.49),

$$\begin{aligned} &\|\chi_T(\alpha,\beta) J_\alpha R_\alpha J_\beta uv\|_{b,b} \\ &\quad \lesssim \big\|v_F R_\alpha \chi_{2T}(\beta) J_\beta\big(u\chi_T(\beta)\big)\big\|_{H^{b-1}_\alpha} + \|R_\alpha \chi_{2T}(\beta) J_\beta(uv_1)\|_{H^{b-1}_\alpha}. \end{aligned} \tag{2.73}$$

The second last term is estimated by Lemma 2.2:

$$\big\|v_F R_\alpha \chi_{2T}(\beta) J_\beta\big(u\chi_T(\beta)\big)\big\|_{H^{b-1}_\alpha} \lesssim \|v_F\|_{H^{-s}_\alpha}\big\|R_\alpha \chi_{2T}(\beta) J_\beta\big(u\chi_T(\beta)\big)\big\|_{H^{b'}}, \tag{2.74}$$

where the last norm is bounded by

$$\big\|\chi_{2T}(\beta)J_\beta\big(u\chi_T(\beta)\big)\big\|_{H^b_\alpha H^{1-s}_\beta+H^{b'}_\alpha H^{1+b}_\beta} \lesssim \|u\|_{H^b_\alpha H^{-s}_\beta+H^{b'}_\alpha H^b_\beta}, \tag{2.75}$$

where we used Lemmas 2.4 and 2.1 with $b>1/2>b'>s>0$ and

$$b-1 \prec \{b',-s\},\ b' \prec \{b,1-s\},\ b' \prec \{b',1+b\}. \tag{2.76}$$

The last term in (2.73) is dominated by

$$\begin{aligned}&\|\chi_{2T}(\beta)J_\beta(uv_1)\|_{H^{b'+r-1/2}_\alpha H^b_\beta}\\ &\quad\lesssim \|uv_1\|_{H^{b'+r-1/2}_\alpha H^{b-1}_\beta} \lesssim \|u\|_{H^b_\alpha H^{-s}_\beta+H^{b'}_\alpha H^b_\beta}\|v_1\|_{H^r_\alpha H^b_\beta+H^b_\alpha H^{b'}_\beta},\end{aligned} \tag{2.77}$$

where we used Lemmas 2.4, 2.1, and 2.2 together with

$$\begin{aligned}&b-1 \prec \Big\{b'+r-\frac{1}{2},b\Big\}, \qquad b'+r-\frac{1}{2} \prec \{b',r\}, \qquad b-1 \prec \{b',-s\},\\ &b>\frac{1}{2}>b'>s>r>0.\end{aligned} \tag{2.78}$$

Thus we obtain the desired estimate on $\phi^\sharp$ and so a unique solution by the iteration argument.

It remains to check the time continuity in (2.59) by using P_t given in (2.54). The free parts are easy and so omitted. First, we estimate u_1,

$$\|u_1(t,x)\|_{H^a_x} \sim \|R_\alpha P_t u_1\|_{H^a_\alpha} \lesssim \|u_1\|_{H^{1-s}_\alpha H^r_\beta+H^{b'}_\alpha H^b_\beta}, \tag{2.79}$$

where we used Lemma 2.4 with the conditions

$$a \prec \{1-s,r\}, \qquad a \prec \{b',b\}, \quad r-s>-\frac{1}{2},\ b'+b>\frac{1}{2}. \tag{2.80}$$

Thus we get $u_1 \in C_t(H^a_x)$. The estimates for v_1 and ϕ_1 are similar.

The estimate for $\partial_t\phi \in C_t(H^{s-1}_x)$ is reduced to estimates similar to (2.74)–(2.77) by the same argument as in (2.56)–(2.57). Thus we obtain (2.59). □

2.5. DKG for $s=a=0$

Finally, we give local well-posedness at the endpoint $s=a=0$.

THEOREM 2.10

For any initial data $\mathfrak{u}(0) \in \mathcal{H}^{0,0}$, there exists $T=T(\|\mathfrak{u}(0)\|_{\mathcal{H}^{0,0}})>0$ such that (2.9) has a unique solution (u,v,ϕ) satisfying

$$\begin{aligned}&u,v \in C(L^2), \qquad \phi \in C(L^2), \qquad \phi_t \in C(H^{-1}),\\ &u \in L^2_\beta L^\infty_\alpha, \qquad v \in L^2_\alpha L^\infty_\beta, \qquad \phi \in L^2_\beta L^\infty_\alpha + L^2_\alpha L^\infty_\beta.\end{aligned} \tag{2.81}$$

Proof

We estimate the iteration map (2.43) in the above function spaces. The bounds on the free parts are obvious. For $u_1^\sharp$, we have, by Hölder and Minkowski,

$$\begin{aligned}\|\chi_T(\alpha,\beta)I_\alpha(\phi v)\|_{L^2_\beta L^\infty_\alpha} &\lesssim \|\chi_{2T}(\alpha,\beta)\phi v\|_{L^2_\beta L^1_\alpha}\\ &\lesssim \|\chi_{2T}(\alpha,\beta)\phi\|_{L^2_\alpha L^2_\beta}\|v\|_{L^\infty_\beta L^2_\alpha}\\ &\lesssim \sqrt{T}\|\phi\|_{L^2_\beta L^\infty_\alpha + L^2_\alpha L^\infty_\beta}\|v\|_{L^2_\alpha L^\infty_\beta},\end{aligned} \tag{2.82}$$

and $v_1^\sharp$ is estimated in the same way. Similarly, for $\phi_1^\sharp$,

$$\begin{aligned}\|\chi_T(\alpha,\beta)I_{\alpha,\beta}(uv)\|_{L^2_\beta L^\infty_\alpha} &\lesssim \|\chi_{2T}(\alpha,\beta)uv\|_{L^1_\alpha L^1_\beta}\\ &\le \|\chi_{2T}(\alpha)u\|_{L^2_\alpha L^2_\beta}\|\chi_{2T}(\beta)v\|_{L^2_\alpha L^2_\beta}\\ &\lesssim T\|u\|_{L^2_\beta L^\infty_\alpha}\|v\|_{L^2_\alpha L^\infty_\beta}.\end{aligned} \tag{2.83}$$

Thus we obtain a unique local solution by iteration.

It remains to show (2.81). For u, we have

$$\|u(t,x)\|_{L^2_x} \le \|\sup_\alpha u(\alpha, t-x)^\times\|_{L^2_x} \lesssim \|u\|_{L^2_\beta L^\infty_\alpha}, \tag{2.84}$$

and the estimate for v is obtained in the same way; the same is true for ϕ after decomposition into pieces in $L^2_\beta L^\infty_\alpha$ and in $L^2_\alpha L^\infty_\beta$. For $\partial_t\phi$, it suffices to estimate $\partial_\alpha\phi$ and $\partial_\beta\phi$. We have

$$\begin{aligned}\|\partial_\alpha\phi_1\|_{L^\infty_t L^2_x} &\lesssim \|\partial_\alpha\phi_1\|_{L^2_\alpha L^\infty_\beta + L^2_\beta L^\infty_\alpha}\\ &\lesssim \|\chi_T'(\alpha)\chi_T(\beta)I_{\alpha,\beta}(uv)\|_{L^2_\beta L^\infty_\alpha} + \|\chi_T(\alpha,\beta)I_\beta(uv)\|_{L^2_\alpha L^\infty_\beta},\end{aligned} \tag{2.85}$$

and then the last term is estimated as in (2.82), and the second last one as in (2.83). Thus we obtain

$$\|\partial_\alpha\phi_1\|_{L^\infty_t L^2_x} \lesssim \|u\|_{L^2_\beta L^\infty_\alpha}\|v\|_{L^2_\alpha L^\infty_\beta}, \tag{2.86}$$

and the continuity in (2.81) follows from the strong continuity of time translation in the completion of $C_0^\infty(|x|+|t|<2T)$ in each space. □

2.6. Global well-posedness of DKG

Now we prove the global well-posedness part of Theorem 1.1 using the local estimates obtained in Sections 2.3 and 2.5.

First, we consider the endpoint $a = s = 0$. We have the charge conservation $\|\psi(t)\|_{L^2_x} = \|\psi(0)\|_{L^2_x}$, and by the energy estimate for the Klein-Gordon equation,* we have, for any $\varepsilon > 0$ and any $0 < T \lesssim 1$,

$$\begin{aligned}\|\phi_1(T)\|_{H_x^{1/2-\varepsilon}} + \|\partial_t\phi_1(T)\|_{H_x^{-1/2-\varepsilon}} &\lesssim \|\psi^*\gamma_0\psi\|_{L^1_t(0,T;H^{-1/2-\varepsilon})}\\ &\lesssim \int_0^T \|\psi(t)\|^2_{L^2_x}\,dt \lesssim T\|\psi(0)\|^2_{L^2_x},\end{aligned} \tag{2.87}$$

*In the massless case $M = 0$, we use $|\sin(t|\xi|)/|\xi|| \lesssim |t|$.

where $\phi_1 := \phi - \phi_F$ denotes the nonlinear part. Thus we obtain an a priori bound on $\mathfrak{u}(t)$ in $\mathcal{H}^{0,0}$. Since the local existence time is bounded below in terms of the $\mathcal{H}^{0,0}$-norm, we can extend the solution globally by the standard argument. Thus we obtain the global well-posedness for $a = s = 0$.

It remains to show global persistence of regularity. In the following argument, all norms should be considered locally in time (but without any restriction in extent). By (2.81), we have also $u \in L^2_\beta L^\infty_\alpha$, $v \in L^2_\alpha L^\infty_\beta$, and hence $uv \in L^2_{\alpha,\beta}$. Then by the same argument as in Section 2.3, we obtain $\phi_1 \in H^1_\alpha H^1_\beta \subset C_t(H^1)$, and so $\partial_t \phi_1 \in C_t(L^2)$. In particular, if $\mathfrak{u}(0) \in \mathcal{H}^{0,s}$ for some $0 < s \le 1$, then $\phi \in H^s_\alpha H^s_\beta \cap C_t(H^s)$ and $\phi_t \in C_t(H^{s-1})$, which implies global well-posedness at $(0, s)$.

Next, we consider the case $a = s \in (0, 1]$. We already know that the solution is global in $\mathcal{H}^{0,a}$, and moreover, $\phi \in H^a_\alpha H^a_\beta$. This is sufficient for the estimate on ψ in Section 2.3, and since the equation of ψ is linear in ψ, the estimate does not blow up. Thus we obtain the global well-posedness in $\mathcal{H}^{a,a}$ for $a \in [0, 1]$. Then the estimates in Section 2.3 imply that $\phi_1 \in H^{a+1}_\alpha H^{a+1}_\beta \subset C_t(H^{a+1})$, and so $\partial_t \phi_1 \in C_t(H^a)$. This gives the global well-posedness in $\mathcal{H}^{a,s}$ for $s \in [a, a+1]$ as well.

Thus we have obtained global well-posedness in the regions $0 \le a \le 1$ and $a \le s \le a+1$. We can extend this to $k \le a \le k+1$ for all $k \in \mathbb{N}$ by induction or by repeating the argument in the previous paragraph. □

3. Bilinear estimates

In this section we prove bilinear estimates that allow us to prove the local well-posedness of DKG, QD, and WM. As explained in Remark 2.6, the Sobolev space $H^a_\alpha H^b_\beta$ is not sufficient by itself for the lower regularity because it fails to control restriction to a fixed t, which is included in the integral equation. However, the product or the quadratic nonlinearity actually behaves better. In fact, if we do not separate the estimate into the product part and the integral part, then we can still control the restriction even in the lower regularity (except for the endpoint). Hence if we add the information about the t trace to our norm, then we can separate the estimate into the product and the integration. The minimal addition in terms of $|\widetilde{u}|$ is obviously given by

$$\|\langle\xi\rangle^a \widetilde{u}\|_{L^2_\xi L^1_\tau} \sim \|\mathcal{F}^{-1}_{t,x}|\widetilde{u}|\|_{L^\infty_t H^a_x}. \tag{3.1}$$

Note that the supremum on the right is achieved by the trace to $t = 0$. We define Banach space $Y^{s,a,b} \subset \mathcal{S}'(\mathbb{R}^2)$ for any $s, a, b \in \mathbb{R}$ by the norm

$$\|f\|_{Y^{s,a,b}} := \|\langle\xi\rangle^s \langle\tau+\xi\rangle^a \langle\tau-\xi\rangle^b \widetilde{u}\|_{L^2_\xi L^1_\tau}. \tag{3.2}$$

This norm has been used to supplement the $X^{s,b}$-spaces for $b \le 1/2$ already by Bourgain [1] for the periodic KdV implicitly and also for the wave equation (the Zakharov system) in [14] more explicitly.

The following embedding is clear from the Fourier transform:

$$\|u\|_{L^\infty_t H^s_x} \lesssim \|u\|_{Y^{s,0,0}} \quad (\forall s \in \mathbb{R}), \tag{3.3}$$

and the proof of Lemma 2.4 readily implies the following embedding.

LEMMA 3.1
Let $a, b, c, a_0, b_0 \in \mathbb{R}^3$. Then we have the linear estimate

$$\|u\|_{Y^{c,a,b}} \lesssim \|u\|_{H_\alpha^{a_0} H_\beta^{b_0}} \tag{3.4}$$

if and only if $c \prec \{a_0 - a, b_0 - b\}$ and $a_0 + b_0 > a + b + 1/2$.

Incarnating the Y norm, we can recover the bilinear estimates for DKG, QD, and WM down to the lowest optimal regularity. For clarity and future use, we give the linear and bilinear estimates by those norms in more general forms than needed for the proof of well-posedness.

3.1. Linear estimates for integrals

First, we have the multiplication estimate by smooth functions:

$$\|\lambda u\|_{Y^{s,a,b}} \lesssim \|\langle\tau\rangle^N \langle\xi\rangle^N \widetilde{\lambda}(\tau,\xi)\|_{\mathcal{M}(\mathbb{R}^2)} \|u\|_{Y^{s,a,b}} \tag{3.5}$$

for any $N \geq |s| + |a| + |b|$. We omit the proof since it is the same as for (2.15). We will use two cutoff functions $\lambda(t), \overline{\lambda}(t)$ satisfying

$$\begin{gathered} \lambda, \overline{\lambda} \in C_0^\infty(\mathbb{R}), \quad \exists t_- < 0 < \exists t_+, \quad \text{s.t.} \\ \operatorname{supp} \lambda \subset [t_-, t_+], \quad \inf_{t_- < t < t_+} \overline{\lambda}(t) > 0. \end{gathered} \tag{3.6}$$

For the Dirac equation, we have the following.

LEMMA 3.2
Let $\lambda(t), \overline{\lambda}(t)$ be any functions satisfying (3.6). Then for any $a \in \mathbb{R}$ and any space-time function $u(t,x)$, we have

$$\begin{aligned} \|\lambda(t)u\|_{Y^{a,0,0}} &\lesssim \|u(0,x)\|_{H^a} + \|\overline{\lambda}(t)\partial_\alpha u\|_{Y^{a,-1,0}}, \\ \|\lambda(t)u\|_{Y^{a,0,0}} &\lesssim \|u(0,x)\|_{H^a} + \|\overline{\lambda}(t)\partial_\beta u\|_{Y^{a,0,-1}}, \end{aligned} \tag{3.7}$$

and for any $s_1, s_2 \in \mathbb{R}$, we have

$$\begin{aligned} \|\lambda(t)u\|_{H_\alpha^{s_1} H_\beta^{s_2}} &\lesssim \|\overline{\lambda}(t)u\|_{L_t^2 H_x^{s_2}} + \|\overline{\lambda}(t)\partial_\alpha u\|_{H_\alpha^{s_1-1} H_\beta^{s_2}}, \\ \|\lambda(t)u\|_{H_\alpha^{s_1} H_\beta^{s_2}} &\lesssim \|\overline{\lambda}(t)u\|_{L_t^2 H_x^{s_1}} + \|\overline{\lambda}(t)\partial_\beta u\|_{H_\alpha^{s_1} H_\beta^{s_2-1}}. \end{aligned} \tag{3.8}$$

Note that the $L_t^2 H_x^{s_j}$-norms are dominated by $Y^{s_j,0,0}$ because of the cutoff $\overline{\lambda}(t)$ and that there is no restriction on the exponents. Conditions on the exponents arise when we try to bound the Y-norm by the Sobolev norm, and also to estimate products.

For the Klein-Gordon equation, we have the following.

COROLLARY 3.3
Let $\lambda(t), \overline{\lambda}(t)$ be any functions satisfying (3.6). Then for any $s \in \mathbb{R}$ and any

space-time function $w(t,x)$ we have

$$\begin{aligned}&\|\lambda(t)w\|_{Y^{s,0,0}}+\|\lambda(t)(\partial_t w,\partial_x w)\|_{Y^{s-1,0,0}}\\ &\quad\lesssim\|w(0,x)\|_{H^s_x}+\|\partial_t w(0,x)\|_{H^{s-1}_x}\\ &\qquad+\|\overline{\lambda}(t)(\partial_t^2-\partial_x^2)w\|_{Y^{s-1,-1,0}\cap Y^{s-1,0,-1}},\end{aligned}\tag{3.9}$$

and for any s_1,s_2, we have

$$\begin{aligned}\|\lambda(t)w\|_{H^{s_1}_\alpha H^{s_2}_\beta}&\lesssim\|\overline{\lambda}(t)w\|_{L^2_tH^{\max(s_1,s_2)}_x}+\|\overline{\lambda}(t)(\partial_t w,\partial_x w)\|_{L^2_tH^{\max(s_1,s_2)-1}_x}\\ &\quad+\|\overline{\lambda}(t)(\partial_t^2-\partial_x^2)w\|_{H^{s_1-1}_\alpha H^{s_2-1}_\beta}.\end{aligned}\tag{3.10}$$

We first derive the corollary from the lemma.

Proof
Applying (3.7) to $(\partial_t\pm\partial_x)w$, we get the estimate (3.9) except for the first term (w itself). For that term, we divide w in frequencies of x:

$$w=w_H+w_L,\quad \widehat{w}_L(t,\xi)=\chi_1(\xi)\widehat{w}(t,\xi).\tag{3.11}$$

Then we have

$$\|\lambda(t)w_H\|_{Y^{s,0,0}}\sim\|\lambda(t)\partial_x w_H\|_{Y^{s-1,0,0}}\lesssim\|\lambda(t)\partial_x w\|_{Y^{s-1,0,0}},\tag{3.12}$$

and by using (3.7) again,

$$\begin{aligned}&\|\lambda(t)w_L\|_{Y^{s,0,0}}\sim\|\lambda(t)w_L\|_{Y^{s-1,0,0}}\\ &\quad\lesssim\|w_L(0)\|_{H^{s-1}_x}+\|\overline{\lambda}(t)\partial_\alpha w_L\|_{Y^{s-1,-1,0}}\\ &\quad\lesssim\|w(0)\|_{H^s_x}+\|\overline{\lambda}(t)\partial_\alpha w\|_{Y^{s-1,0,0}},\end{aligned}\tag{3.13}$$

so the bound on w is reduced to that on the other terms, which is already obtained.

Next, using (3.8) twice, we get

$$\begin{aligned}\|\lambda(t)w\|_{H^{s_1}_\alpha H^{s_2}_\beta}&\lesssim\|\check{\lambda}(t)w\|_{Y^{s_2,0,0}}+\|\check{\lambda}(t)\partial_\alpha w\|_{H^{s_1-1}_\alpha H^{s_2}_\beta}\\ &\lesssim\|\check{\lambda}(t)w\|_{Y^{s_2,0,0}}+\|\overline{\lambda}(t)\partial_\alpha w\|_{Y^{s_1-1,0,0}}\\ &\quad+\|\overline{\lambda}(t)\partial_\beta\partial_\alpha w\|_{H^{s_1-1}_\alpha H^{s_2-1}_\beta},\end{aligned}\tag{3.14}$$

where we chose an intermediate $\check{\lambda}(t)\in C_0^\infty(\mathbb{R})$ satisfying $\inf_{t_-<t<t_+}\check{\lambda}(t)>0$ and

$$\exists t'_-<t_-<t_+<\exists t'_+,\quad \text{s.t. } \operatorname{supp}\check{\lambda}\subset[t'_-,t'_+],\ \inf_{t'_-<t<t'_+}\overline{\lambda}(t)>0.\tag{3.15}$$

Since $\check{\lambda}/\overline{\lambda}\in C_0^\infty(\mathbb{R})$, we can replace $\check{\lambda}$ with $\overline{\lambda}$ in the last line of (3.14) by (3.5). □

Proof of Lemma 3.2
By symmetry, it suffices to show the first line of each group of estimates. Let $v(t,x)=u(t,x+t)$. Then we have $v(0,x)=u(0,x)$, $\partial_t v(t,x)=2\partial_\alpha u(t,x+t)$,

$\widetilde{v}(\tau,\xi)=\widetilde{u}(\tau-\xi,\xi)$, and so

$$\begin{aligned}&\|u\|_{Y^{s,a,b}}=\|\langle\xi\rangle^s\langle\tau\rangle^a\langle\tau-2\xi\rangle^b\widetilde{v}\|_{L^2_\xi L^1_\tau},\\ &\|u\|_{H^{s_1}_\alpha H^{s_2}_\beta}=\|\langle\tau\rangle^{s_1}\langle\tau-2\xi\rangle^{s_2}\widetilde{v}\|_{L^2_\xi L^2_\tau}\end{aligned}\tag{3.16}$$

by changing $\tau\mapsto\tau+\xi$.

First we prove (3.7), for which it suffices to show, fixing ξ,

$$\|\mathcal{F}(\lambda v)\|_{L^1_\tau}\lesssim|v(0)|+\|\langle\tau\rangle^{-1}\mathcal{F}(\overline{\lambda}v_t)\|_{L^1_\tau}.\tag{3.17}$$

Regarding v as a function of t only, we have

$$\begin{aligned}\lambda(t)v(t)&=\lambda(t)v(0)+\lambda(t)\int_0^t w(s)\,ds\\ &=\lambda(t)v(0)+\mathcal{F}^{-1}\int\widehat{w}(\sigma)\frac{\widehat{\lambda}(\tau-\sigma)-\widehat{\lambda}(\tau)}{i\sigma}\,d\sigma,\end{aligned}\tag{3.18}$$

where $w(t):=\check{\lambda}(t)v_t(t)$ with a cutoff $\check{\lambda}(t)\in C_0^\infty(\mathbb{R})$ chosen such that

$$t_-<t<t_+\implies\check{\lambda}(t)=1,\qquad\inf_{t\in\operatorname{supp}\check{\lambda}}\overline{\lambda}>0.\tag{3.19}$$

Hence we have

$$\|\mathcal{F}(\lambda v)\|_{L^1_\tau}\lesssim\|\widehat{\lambda}\|_{L^1_\tau}|v(0)|+\iint|\widehat{w}(\sigma)|\frac{|\widehat{\lambda}(\tau-\sigma)-\widehat{\lambda}(\tau)|}{|\sigma|}\,d\sigma\,d\tau,\tag{3.20}$$

where the last integral is bounded by

$$\begin{aligned}&\int_{|\sigma|>1}|\widehat{w}(\sigma)|\frac{\|\widehat{\lambda}\|_{L^1_\tau}}{|\sigma|}\,d\sigma+\iint_{|\sigma|<1}\int_0^1|\widehat{w}(\sigma)||\widehat{\lambda}_\tau(\tau-\theta\sigma)|\,d\theta\,d\sigma\,d\tau\\ &\lesssim(\|\widehat{\lambda}\|_{L^1_\tau}+\|\widehat{\lambda}_\tau\|_{L^1_\tau})\|\langle\sigma\rangle^{-1}\widehat{w}(\sigma)\|_{L^1_\sigma}.\end{aligned}\tag{3.21}$$

Since $\check{\lambda}/\overline{\lambda}\in C_0^\infty$, we have

$$\|\langle\tau\rangle^{-1}\widehat{w}(\tau)\|_{L^1_\tau}\lesssim\|\langle\tau\rangle^{-1}\mathcal{F}(\overline{\lambda}v_t)\|_{L^1_\tau}\tag{3.22}$$

by the same argument as in (2.15). Thus we obtain (3.17) and so (3.7).

Next we prove (3.8). We decompose the Fourier transform

$$\begin{aligned}&\langle\tau\rangle^{s_1}\langle\tau-2\xi\rangle^{s_2}\mathcal{F}(\lambda v)\\ &\quad=\langle\tau\rangle^{s_1}\langle\tau-2\xi\rangle^{s_2}\chi_1(\tau)\mathcal{F}(\lambda v)\\ &\qquad+\big(1-\chi_1(\tau)\big)\frac{\langle\tau\rangle}{i\tau}\langle\tau\rangle^{s_1-1}\langle\tau-2\xi\rangle^{s_2}\mathcal{F}(\lambda'v+\lambda v_t).\end{aligned}\tag{3.23}$$

The first term on the right-hand side is bounded by $\langle\xi\rangle^{s_2}|\mathcal{F}(\lambda v)|$ since $|\tau|\lesssim1$ on the support. Hence its $L^2_{\tau,\xi}$-norm is bounded by

$$\|\lambda v\|_{L^2_tH^{s_2}_x}\sim\|\lambda u\|_{L^2_tH^{s_2}_x}\lesssim\|\overline{\lambda}u\|_{L^2_tH^{s_2}_x},\tag{3.24}$$

where we used $\lambda/\overline{\lambda}\in C_0^\infty(\mathbb{R})$ and (2.15). The last term in (3.23) is bounded by $\langle\tau\rangle^{s_1-1}\langle\tau-2\xi\rangle^{s_2}|\mathcal{F}(\lambda v_t)|$ since $|\tau|\gtrsim1$ on the support. Hence its $L^2_{\tau,\xi}$-norm is bounded by

$$\|\langle\tau\rangle^{s_1-1}\langle\tau-2\xi\rangle^{s_2}\mathcal{F}(\lambda v_t)\|_{L^2_{\tau,\xi}} \sim \|\lambda u_\alpha\|_{H^{s_1-1}_\alpha H^{s_2}_\beta} \lesssim \|\overline{\lambda} u_\alpha\|_{H^{s_1-1}_\alpha H^{s_2}_\beta}, \tag{3.25}$$

where we used (2.15) again. It remains to estimate the second last term of (3.23), bounded by

$$\langle\tau\rangle^{s_1-1}\langle\tau-2\xi\rangle^{s_2}|\mathcal{F}(\lambda' v)|, \tag{3.26}$$

for which we use an induction on s_1. If $s_1+|s_2|\le 1$, then using (2.12) we have

$$\langle\tau\rangle^{s_1-1}\langle\tau-2\xi\rangle^{s_2} \lesssim \langle\tau\rangle^{s_1+|s_2|-1}\langle\xi\rangle^{s_2} \lesssim \langle\xi\rangle^{s_2}, \tag{3.27}$$

and so the L^2-norm of (3.26) is bounded by

$$\|\lambda' v\|_{L^2_t H^{s_2}_x} \lesssim \|\overline{\lambda} u\|_{L^2_t H^{s_2}_x} \tag{3.28}$$

since $\operatorname{supp}\lambda' \subset \operatorname{supp}\lambda$; thus we obtain (3.8) for $s_1+|s_2|\le 1$. Assume that for some $k\in\mathbb{N}$ we have (3.8) for all $s_1+|s_2|\le k$, and let $s_1+|s_2|\le k+1$. Then by the above argument, we have

$$\|\lambda u\|_{H^{s_1}_\alpha H^{s_2}_\beta} \lesssim \|\overline{\lambda} u\|_{L^2_t H^{s_2}} + \|\overline{\lambda} u_\alpha\|_{H^{s_1-1}_\alpha H^{s_2}_\beta} + \|\lambda' u\|_{H^{s_1-1}_\alpha H^{s_2}_\beta}. \tag{3.29}$$

Since $\operatorname{supp}\lambda' \subset \operatorname{supp}\lambda$ and $s_1-1+|s_2|\le k$, the last term is bounded by

$$\|\overline{\lambda} u\|_{L^2_t H^{s_2}_x} + \|\overline{\lambda} u_\alpha\|_{H^{s_1-2}_\alpha H^{s_2}_\beta} \tag{3.30}$$

by using the assumption. Hence by induction on $k\in\mathbb{N}$, we obtain (3.8) for all $s_1, s_2\in\mathbb{R}$. □

3.2. Bilinear estimate for product

To close the bilinear estimates for the well-posedness proof, it remains to bound the Y-norm for product. The following estimate is the main ingredient of this section.

LEMMA 3.4

Let $a_1, a_2, b_1, b_2, a, b, s \in \mathbb{R}$ satisfy the following conditions: there exist $a_0, b_0\in\mathbb{R}$ such that

$$\begin{aligned} &a_0 \prec \{a_1, a_2\}, \qquad b_0 \prec \{b_1, b_2\}, \qquad s \prec \{a_0-a, b_0-b\}, \\ &a_1+b_1 > a+b+\frac{1}{2}, \qquad a_2+b_2 > a+b+\frac{1}{2}. \end{aligned} \tag{3.31}$$

Then we have

$$\|fg\|_{Y^{s,a,b}} \lesssim \|f\|_{H^{a_1}_\alpha H^{b_1}_\beta}\|g\|_{H^{a_2}_\alpha H^{b_2}_\beta}. \tag{3.32}$$

REMARK 3.5

If we derive the above estimate by combining Lemmas 2.2 and 3.1,

$$\|fg\|_{Y^{s,a,b}} \lesssim \|fg\|_{H^{a_0}_\alpha H^{b_0}_\beta} \lesssim \|f\|_{H^{a_1}_\alpha H^{b_1}_\beta}\|g\|_{H^{a_2}_\alpha H^{b_2}_\beta}, \tag{3.33}$$

then we need

$$(3.34)\qquad \begin{aligned} &a_0 \prec \{a_1, a_2\}, \qquad b_0 \prec \{b_1, b_2\}, \qquad s \prec \{a_0 - a, b_0 - b\},\\ &a_0 + b_0 > a + b + \frac{1}{2}. \end{aligned}$$

The first three conditions are the same as in (3.31), but the last one is stronger than the last two of (3.31) because $a_0 \le \min(a_1, a_2)$ and $b_0 \le \min(b_1, b_2)$, but they are not necessarily equal.

REMARK 3.6

The condition (3.31) is almost optimal, but we could still extend it to some of the borderline cases, where some of the inequalities are replaced with equality. We do not pursue the optimal condition here because even just stating it could be very complicated and much more for the proof (one can see below that treating all the cases in (3.31) is already quite cumbersome), and anyway they would not contribute to the well-posedness proof.

Proof of Lemma 3.4

By the Fourier transform and the duality argument, and after appropriate linear changes of coordinates, the desired estimate is reduced to

$$(3.35)\qquad \begin{aligned} &|T(F, G, \varphi)| \lesssim \|F\|_{L^2L^2} \|G\|_{L^2L^2} \|\varphi\|_{L^2},\\ &T(F, G, \varphi) := \iiiint_{\substack{\zeta+\xi+\eta=0,\\ \xi+\xi_1+\xi_2=0,\\ \eta+\eta_1+\eta_2=0}} wF(\xi_1, \eta_1) G(\xi_2, \eta_2) \varphi(\zeta)\, dv,\\ &w := \langle\zeta\rangle^s \langle\xi\rangle^a \langle\eta\rangle^b \langle\xi_1\rangle^{-a_1} \langle\eta_1\rangle^{-b_1} \langle\xi_2\rangle^{-a_2} \langle\eta_2\rangle^{-b_2} \end{aligned}$$

for arbitrary nonnegative functions $F(\xi, \eta)$, $G(\xi, \eta)$, and $\varphi(\xi)$, where dv denotes the volume element on the 4-dimensional hyperplane given by the 3 linear constraints in $\mathbb{R}^7$. Hence we can arbitrarily choose 4 independent variables to integrate from the 7 variables.

Actually, we choose the 3 smallest variables to optimize the Hölder inequality. (Here and after, smallness of variables means that in the absolute values.) We decompose the integral region according to which is the smallest variable in each constraint. To express such domains in short, we introduce the following notation. For any variables x, y, z, we denote by $[x : y, z]$ the following constraint:

$$(3.36)\qquad x + y + z = 0 \qquad \text{and} \qquad |x| \le \min(|y|, |z|).$$

Then we have $|y| \sim |z|$. Moreover, we denote the smallest variables by

$$(3.37)\qquad \zeta_0 \in \{\zeta, \xi, \eta\}, \qquad \xi_0 \in \{\xi, \xi_1, \xi_2\}, \qquad \eta_0 \in \{\eta, \eta_1, \eta_2\}$$

among each set, and we let $m := (\zeta_0, \xi_0, \eta_0)$. The integral region is decomposed into $3^3 = 27$ regions corresponding to the combination in m. By using the symmetry under the two exchanges

$$(\xi,\xi_1,\xi_2,a,a_1,a_2)\leftrightarrow(\eta,\eta_1,\eta_2,b,b_1,b_2),$$
$$(\xi_1,a_1,\eta_1,b_1)\leftrightarrow(\xi_2,a_2,\eta_2,b_2), \tag{3.38}$$

we can reduce the number of domains to be considered. The remaining task is a rather routine sequence of estimates. In order to treat the critical cases (i.e., when we have equalities in (3.31)), we also introduce the L^2-norms on dyadic pieces. For any functions $\varphi(\zeta)$, $f(\xi,\eta)$, and $(j,k)\in\mathbb{N}^2$, we denote

$$\begin{aligned} &f_{j\cdot}(\eta)=\|f\|_{L^2_\xi(2^{j-2}-1<|\xi|<2^{j+2})}, \qquad f_{\cdot k}(\xi)=\|f\|_{L^2_\eta(2^{k-2}-1<|\eta|<2^{k+2})},\\ &\varphi_j=\|\varphi\|_{L^2_\zeta(2^{j-2}-1<|\zeta|<2^{j+2})},\\ &f_{j,k}=\|f\|_{L^2_{\xi,\eta}(2^{j-2}-1<|\xi|<2^{j+2},\,2^{k-2}-1<|\eta|<2^{k+2})},\end{aligned} \tag{3.39}$$

and we assign the following dyadic parameters throughout the proof:

$$\begin{aligned} &|\xi|\sim 2^j,\quad |\eta|\sim 2^k,\quad |\zeta|\sim 2^l,\quad |\xi_1|\sim 2^{j_1},\quad |\xi_2|\sim 2^{j_2},\\ &|\eta_1|\sim 2^{k_1},\quad |\eta_2|\sim 2^{k_2}.\end{aligned} \tag{3.40}$$

More precisely, j_1 is the least positive integer such that $|\xi_1|\le 2^{j_1}$, and the other numbers are defined in the same way.

Now we start with the cases where the 3 smallest variables are independent.

(I): *The domain* $m=(\zeta,\xi,\eta_1)$. By symmetry, (ζ,ξ,η_2), (ζ,ξ_1,η), and (ζ,ξ_2,η) are also reduced to this case. The above m implies that

$$\begin{aligned} &|\xi_1|\sim|\xi_2|\gtrsim|\xi|\sim|\eta|\sim|\eta_2|\gtrsim|\zeta|\vee|\eta_1|,\\ &w\sim\langle\zeta\rangle^s\langle\xi\rangle^{a+b-b_2}\langle\xi_1\rangle^{-a_1-a_2}\langle\eta_1\rangle^{-b_1}\end{aligned} \tag{3.41}$$

in this region, and we choose ξ_1,ξ,ζ,η_1 as the integral variables. First, we apply the Schwarz inequality to F and G for the integral in ξ_1 on each dyadic piece $2^{j_1}\sim|\xi_1|\sim|\xi_2|\gtrsim|\xi|$. Then we obtain

$$\begin{aligned} &\iiiint_{[\zeta:\xi,\eta][\xi:\xi_1,\xi_2][\eta_1:\eta,\eta_2]} wFG\varphi\, d\xi_1\, d\xi\, d\zeta\, d\eta_1\\ &\lesssim\iiint_{[\zeta:\xi,\eta][\eta_1:\eta,\eta_2]}\langle\zeta\rangle^s\langle\xi\rangle^{a+b-b_2}\langle\eta_1\rangle^{-b_1}\\ &\qquad\times\sum_{2^{j_1}\gtrsim|\xi|}2^{(-a_1-a_2)j_1}F_{j_1\cdot}(\eta_1)G_{j_1\cdot}(\eta_2)\varphi(\zeta)\, d\xi\, d\zeta\, d\eta_1.\end{aligned} \tag{3.42}$$

For the integral in ξ, we apply Schwarz to G and 1 on each dyadic piece $2^j\sim|\xi|\sim|\eta|\sim|\eta_2|\gtrsim|\zeta|\vee|\eta_1|$. Then (3.42) is bounded by

$$\begin{aligned} &\lesssim\iint_{\mathbb{R}^2}\langle\zeta\rangle^s\langle\eta_1\rangle^{-b_1}\\ &\qquad\times\sum_{2^{j_1}\gtrsim 2^j\gtrsim|\zeta|\vee|\eta_1|}2^{(1/2+a+b-b_2)j-(a_1+a_2)j_1}F_{j_1\cdot}(\eta_1)G_{j_1,j}\varphi\, d\zeta\, d\eta_1.\end{aligned} \tag{3.43}$$

Similarly, we apply Schwarz to φ and 1 for the integral on $2^l \sim |\zeta|$, and to F and 1 on $2^{k_1} \sim |\eta_1|$. Then the above is bounded by

$$\sum_{k_1 \vee l \le j+2 \le j_1+4} 2^{(1/2+s)l+(1/2-b_1)k_1+(1/2+a+b-b_2)j-(a_1+a_2)j_1} F_{j_1,k_1} G_{j_1,j} \varphi_l. \tag{3.44}$$

The exponent on 2 can be rearranged as

$$\begin{aligned}
&k_1 \ge l \implies -\sigma_1(j_1 - j) - \sigma_2(j - k_1) - \sigma_3(k_1 - l) - \sigma_4 l, \\
&l \ge k_1 \implies -\sigma_1(j_1 - j) - \sigma_2(j - l) - \sigma_5(l - k_1) - \sigma_4 k_1, \\
&\sigma_1 = a_1 + a_2, \\
&\sigma_2 = a_1 + a_2 - \frac{1}{2} - a - b + b_2, \\
&\sigma_3 = a_1 + a_2 - \frac{1}{2} - a - b + b_1 + b_2 - \frac{1}{2}, \\
&\sigma_5 = a_1 + a_2 - \frac{1}{2} - a - b + b_2 - s - \frac{1}{2}, \\
&\sigma_4 = a_1 + a_2 - \frac{1}{2} - a - b + b_1 + b_2 - \frac{1}{2} - s - \frac{1}{2}.
\end{aligned} \tag{3.45}$$

By the assumption, we have $\sigma_1 \ge 0$, $\sigma_2 \wedge \sigma_3 \ge a_0 + b_0 - a - b \ge 0$, and $\sigma_5 \wedge \sigma_4 \ge a_0 + b_0 - a - b - s - 1/2 \ge 0$. Moreover, we can observe that in each case at most one coefficient σ_* can be zero because equality after adding $1/2$ implies strict inequalities in the preceding steps, thanks to the exclusion rule (1.22) in the product relation. For example, $\sigma_5 = 0$ implies the exclusion rule for $a_0 \prec \{a_1, a_2\}$ and $s \prec \{a_0 - a, b_0 - b\}$, and so $\sigma_1, \sigma_2 > 0$. Thus we can bound the sum (3.44) by

$$\begin{aligned}
\|F_{j_1,k_1} G_{j_1,j} \varphi_l\|_{\ell^1_{j_1} \ell^2_j \ell^2_{k_1} \ell^2_l} &\sim \|F_{j,k}\|_{\ell^2_j \ell^2_k} \|G_{j,k}\|_{\ell^2_j \ell^2_k} \|\varphi_l\|_{\ell^2_l} \\
&\sim \|F\|_{L^2_\xi L^2_\eta} \|G\|_{L^2_\xi L^2_\eta} \|\varphi\|_{L^2_\xi}
\end{aligned} \tag{3.46}$$

by applying Hölder in some appropriate order for each discrete variable. More precisely, if none of σ_* vanishes, then we can use Hölder in arbitrary order. If one of σ_* vanishes, then we should start with the index among $\{j, k_1, l\}$ for which we have only one exponential factor. For example, if $\sigma_2 = 0$, then we should start with Schwarz in j, but the remaining order is free. Hence (3.44) is bounded by, for example,

$$\|W\|_{\ell^1_{k_1} \ell^1_l \ell^\infty_{j_1} \ell^2_j} \|F_{j_1,k_1} G_{j_1,j} \varphi_l\|_{\ell^2_{k_1} \ell^2_l \ell^1_{j_1} \ell^2_j} \lesssim \|F_{j_1,k_1} G_{j_1,j} \varphi_l\|_{\ell^1_{j_1} \ell^2_j \ell^2_{k_1} \ell^2_l}, \tag{3.47}$$

where W denotes the weight part in (3.44).

(II): *The domain* $m = (\zeta, \xi_1, \eta_1)$. This includes the case (ζ, ξ_2, η_2) by symmetry. Here we have

$$\begin{aligned}
&|\xi| \sim |\eta| \sim |\xi_2| \sim |\eta_2| \gtrsim |\zeta| \vee |\xi_1| \vee |\eta_1|, \\
&w \sim \langle\zeta\rangle^s \langle\xi\rangle^{a+b-a_2-b_2} \langle\xi_1\rangle^{-a_1} \langle\eta_1\rangle^{-b_1}.
\end{aligned} \tag{3.48}$$

Choosing $\xi, \xi_1, \eta_1, \zeta$ as the integral variables, we want to apply Schwarz inequality as in case (I). However, in this case we cannot start with the largest variable

ξ because it is contained in the two variables of the same function G. Thus we are forced to integrate first on the largest variable among ξ_1, η_1, ζ, for which we apply Schwarz to one variable of G and either F or φ. Then we apply Schwarz for the integrals in ξ and the remaining two from ξ_1, η_1, ζ, to the function 1 and some of F, G, φ. Thus we get

$$\begin{aligned}
&\iiiint_{[\zeta:\xi,\eta],[\xi_1:\xi,\xi_2],[\eta_1:\eta,\eta_2]} wFG\varphi\, d\xi\, d\xi_1\, d\eta_1\, d\zeta \\
&\lesssim \sum_{j_1\vee k_1\le l+2\le j+4} 2^{sl+(1/2-a_1)j_1+(1/2-b_1)k_1+(1/2+a+b-a_2-b_2)j} F_{j_1,k_1} G_{j,l} \\
&\quad + \sum_{k_1\vee l\le j_1+2\le j+4} 2^{(1/2+s)l+(-a_1)j_1+(1/2-b_1)k_1+(1/2+a+b-a_2-b_2)j} \\
&\qquad\qquad \times F_{j_1,k_1} G_{j_1,j} \varphi_l \\
&\quad + \sum_{l\vee j_1\le k_1+2\le j+4} 2^{(1/2+s)l+(1/2-a_1)j_1+(-b_1)k_1+(1/2+a+b-a_2-b_2)j} \\
&\qquad\qquad \times F_{j_1,k_1} G_{j,k_1} \varphi_l.
\end{aligned} \tag{3.49}$$

To bound the sum, we rearrange the exponent on 2 as in the previous domain. For example, if $j_1 \le k_1 \le l \le j$, then we can rewrite it as

$$\begin{aligned}
&-\sigma_6(j-l) - \sigma_7(l-k_1) - \sigma_8(k_1-j_1) - \sigma_4 j_1, \\
&\sigma_6 = a_2+b_2-a-b-\frac{1}{2}, \quad \sigma_7 = a_2+b_2-a-b-s-\frac{1}{2}, \\
&\sigma_8 = a_2+b_1+b_2-\frac{1}{2}-a-b-s-\frac{1}{2},
\end{aligned} \tag{3.50}$$

and by the assumption, we have $\sigma_6 > 0$, $\sigma_7 \wedge \sigma_8 \ge a_0 - a + b_0 - b - s - 1/2 \ge 0$. In the other cases, we get a new coefficient $\sigma_9 = a_2 + b_1 + b_2 - a - b - 1/2$.

In all cases, the coefficients are all nonpositive and negative except for at most one of the four, and hence the above sum is bounded, as desired by Hölder, in the same way as the previous domain.

(III): *The domain* $m = (\zeta, \xi_2, \eta_1)$. This covers the case (ζ, ξ_1, η_2) by symmetry. Here we have

$$\begin{aligned}
&|\xi| \sim |\eta| \sim |\xi_1| \sim |\eta_2| \gtrsim |\zeta| \vee |\xi_2| \vee |\eta_1|, \\
&w \sim \langle\zeta\rangle^s \langle\xi\rangle^{a+b-a_1-b_2} \langle\xi_2\rangle^{-a_2} \langle\eta_1\rangle^{-b_1}.
\end{aligned} \tag{3.51}$$

Choosing $\xi, \xi_2, \eta_1, \zeta$ as the integral variable, we apply Schwarz as in case (I) to F and G for the integral in ξ, to G and 1 in ξ_2, to F and 1 in η_1, and then to φ and 1 in ζ, on each dyadic piece. Thus we obtain

$$\begin{aligned}
&\iiiint_{[\zeta:\xi,\eta],[\xi_2:\xi,\xi_1],[\eta_1:\eta,\eta_2]} wFG\varphi\, d\xi\, d\xi_2\, d\eta_1\, d\zeta \\
&\lesssim \sum_{j_2\vee k_1\vee l\le j+2} 2^{(1/2+s)l+(1/2-a_2)j_2+(1/2-b_1)k_1+(a+b-a_1-b_2)j} F_{j,k_1} G_{j_2,j} \varphi_l.
\end{aligned} \tag{3.52}$$

Rearranging the exponent on 2 as in domains (I) and (II), we get the following new coefficients:

$$\begin{aligned}&\sigma_{11} = a_1 + b_2 - a - b, \qquad \sigma_{12} = a_1 + b_2 - a - b - s - \frac{1}{2},\\ &\sigma_{13} = a_1 + b_1 + b_2 - \frac{1}{2} - a - b,\\ &\sigma_{14} = a_1 + b_1 + b_2 - \frac{1}{2} - a - b - s - \frac{1}{2},\end{aligned} \tag{3.53}$$

and the summation estimate proceeds just as before.

(IV): *The domain* $m = (\xi, \xi_1, \eta)$. This includes (ξ, ξ_2, η), (η, ξ, η_1), and (η, ξ, η_2) by symmetry. We have

$$\begin{aligned}&|\eta_1| \sim |\eta_2| \gtrsim |\eta| \sim |\zeta| \gtrsim |\xi| \sim |\xi_2| \gtrsim |\xi_1|,\\ &w \sim \langle \xi \rangle^{a-a_2} \langle \eta \rangle^{s+b} \langle \xi_1 \rangle^{-a_1} \langle \eta_1 \rangle^{-b_1-b_2}.\end{aligned} \tag{3.54}$$

Choosing η_1, η, ξ, ξ_1 as the integral variables, we apply Schwarz to F and G in η_1, to φ and 1 in η, to G and 1 in ξ, and then to F and 1 in ξ_1, respectively, on each dyadic piece. Thus we obtain

$$\begin{aligned}&\iiiint_{[\xi:\eta,\zeta],[\xi_1:\xi,\xi_2],[\eta:\eta_1,\eta_2]} wFG\varphi \, d\eta_1 \, d\eta \, d\xi \, d\xi_1\\ &\lesssim \sum_{j_1 \le j+2 \le k+4 \le k_1+6} 2^{(1/2+a-a_2)j+(1/2+s+b)k+(1/2-a_1)j_1-(b_1+b_2)k_1}\\ &\qquad \times F_{j_1,k_1} G_{j,k_1} \varphi_k.\end{aligned} \tag{3.55}$$

We can rearrange the exponent on 2 as

$$\begin{aligned}&-\sigma_{15}(k_1 - k) - \sigma_{16}(k - j) - \sigma_8(j - j_1) - \sigma_4 j_1,\\ &\sigma_{15} = b_1 + b_2 \ge 0, \;\; \sigma_{16} = b_1 + b_2 - 1/2 - b - s \ge b_0 - b - s \ge 0,\end{aligned} \tag{3.56}$$

and the rest of the argument is the same as before.

(V): *The domain* $m = (\xi, \xi_1, \eta_1)$. By symmetry, we have the same for (ξ, ξ_2, η_2), (η, ξ_1, η_1), and (η, ξ_2, η_2). Here we have

$$\begin{aligned}&|\zeta| \sim |\eta| \sim |\eta_2| \gtrsim |\eta_1| \vee |\xi|, \;\; |\xi| \sim |\xi_2| \gtrsim |\xi_1|,\\ &w \sim \langle \xi \rangle^{a-a_2} \langle \eta \rangle^{s+b-b_2} \langle \xi_1 \rangle^{-a_1} \langle \eta_1 \rangle^{-b_1}.\end{aligned} \tag{3.57}$$

Choosing η, η_1, ξ, ξ_1 as the integral variables, we apply Schwarz to G and φ in η, to F and 1 in η_1, to G and 1 in ξ, and to F and 1 in ξ_1. Thus we obtain

$$\begin{aligned}&\iiiint_{[\xi:\eta,\zeta],[\xi_1:\xi,\xi_2],[\eta_1:\eta,\eta_2]} wFG\varphi \, d\eta \, d\eta_1 \, d\xi \, d\xi_1\\ &\lesssim \sum_{j_1 \le j+2, j \vee k_1 \le k+2} 2^{(1/2+a-a_2)j+(s+b-b_2)k+(1/2-a_1)j_1+(1/2-b_1)k_1}\\ &\qquad \times F_{j_1,k_1} G_{j,k} \varphi_k.\end{aligned} \tag{3.58}$$

The exponent is rearranged as follows. If $j_1 \lesssim j \lesssim k_1$, then

$$-\sigma_{17}(k-k_1)-\sigma_{16}(k_1-j)-\sigma_8(j-j_1)-\sigma_4 j_1, \tag{3.59}$$

where

$$\sigma_{17} := b_2-b-s \geq b_0-b-s \geq 0. \tag{3.60}$$

If $j_1 \lesssim k_1 \lesssim j$, then

$$-\sigma_{17}(k-j)-\sigma_7(j-k_1)-\sigma_8(k_1-j_1)-\sigma_4 j_1, \tag{3.61}$$

and if $k_1 \lesssim j_1 \lesssim j$, then

$$-\sigma_{17}(k-j)-\sigma_7(j-j_1)-\sigma_5(j_1-k_1)-\sigma_4 k_1. \tag{3.62}$$

In any case, we can bound the sum as in the previous domains.

(VI): *The domain* $m=(\xi,\xi_2,\eta_1)$. From it the symmetry generates the cases (ξ,ξ_1,η_2), (η,ξ_1,η_2), (η,ξ_2,η_1). Here we have

$$\begin{aligned} &|\zeta|\sim|\eta|\sim|\eta_2|\gtrsim|\eta_1|\vee|\xi|,\ |\xi|\sim|\xi_1|\gtrsim|\xi_2|,\\ &w\sim\langle\xi\rangle^{a-a_1}\langle\eta\rangle^{s+b-b_2}\langle\xi_2\rangle^{-a_2}\langle\eta_1\rangle^{-b_1}. \end{aligned} \tag{3.63}$$

Choosing η,η_1,ξ,ξ_2 as the integral variables, we obtain, in the same way as above,

$$\begin{aligned} &\iiiint_{[\xi:\eta,\zeta],[\xi_2:\xi,\xi_1],[\eta_1:\eta,\eta_2]} wFG\varphi\, d\eta\, d\eta_1\, d\xi\, d\xi_2\\ &\lesssim \sum_{j_2\le j+2,\, j\vee k_1\le k+2} 2^{(1/2+a-a_1)j+(s+b-b_2)k+(1/2-a_2)j_2+(1/2-b_1)k_1}\\ &\qquad\times F_{j_2,k}G_{j,k_1}\varphi_k. \end{aligned} \tag{3.64}$$

The exponent is rearranged as

$$\begin{aligned} j_2\lesssim j\lesssim k_1 &\implies -\sigma_{17}(k-k_1)-\sigma_{16}(k_1-j)-\sigma_{13}(j-j_2)-\sigma_4 j_2,\\ j_2\lesssim k_1\lesssim j &\implies -\sigma_{17}(k-k_1)-\sigma_{12}(j-k_1)-\sigma_{13}(k_1-j_2)-\sigma_4 j_2,\\ k_1\lesssim j_2\lesssim j &\implies -\sigma_{17}(k-j)-\sigma_{12}(j-j_2)-\sigma_5(j_2-k_1)-\sigma_4 k_1, \end{aligned} \tag{3.65}$$

and so we can bound the sum as before.

Next we consider those cases where m is linearly dependent.

(VII): *The domain* $m=(\zeta,\xi,\eta)$. The symmetry does not produce any other case from it. Here we have

$$\begin{aligned} &|\zeta|\lesssim|\xi|\sim|\eta|\lesssim\begin{cases}|\xi_1|\sim|\xi_2|,\\ |\eta_1|\sim|\eta_2|,\end{cases}\\ &w\sim\langle\zeta\rangle^{s}\langle\xi\rangle^{a+b}\langle\xi_1\rangle^{-a_1-a_2}\langle\eta_1\rangle^{-b_1-b_2}. \end{aligned} \tag{3.66}$$

Choosing η_1,ξ_1,ξ,ζ as the integral variables, we apply Schwarz to F and G for the integrals in η_1 and ξ_1, and we simply integrate 1 for the integral in ξ, and then apply Schwarz to φ and 1 in ζ, on each dyadic piece. Then we get

$$\iiiint_{[\zeta:\xi,\eta],[\xi:\xi_1,\xi_2],[\eta:\eta_1,\eta_2]} wFG\varphi\, d\eta_1\, d\xi_1\, d\xi\, d\zeta \tag{3.67}$$

$$\lesssim \sum_{l\le j+2\le j_1+4\le k_1+6} 2^{(1/2+s)l+(1+a+b)j-(a_1+a_2)j_1-(b_1+b_2)k_1} \times F_{j_1,k_1}G_{j_1,k_1}\varphi_l.$$

The exponent is rearranged as

$$-\sigma_{15}(k_1-j)-\sigma_1(j_1-j)-\sigma_3(j-l)-\sigma_4 l, \tag{3.68}$$

and the function part is bounded in

$$\begin{aligned}\|F_{j_1,k_1}G_{j_1,k_1}\varphi_l\|_{\ell^1_{k_1}\ell^1_{j_1}\ell^2_l\ell^\infty_j} &\lesssim \|F_{j_1,k_1}\|_{\ell^2_{j_1}\ell^2_{k_1}}\|G_{j_1,k_1}\|_{\ell^2_{j_1}\ell^2_{k_1}}\|\varphi_l\|_{\ell^2_l}\\ &\sim \|F\|_{L^2_{\xi,\eta}}\|G\|_{L^2_{\xi,\eta}}\|\varphi\|_{L^2_\xi}.\end{aligned} \tag{3.69}$$

Hence we need exponential decay factors only for j and l. Here $\sigma_3>0$ or $\sigma_4>0$, but we may have $\sigma_{15}=\sigma_1=0$. In that case, $\sigma_3>0$ and $\sigma_4>0$, so we start with Schwarz in j. Otherwise, we start with Schwarz in l. The remaining argument is the same as before.

(VIII): *The domain* $m=(\xi,\xi,\eta_1)$. The symmetry reduces (ξ,ξ,η_2), (η,ξ_1,η), and (η,ξ_2,η) to this case. Here we have

$$\begin{aligned}&|\xi_1|\sim|\xi_2|, \qquad |\eta_2|\sim|\eta|\sim|\zeta|\gtrsim|\eta_1|, \qquad |\xi_2|\wedge|\zeta|\gtrsim|\xi|,\\ &w\sim\langle\zeta\rangle^{s+b-b_2}\langle\xi\rangle^a\langle\xi_2\rangle^{-a_1-a_2}\langle\eta_1\rangle^{-b_1}.\end{aligned} \tag{3.70}$$

Choosing ξ_2,ζ,η_1,ξ as integral variables, we apply Schwarz to F and G in ξ_2, to G and φ in ζ, to F and 1 in η_1, and then simply integrate 1 in ξ. Thus we obtain

$$\begin{aligned}&\iiiint_{[\xi:\zeta,\eta],[\xi:\xi_1,\xi_2],[\eta_1:\eta,\eta_2]} wFG\varphi\,d\xi_2\,d\zeta\,d\eta_1\,d\xi\\ &\lesssim \sum_{j\le(l\wedge j_2)+2,k_1\le l+2} 2^{(s+b-b_2)l+(1+a)j-(a_1+a_2)j_2+(1/2-b_1)k_1}\\ &\qquad\times F_{j_2,k_1}G_{j_2,l}\varphi_l.\end{aligned} \tag{3.71}$$

The exponent is rearranged as

$$\begin{aligned}&k_1\lesssim j \implies -\sigma_1(j_2-j)-\sigma_{17}(l-j)-\sigma_5(j-k_1)-\sigma_4 k_1,\\ &k_1\gtrsim j \implies -\sigma_1(j_2-j)-\sigma_{17}(l-k_1)-\sigma_{16}(k_1-j)-\sigma_4 j,\end{aligned} \tag{3.72}$$

while the function part belongs to $\ell^1_j\ell^1_{j_2}\ell^2_{k_1}\ell^\infty_j$. Hence we can bound the sum as in the previous domain.

(IX): *The domain* $m=(\xi,\xi,\eta)$. The case $m=(\eta,\xi,\eta)$ is the same by symmetry. Here we have

$$\begin{aligned}&\left.\begin{matrix}|\eta_1|\sim|\eta_2|\gtrsim|\eta|\sim|\zeta|\\ |\xi_1|\sim|\xi_2|\end{matrix}\right\}\gtrsim|\xi|,\\ &w\sim\langle\xi\rangle^a\langle\eta\rangle^{s+b}\langle\xi_1\rangle^{-a_1-a_2}\langle\eta_2\rangle^{-b_1-b_2}.\end{aligned} \tag{3.73}$$

Choosing η_2, ξ_1, η, ξ as integral variables, we apply Schwarz to F and G in η_2 and ξ_1, to φ and 1 in η, and then simply integrate 1 in ξ. Then we get

$$\begin{aligned}&\iiiint_{[\xi:\zeta,\eta],[\xi:\xi_1,\xi_2],[\eta:\eta_1,\eta_2]} wFG\varphi\, d\eta_2\, d\xi_1\, d\eta\, d\xi \\ &\lesssim \sum_{j\le(k\wedge j_1)+2,k\le k_2+2} 2^{(1+a)j+(1/2+s+b)k-(a_1+a_2)j_1-(b_1+b_2)k_2} \\ &\qquad \times F_{j_1,k_2}G_{j_1,k_2}\varphi_k.\end{aligned} \tag{3.74}$$

The exponent is rearranged as

$$-\sigma_1(j_1-j)-\sigma_{15}(k_2-k)-\sigma_{16}(k-j)-\sigma_4 j, \tag{3.75}$$

and the function part belongs to $\ell^1_{j_1}\ell^1_{k_2}\ell^2_k\ell^\infty_j$. Hence we can bound the sum as in the previous domains. □

4. Well-posedness by bilinear estimates

In this section, we prove the local well-posedness for DKG, QD, and WM by using the bilinear estimates derived in Section 3.

4.1. Local well-posedness for DKG

First, we give another proof of Theorem 1.1 except for $(a,s)=(0,0)$, stating it for the iteration map. The actual proof is immediate by the standard fixed point theorem after the rescaling argument in Section 2.1.

THEOREM 4.1

Let $s>0$, $a>-1/2$, $a+1\ge s\ge|a|$. We take b as follows:

$$b=\begin{cases} a+1-\varepsilon & (s=1/2),\\ 1/2-\varepsilon & (a+s=0),\\ \min\{a+1,a+s+1/2\} & (otherwise),\end{cases} \tag{4.1}$$

where $\varepsilon>0$ is a sufficiently small number satisfying $\varepsilon<\min\{1/2,a+1/2,s\}$. Assume that $u\in H^b_\alpha H^a_\beta\cap Y^{a,0,-1}$, $v\in H^a_\alpha H^b_\beta\cap Y^{a,-1,0}$, $\phi\in H^s_\alpha H^s_\beta\cap Y^{s-1,0,0}$, and $\mathfrak{u}(0)\in\mathcal{H}^{a,s}$. Let $(u^\sharp,v^\sharp,\phi^\sharp)$ be given by

$$\begin{aligned} &u^\sharp=u_F+I_\alpha(c_1v+c_2\phi v), \qquad v^\sharp=v_F+I_\beta(c_3u+c_4\phi u),\\ &\phi^\sharp=\phi_F+I_{\alpha,\beta}(c_5\phi+c_6uv),\end{aligned} \tag{4.2}$$

using the same notation as in (2.43). Then for any $T>0$, we have

$$\begin{aligned} &\|\chi_T(t)u^\sharp\|_{H^b_\alpha H^a_\beta\cap Y^{a,0,0}}\\ &\quad\lesssim\|u(0)\|_{H^a_x}+|c_1|\|v\|_{Y^{a,-1,0}\cap H^a_\alpha H^b_\beta}+|c_2|\|\phi\|_{H^s_\alpha H^s_\beta}\|v\|_{H^a_\alpha H^b_\beta},\\ &\|\chi_T(t)v^\sharp\|_{H^a_\alpha H^b_\beta\cap Y^{a,0,0}}\\ &\quad\lesssim\|v(0)\|_{H^a_x}+|c_3|\|u\|_{Y^{a,0,-1}\cap H^b_\alpha H^a_\beta}+|c_4|\|\phi\|_{H^s_\alpha H^s_\beta}\|u\|_{H^b_\alpha H^a_\beta},\end{aligned} \tag{4.3}$$

$$\begin{aligned}
&\|\chi_T(t)\phi^\sharp\|_{H^s_\alpha H^s_\beta \cap Y^{s,0,0}} + \|\chi_T(t)(\partial_t\phi^\sharp, \partial_x\phi^\sharp)\|_{Y^{s-1,0,0}} \\
&\quad \lesssim \|\phi(0)\|_{H^s_x} + \|\partial_t\phi(0)\|_{H^{s-1}_x} + |c_5|\|\phi\|_{Y^{s-1,0,0}\cap H^{s-1}_\alpha H^{s-1}_\beta} \\
&\qquad + |c_6|\|u\|_{H^b_\alpha H^a_\beta}\|v\|_{H^a_\alpha H^b_\beta}.
\end{aligned}$$

Note that the coefficients c_1–c_6 are determined as in (1.13) from the original Dirac-Klein-Gordon system such that $|c_1| + |c_3| \lesssim m$ and $|c_5| \lesssim M^2$.

Proof
Note that $b > 1/2$ if $a + s > 0$. We first estimate $u^\sharp$. The estimate on $v^\sharp$ is the same by symmetry. Thanks to Lemmas 3.2 and 3.4, we have only to find $a_0, b_0 \in \mathbb{R}$ such that

$$\begin{aligned}
&b - 1 \prec \{s, a\}, \qquad a \prec \{s, b\}, \\
&a_0 \prec \{s, a\}, \qquad b_0 \prec \{s, b\}, \qquad a \prec \{a_0 + 1, b_0\}, \\
&2s > -\frac{1}{2}, \ a + b > -\frac{1}{2}.
\end{aligned} \tag{4.4}$$

We can choose a_0, b_0 as follows:

$$a_0 = b - 1, \qquad b_0 = \begin{cases} 1/2 & (s = 1/2), \\ s - \varepsilon & (a + s = 0), \\ s & (\text{otherwise}). \end{cases} \tag{4.5}$$

Then the inequalities in the second line of (4.4) hold, while the others follow from the assumptions.

Using (3.7), we get

$$\|\chi_T(t)u^\sharp\|_{Y^{a,0,0}} \lesssim \|u(0)\|_{H^a_x} + \|c_1 v + c_2\phi v\|_{Y^{a,-1,0}}, \tag{4.6}$$

and the last term is bounded by using (3.32) together with the condition (4.4):

$$|c_1|\|v\|_{Y^{a,-1,0}} + |c_2|\|\phi\|_{H^s_\alpha H^s_\beta}\|v\|_{H^a_\alpha H^b_\beta}. \tag{4.7}$$

Similarly, we get from (3.8),

$$\|\chi_T(t)u^\sharp\|_{H^b_\alpha H^a_\beta} \lesssim \|\chi_{2T}(t)u^\sharp\|_{Y^{a,0,0}} + \|c_1 v + c_2\phi v\|_{H^{b-1}_\alpha H^a_\beta}. \tag{4.8}$$

The second last term satisfies the same estimate as (4.6), and the last term is bounded by using (2.14) together with (4.4) and the fact that $b - 1 \le a \le b$:

$$|c_1|\|v\|_{H^a_\alpha H^b_\beta} + |c_2|\|\phi\|_{H^s_\alpha H^s_\beta}\|v\|_{H^a_\alpha H^b_\beta}. \tag{4.9}$$

Note that we could use the product estimate under the stronger condition (3.34) in the above argument since $a_0 + b_0 > -1/2$. The difference from the weaker condition (3.31) appears in the following estimate on $\phi^\sharp$.

For the estimate on $\phi^\sharp$, we have only to find $a_0, b_0 \in \mathbb{R}$ such that

$$\begin{aligned}
&s - 1 \prec \{a, b\}, \\
&a_0 \prec \{a, b\}, \qquad b_0 \prec \{a, b\},
\end{aligned} \tag{4.10}$$

$$s-1 \prec \{a_0+1, b_0\}, \qquad s-1 \prec \{a_0, b_0+1\},$$
$$a+b > -1/2.$$

We can choose a_0, b_0 as follows:

$$a_0 = b_0 = \begin{cases} b-1 & (s=1/2), \\ a-\varepsilon & (a+s=0), \\ a & (\text{otherwise}). \end{cases} \tag{4.11}$$

Then the inequalities in the second line also hold, while the others follow from the assumption.

By (3.9), we get

$$\begin{aligned} &\|\chi_T(t)\phi^\sharp\|_{Y^{s,0,0}} + \|\chi_T(t)(\partial_t\phi^\sharp, \partial_x\phi^\sharp)\|_{Y^{s-1,0,0}} \\ &\lesssim \|\phi(0)\|_{H^s_x} + \|\partial_t\phi(0)\|_{H^{s-1}_x} + \|c_5\phi + c_6 uv\|_{Y^{s-1,-1,0}\cap Y^{s-1,0,-1}}, \end{aligned} \tag{4.12}$$

and the last term is bounded by using (3.32) together with the condition (4.10):

$$\lesssim |c_5|\|\phi\|_{Y^{s-1,0,0}} + |c_6|\|u\|_{H^b_\alpha H^a_\beta}\|v\|_{H^a_\alpha H^b_\beta}. \tag{4.13}$$

Note that if we used (3.34), then we would need

$$a_0 + b_0 > -\frac{1}{2}, \tag{4.14}$$

which requires $a > -1/4$ since $a_0 \le a$. Thus we encounter the essential advantage of (3.31).

Similarly, we have from (3.10),

$$\begin{aligned} \|\chi_T(t)\phi^\sharp\|_{H^s_\alpha H^s_\beta} &\lesssim \|\chi_{2T}(t)\phi^\sharp\|_{Y^{s,0,0}} + \|\chi_{2T}(t)(\partial_t\phi^\sharp, \partial_x\phi^\sharp)\|_{Y^{s-1,0,0}} \\ &\quad + \|c_5\phi + c_6 uv\|_{H^{s-1}_\alpha H^{s-1}_\beta}, \end{aligned} \tag{4.15}$$

where the Y-norms are bounded in the same way as (4.12), while the last term is estimated by using (3.32) together with the condition (4.10):

$$\lesssim |c_5|\|\phi\|_{H^{s-1}_\alpha H^{s-1}_\beta} + |c_6|\|u\|_{H^b_\alpha H^a_\beta}\|v\|_{H^a_\alpha H^b_\beta}. \tag{4.16}$$

□

4.2. Local well-posedness of QD

Now we prove Theorems 1.5 and 1.6. We state them in iteration form.

THEOREM 4.2

Let $a > -1/2$. Assume that $u \in H^{a+1}_\alpha H^a_\beta \cap Y^{a,0,0}$, $v \in H^a_\alpha H^{a+1}_\beta \cap Y^{a,0,0}$, $u(0,x) \in H^a$, and $v(0,x) \in H^a$. Define $u^\sharp(t,x)$, $v^\sharp(t,x)$ by

$$\begin{aligned} u^\sharp &= u_F + I_\alpha(c_3 v + c_7 uv), \\ v^\sharp &= v_F + I_\beta(c_5 u + c_8 uv) \end{aligned} \tag{4.17}$$

using the same notation as in (2.43) and some constants $c_7, c_8 \in \mathbb{C}$. Then for any $T > 0$, we have

$$\begin{aligned} \|\chi_T(t)u^\sharp\|_{H_\alpha^{a+1}H_\beta^a \cap Y^{a,0,0}} &\lesssim \|u(0)\|_{H_x^a} + |c_3|\|v\|_{Y^{a,-1,0}\cap H_\alpha^a H_\beta^a} \\ &\quad + |c_7|\|u\|_{H_\alpha^{a+1}H_\beta^a}\|v\|_{H_\alpha^a H_\beta^{a+1}}, \\ \|\chi_T(t)v^\sharp\|_{H_\alpha^a H_\beta^{a+1} \cap Y^{a,0,0}} &\lesssim \|v(0)\|_{H_x^a} + |c_5|\|u\|_{Y^{a,0,-1}\cap H_\alpha^a H_\beta^a} \\ &\quad + |c_8|\|u\|_{H_\alpha^{a+1}H_\beta^a}\|v\|_{H_\alpha^a H_\beta^{a+1}}. \end{aligned} \tag{4.18}$$

Proof
By (3.7), we have

$$\|\chi_T(t)u^\sharp\|_{Y^{a,0,0}} \lesssim \|u(0)\|_{H_x^a} + \|c_3 v + c_7 uv\|_{Y^{a,-1,0}}, \tag{4.19}$$

and the last term is bounded by

$$|c_3|\|v\|_{Y^{a,-1,0}} + |c_7|\|u\|_{H_\alpha^{a+1}H_\beta^a}\|v\|_{H_\alpha^a H_\beta^{a+1}}, \tag{4.20}$$

where we used (3.32) together with the conditions

$$a \prec \{a, a+1\}, \quad a + a + 1 > -\frac{1}{2}, \tag{4.21}$$

which follow from $a > -1/2$. Similarly, from (3.8) we have

$$\|\chi_T(t)u^\sharp\|_{H_\alpha^{a+1}H_\beta^a} \lesssim \|\chi_{2T}(t)u^\sharp\|_{Y^{a,0,0}} + \|c_3 v + c_7 uv\|_{H_\alpha^a H_\beta^a}, \tag{4.22}$$

and the first term on the right is estimated in the same way as above, while the last term is bounded by

$$|c_3|\|v\|_{H_\alpha^a H_\beta^a} + |c_7|\|u\|_{H_\alpha^{a+1}H_\beta^a}\|v\|_{H_\alpha^a H_\beta^{a+1}}, \tag{4.23}$$

where we used (2.14) and the condition $a \prec \{a, a+1\}$. The estimates for $v^\sharp$ are done in the same way by symmetry. □

4.3. Local well-posedness of WM

It is convenient to rewrite WM in a system similar to QD for $u := \partial_\beta \phi$ and $v := \partial_\alpha \phi$. The nonlinear term is given by

$$g(\phi)(u, v) := \Big(\sum_{k,l=1}^{N} g(\phi)_j^{k,l} u_k v_l \Big)_{j=1,\dots,N}, \tag{4.24}$$

and WM is rewritten by

$$\begin{aligned} u &= u_F + I_\alpha\big(g_0 + g^\triangle(\phi^\triangle)\big)(u, v), \\ v &= v_F + I_\beta\big(g_0 + g^\triangle(\phi^\triangle)\big)(u, v), \end{aligned} \qquad \phi^\triangle = J_\beta u|_{\alpha=0} + J_\alpha v, \tag{4.25}$$

where g_0 and $g^\triangle$ are defined by

$$g_0 = g\big(\phi(0,0)\big), \qquad g^\triangle = g\big(\phi(0,0) + \phi^\triangle\big) - g\big(\phi(0,0)\big). \tag{4.26}$$

Obviously ϕ is reconstructed by $\phi = \phi(0,0) + \phi^\triangle$.

The initial data for u, v are made small in H^{s-1} by scaling,* and the estimate on $\phi^\triangle$ is trivial from Lemma 2.1 after the space-time localization as in Section 2.1 (where the point $(0,0)$ should be shifted to the center of each spatial localization). Hence the only new ingredient (compared with QD) is the multiplication by $g^\triangle(\phi^\triangle)$, for which we need the following lemmas.

LEMMA 4.3
Let $1/2 < s \le r \in \mathbb{N}$, $N \in \mathbb{N}$, $g \in C^{2r}(\mathbb{R}^N;\mathbb{R})$, and $g(0) = 0$. Then we have, for any space-time function $u : \mathbb{R}^2 \to \mathbb{R}^N$,

$$\|g(u)\|_{H^s_\alpha H^s_\beta} \lesssim C(\|u\|_{H^s_\alpha H^s_\beta})\|u\|_{H^s_\alpha H^s_\beta}, \tag{4.27}$$

where C is a nondecreasing continuous function determined by g and s.

The estimate on the difference follows from this together with the mean value theorem and the algebraic property of $H^s_\alpha H^s_\beta$ if $g \in C^{2r+1}$:

$$\begin{aligned} &\|g(u_0) - g(u_1)\|_{H^s_\alpha H^s_\beta} \\ &\le \int_0^1 \|g'(u_\theta)(u_0 - u_1)\|_{H^s_\alpha H^s_\beta}\, d\theta \\ &\le \int_0^1 \big[\|g'(u_\theta) - g'(0)\|_{H^s_\alpha H^s_\beta} + |g'(0)|\big]\|u_0 - u_1\|_{H^s_\alpha H^s_\beta}\, d\theta \\ &\le C(\|u_0\|_{H^s_\alpha H^s_\beta} + \|u_1\|_{H^s_\alpha H^s_\beta})\|u_0 - u_1\|_{H^s_\alpha H^s_\beta}, \end{aligned} \tag{4.28}$$

where $u_\theta := (1-\theta)u_0 + \theta u_1$.

The next lemma is a multiplier property of $H^s_\alpha H^s_\beta$ on the Y^{s-1}-space.

LEMMA 4.4
Let $s > 1/2$ and $Z^s := H^{s-1}_\alpha H^{s-1}_\beta \cap Y^{s-1,-1,0} \cap Y^{s-1,0,-1}$. Then we have, for any space-time functions $f(t,x)$ and $u(t,x)$,

$$\|fu\|_{Z^s} \lesssim \|f\|_{H^s_\alpha H^s_\beta}\|u\|_{Z^s}. \tag{4.29}$$

Using them, we get the well-posedness of WM by iteration for localized (4.25).

THEOREM 4.5
Let $1/2 < s \le r \in \mathbb{N}$ and $g \in C^{2r+1}(\mathbb{R}^N \to \mathbb{R}^{N^3})$. Assume that $u \in H^s_\alpha H^{s-1}_\beta$, $v \in H^{s-1}_\alpha H^s_\beta$, $\phi \in H^s_\alpha H^s_\beta$, and $u(0,x), v(0,x) \in H^{s-1}$. Let $u^\sharp, v^\sharp, \phi^\sharp$ be given by

$$\begin{aligned} u^\sharp &= u_F + I_\alpha\big(g_0 + g^\triangle(\phi^\sharp)\big)(u,v), \\ v^\sharp &= v_F + I_\beta\big(g_0 + g^\triangle(\phi^\sharp)\big)(u,v), \\ \phi^\sharp &= \chi_T(\alpha,\beta)[J_\beta u|_{\alpha=0} + J_\alpha v], \end{aligned} \tag{4.30}$$

*We avoided scaling $\phi(0)$ in H^s, whose low-frequency part is not scaled in a good way.

where u_F, v_F are the same as in (2.43), and $g_0, g^\triangle$ are given by (4.26) for a prescribed $\phi(0,0)$. Then for any $T>0$, we have

$$\begin{aligned}&\|\chi_T(t)u^\sharp\|_{H^s_\alpha H^{s-1}_\beta \cap Y^{s-1,0,0}} + \|\chi_T(t)v^\sharp\|_{H^{s-1}_\alpha H^s_\beta \cap Y^{s-1,0,0}} + \|\phi^\sharp\|_{H^s_\alpha H^s_\beta \cap Y^{s,0,0}} \\ &\lesssim \|u(0)\|_{H^s} + \|v(0)\|_{H^s} + \|u\|_{H^s_\alpha H^{s-1}_\beta} + \|v\|_{H^{s-1}_\alpha H^s_\beta} \\ &\quad + C(\|u\|_{H^s_\alpha H^{s-1}_\beta} + \|v\|_{H^{s-1}_\alpha H^s_\beta})\|u\|_{H^s_\alpha H^{s-1}_\beta}\|v\|_{H^{s-1}_\alpha H^s_\beta},\end{aligned} \tag{4.31}$$

where C is a nondecreasing continuous function determined by T, $\phi(0,0)$, g, and s.

Proof
By the same argument as for (4.18), we have

$$\|\chi_T(t)u^\sharp\|_{Y^{s-1,0,0}\cap H^s_\alpha H^{s-1}_\beta} \lesssim \|u(0)\|_{H^{s-1}} + \big\|\big(g_0 + g^\triangle(\phi^\sharp)\big)(u,v)\big\|_{Z^s}, \tag{4.32}$$

where the last term is bounded by

$$\sum_{j,k,l=1}^{N} \big(|g^{k,l}_{0,j}| + \|g^\triangle(\phi^\sharp)^{k,l}_j\|_{H^s_\alpha H^s_\beta}\big)\|u_k v_l\|_{Z^s}, \tag{4.33}$$

where we used Lemma 4.4. The estimate on $\phi^\sharp$ is simpler. Then the norm of $g^\triangle(\phi^\sharp)$ is estimated by Lemma 4.3, and the last factor is bounded by

$$\|u_k\|_{H^s_\alpha H^{s-1}_\beta}\|v_l\|_{H^{s-1}_\alpha H^s_\beta}, \tag{4.34}$$

where we used (2.14) and (3.32). □

Now we have only to prove the lemmas.

Proof of Lemma 4.3
We first consider the case $1/2 < s < 1$. Then $r \ge 1$, and so $g \in C^2$. By Plancherel or by the standard argument in the usual Besov space, it is easy to see that the following norm is equivalent to $H^s_\alpha H^s_\beta$:

$$\|2^{js+ks}\delta^j_+\delta^k_- u\|_{L^2_{j,k,t,x}(\mathbb{N}^2\times\mathbb{R}^2)} + \sum_\pm \|2^{js}\delta^j_\pm u\|_{L^2_{j,t,x}(\mathbb{N}\times\mathbb{R}^2)} + \|u\|_{L^2_{t,x}(\mathbb{R}^2)}, \tag{4.35}$$

where $\delta^j_\pm$ are the difference operators defined by

$$\delta^j_\pm u(t,x) = u(t+2^{-j}, x\pm 2^{-j}) - u(t,x). \tag{4.36}$$

For $\|g(u)\|_{H^s_\alpha H^s_\beta}$, we estimate only the first component of (4.35) since the others are easier. The double difference can be rewritten as

$$\delta^j_+\delta^k_- g(u) = \iint_{[0,1]^2} \partial_p\partial_q g\big(U^{j,k}(p,q)\big)\,dp\,dq, \tag{4.37}$$

where $U^{j,k}$ is defined by

$$U^{j,k}(p,q) := u + p\delta^j_+ u + q\delta^k_- u + pq\delta^j_+\delta^k_- u. \tag{4.38}$$

Then we can expand it by using the derivatives of g,

$$\delta_+^j \delta_-^k g(u) = \iint_{[0,1]^2} g''(U) U_p U_q + g'(U) U_{pq} \, dp \, dq, \tag{4.39}$$

where we omit the superscripts j, k and

$$U_p = \delta_+^j [u + q\delta_-^k u], \qquad U_q = \delta_-^k [u + p\delta_+^j u], \quad U_{pq} = \delta_+^j \delta_-^k u. \tag{4.40}$$

By the Sobolev embedding $H_\alpha^s H_\beta^s \subset L_{t,x}^\infty$, we may assume that g' and g'' are bounded (on the convex hull of the image of u). Then

$$\begin{aligned} &\|2^{js+ks} \delta_+^j \delta_-^k g(u)\|_{L^2_{j,k,t,x}} \\ &\lesssim \|2^{js} \delta_+^j u\|_{\ell_j^2 L_\alpha^2 L_\beta^\infty} \|2^{ks} \delta_-^k u\|_{\ell_k^2 L_\alpha^\infty L_\beta^2} \\ &\quad + \|2^{js+ks} \delta_+^j \delta_-^k u\|_{L^2_{j,k,t,x}}, \end{aligned} \tag{4.41}$$

where the implicit constant depends on the bound of g'' and g'. By the Sobolev embedding $H^s(\mathbb{R}) \subset L^\infty$, we have

$$\|2^{js} \delta_+^j u\|_{\ell_j^2 L_\alpha^2 L_\beta^\infty} + \|2^{ks} \delta_-^k u\|_{\ell_k^2 L_\alpha^\infty L_\beta^2} \lesssim \|u\|_{H_\alpha^s H_\beta^s}. \tag{4.42}$$

Thus we obtain the desired estimate for $1/2 < s < 1$. The case $s = 1$ is easy by

$$\|u_\alpha\|_{L_\alpha^2 L_\beta^\infty} + \|u_\beta\|_{L_\alpha^\infty L_\beta^2} \lesssim \|u\|_{H_\alpha^1 H_\beta^1}. \tag{4.43}$$

We can extend the desired estimate to higher s by induction. Suppose that it holds for $1/2 < s \le k \in \mathbb{N}$. Then for $k < s \le k+1$, we have

$$\|g(u)\|_{H_\alpha^s H_\beta^s} \lesssim \sum_{p,q=0}^{1} \|\partial_\alpha^p \partial_\beta^q g(u)\|_{H_\alpha^{s-1} H_\beta^{s-1}}. \tag{4.44}$$

We estimate the right-hand side only for $p = q = 1$ since the other terms are easier:

$$\begin{aligned} &\|\partial_\alpha \partial_\beta g(u)\|_{H_\alpha^{s-1} H_\beta^{s-1}} \\ &= \|g''(u) u_\alpha u_\beta + g'(u) u_{\alpha\beta}\|_{H_\alpha^{s-1} H_\beta^{s-1}} \\ &\lesssim \|g''(u)\|_{H_\alpha^k H_\beta^k} \|u_\alpha\|_{H_\alpha^{s-1} H_\beta^s} \|u_\beta\|_{H_\alpha^s H_\beta^{s-1}} \\ &\quad + \|g'(u)\|_{H_\alpha^k H_\beta^k} \|u_{\alpha\beta}\|_{H_\alpha^{s-1} H_\beta^{s-1}}, \end{aligned} \tag{4.45}$$

where we used the product estimate (2.2). Since $g'', g' \in C^{2r-2}$ and $2r - 2 \ge 2k$, by the assumption we have $g''(u), g'(u) \in H_\alpha^k H_\beta^k$. Thus the desired estimate is extended to $s \le k+1$, and so by induction we obtain it for all $s > 1/2$. □

Proof of Lemma 4.4

The estimate on the $H_\alpha^{s-1} H_\beta^{s-1}$-component is immediate from the product estimate (2.2), so it remains to estimate the Y-components, which are stronger than the $H_\alpha^{s-1} H_\beta^{s-1}$-norm only if $s \le 3/4$ (because of the embedding Lemma 3.1) and only in the Fourier region $\{|\tau| \gg \langle\xi\rangle\}$, because in the complement we have

$$\begin{aligned}
&\big\|\langle\xi\rangle^{s-1}(\langle\tau-\xi\rangle^{-1}+\langle\tau+\xi\rangle^{-1})\widetilde{u}\big\|_{L^2_\xi L^1_\tau(|\tau|\lesssim\langle\xi\rangle)} \\
&\lesssim \sum_{\pm} \|\langle\tau\pm\xi\rangle^{s-1}\langle\tau\mp\xi\rangle^{-1}\widetilde{u}\|_{L^2_\xi L^1_\tau(|\tau|\lesssim\langle\xi\rangle,\,\pm\tau\xi>0)} \\
&\lesssim \|\langle\tau\pm\xi\rangle^{s-1}\langle\tau\mp\xi\rangle^{s-1}\widetilde{u}\|_{L^2_\xi L^2_\tau},
\end{aligned} \tag{4.46}$$

where in the last step we used Schwarz in τ and the fact that $s > 1/2$.

We are going to show that

$$\begin{aligned}
&\Big\|\iint_{\mathbb{R}^2} I\, d\sigma\, d\eta\Big\|_{L^2_\xi L^1_\tau(K)} \lesssim \|f\|_{H^s_\alpha H^s_\beta}\|u\|_{Z^s}, \\
&I := \langle\tau\rangle^{-1}\langle\xi\rangle^{s-1}\widetilde{f}(\tau-\sigma,\xi-\eta)\widetilde{u}(\sigma,\eta), \quad K := \big\{|\tau| \gg \langle\xi\rangle\big\},
\end{aligned} \tag{4.47}$$

for $1/2 < s < 1$. We divide the integral of I into 4 regions. In the region where $\langle\sigma\rangle \lesssim \langle\tau\rangle$ and $\langle\xi\rangle \lesssim \langle\eta\rangle$, the above estimate is trivial from Minkowski since the weight is transferred to $\widetilde{u}$ and $\widetilde{f} \in L^1_\tau L^1_\xi$.

In the region $D_1 := \{\langle\sigma\rangle \gg \langle\tau\rangle,\ \langle\xi\rangle \gtrsim \langle\eta\rangle\}$, we have

$$|\tau-\sigma| \sim |\sigma| \gg |\tau| \gg \langle\xi\rangle \gtrsim \langle\xi-\eta\rangle + \langle\eta\rangle. \tag{4.48}$$

Let $F := \langle\tau+\xi\rangle^s\langle\tau-\xi\rangle^s\widetilde{f}$ and $G := \langle\tau\rangle^{-1}\langle\xi\rangle^{s-1}\widetilde{u}$. Then we have

$$\|F\|_{L^2_\xi L^2_\tau} \sim \|f\|_{H^s_\alpha H^s_\beta}, \qquad \|G\|_{L^2_\xi L^1_\tau} \sim \|u\|_{Z^s}, \tag{4.49}$$

and

$$\begin{aligned}
|I| &\lesssim \langle\tau\rangle^{-1}\langle\xi\rangle^{s-1}\langle\sigma\rangle^{-2s}\langle\sigma\rangle\langle\xi\rangle^{1-s}F(\tau-\sigma,\xi-\eta)G(\sigma,\eta) \\
&\lesssim \langle\tau\rangle^{-2s}F(\tau-\sigma,\xi-\eta)G(\sigma,\eta).
\end{aligned} \tag{4.50}$$

Hence by Hölder and Young,

$$\begin{aligned}
\Big\|\iint_{D_1} I\, d\sigma\, d\eta\Big\|_{L^2_\xi L^1_\tau(K)} &\lesssim \|\langle\tau\rangle^{-2s}\|_{L^2_{\xi,\tau}(K)}\|F*G\|_{L^\infty_\xi L^2_\tau} \\
&\lesssim \|F\|_{L^2_{\xi,\tau}}\|G\|_{L^2_\xi L^1_\tau} \sim \|f\|\|u\|.
\end{aligned} \tag{4.51}$$

In the region $D_2 := \{\langle\sigma\rangle \lesssim \langle\tau\rangle,\ \langle\eta\rangle \gg \langle\xi\rangle\}$, we have

$$\sum_{\pm} |\tau-\sigma\pm\xi\mp\eta| \gtrsim |\xi-\eta| \sim |\eta| \gg \langle\xi\rangle, \tag{4.52}$$

and so

$$|I| \lesssim \langle\xi\rangle^{-s}\sum_{\pm}\langle\tau-\sigma\pm\xi\mp\eta\rangle^{-s}F(\tau-\sigma,\xi-\eta)G(\sigma,\eta). \tag{4.53}$$

Then by Hölder and Young,

$$\begin{aligned}
\Big\|\iint_{D_2} I\, d\sigma\, d\eta\Big\|_{L^2_\xi L^1_\tau(K)} &\lesssim \|\langle\xi\rangle^{-s}\|_{L^2_\xi}\Big\|G*\sum_{\pm}\langle\tau\pm\xi\rangle^{-s}F\Big\|_{L^\infty_\xi L^1_\tau} \\
&\lesssim \|G\|_{L^2_\xi L^1_\tau}\Big\|\sum_{\pm}\langle\tau\pm\xi\rangle^{-s}F\Big\|_{L^2_\xi L^1_\tau} \lesssim \|f\|\|u\|.
\end{aligned} \tag{4.54}$$

In the region $D_3 := \{\langle\sigma\rangle \gg \langle\tau\rangle,\ \langle\eta\rangle \gg \langle\xi\rangle\}$, we have

$$\sum_{\pm} |\tau-\sigma \pm \xi \mp \eta| \sim |\tau-\sigma| + |\xi-\eta| \sim |\sigma| + |\eta| \sim \sum_{\pm} |\sigma \pm \eta|. \tag{4.55}$$

Let $\nu_1 := \min_\pm |\tau-\sigma\pm\xi\mp\eta|$ and $\nu_2 := \min_\pm |\sigma\pm\eta|$. Then we have in K,

$$\langle\nu_2\rangle \lesssim \langle\nu_1\rangle + |\tau| + |\xi| \lesssim \langle\nu_1\rangle + \langle\tau\rangle. \tag{4.56}$$

Let $H := \langle\tau+\xi\rangle^{s-1}\langle\tau-\xi\rangle^{s-1}\widetilde{u}$. Then we have $\|H\|_{L^2_\xi L^2_\tau} \lesssim \|u\|_{H^{s-1}_\alpha H^{s-1}_\beta}$, and

$$\begin{aligned} |I| &\lesssim \langle\tau\rangle^{-1}\langle\xi\rangle^{s-1}\langle|\sigma|+|\eta|\rangle^{1-2s}\langle\nu_1\rangle^{-s}\langle\nu_2\rangle^{1-s}F(\tau-\sigma,\xi-\eta)H(\sigma,\eta) \\ &\lesssim \langle\tau\rangle^{-2s}\langle\xi\rangle^{s-1}[1+\langle\nu_1\rangle^{-s}\langle\tau\rangle^{1-s}]F(\tau-\sigma,\xi-\eta)H(\sigma,\eta). \end{aligned} \tag{4.57}$$

Using Hölder and Young as in the previous domains, we obtain

$$\begin{aligned} \Big\| \iint_{D_3} I\, d\sigma\, d\eta \Big\|_{L^2_\xi L^1_\tau(K)} &\lesssim \|\langle\tau\rangle^{-2s}\langle\xi\rangle^{s-1}\|_{L^2_\xi L^1_\tau(K)} \|F*H\|_{L^\infty_\xi L^\infty_\tau} \\ &\quad + \|\langle\tau\rangle^{1-3s}\langle\xi\rangle^{s-1}\|_{L^2_{\xi,\tau}(K)} \Big\| H * \sum_\pm \langle\tau\pm\xi\rangle^{-s}F \Big\|_{L^\infty_\xi L^2_\tau} \\ &\lesssim \|F\|_{L^2_{\tau,\xi}} \|H\|_{L^2_{\tau,\xi}} \lesssim \|f\|\|u\|. \end{aligned} \tag{4.58}$$

□

5. Ill-posedness results

In this section, we prove the ill-posedness results. We use the estimates in the previous arguments for the well-posedness, as well as the notation.

5.1. Instant exit for DKG

We start with ill-posedness by instantaneous exit, Theorem 1.2 for DKG, which is caused by unbalanced regularity. In the following, all estimates should be understood locally in space-time by the finite propagation property. As before, we denote the free solutions by u_F, v_F, ϕ_F and the remaining part by $u_1 = u - u_F$, $v_1 = v - v_F$, and $\phi_1 = \phi - \phi_F$.

5.1.1. DKG for $a > \max(s, 0)$

In this case, the regularity of ϕ is too low for u and v. First we consider the case $a > s > 0$. Then by the well-posedness in $\mathcal{H}^{s,s}$ and the proofs in the previous sections, we have

$$\begin{aligned} &u \in H^b_\alpha H^s_\beta, \qquad v \in H^s_\alpha H^b_\beta, \qquad \phi \in H^s_\alpha H^s_\beta, \\ &\phi_1 \in H^{s+1}_\alpha H^{s+1}_\beta, \qquad I_\alpha(\phi_1 v) \in H^{s+1}_\alpha H^b_\beta, \qquad I_\beta(\phi_1 u) \in H^b_\alpha H^{s+1}_\beta, \end{aligned} \tag{5.1}$$

where b satisfies (2.42), and so the last three terms are bounded in $L^\infty_{t,x}$ by the Sobolev embedding. For any $\varphi \in H^s$, we can choose the initial data of ϕ such that $\phi_F = \varphi(\beta)$. Then we have

$$\phi, u \in L^p_\beta L^\infty_\alpha, \quad |I_\beta(\phi_F u)(t,x)| \lesssim |t|^\delta, \tag{5.2}$$

where $p > 2$ is such that $H^s \subset L^p$, and $\delta = 2(1/2 - 1/p) > 0$. Hence we have $\|v_1\|_{L^\infty_{t,x}} < 1/2$ for small t. Now we choose the initial data of v to be smooth and 1 for $|x| < 1$. Then we have $v_F(t,x) = 1$, and hence $v(t,x) > 1/2$ if $|t| + |x| < 1$ and $t \ll 1$. Then in this region we have

$$I_\alpha(\phi_0 v) = \varphi(\beta)V, \quad V := I_\alpha v, \tag{5.3}$$

$v \in H^s_\alpha H^b_\beta$, and $\mathrm{Re}[v(t,x)] > 1/2$. Therefore we have

$$V \in H^{s+1}_\alpha H^b_\beta \hookrightarrow L^\infty_t H^b_x, \quad |V(t,x)| > \frac{t}{2}. \tag{5.4}$$

Hence we can divide $I_\alpha(\phi_0 v)$ by $V(\alpha,\beta)^\times$, which implies that the regularity of $I_\alpha(\phi_0 v)$ cannot be better than that of $\varphi(\beta)$. Thus if we choose $\varphi \notin H^a$, then $I_\alpha(\phi_0 v) \notin H^a_x$ for any $t \neq 0$ (all in the region $|t| + |x| < 1$). Since the other part of u_1 is more regular, this implies that $u(t)$ instantly exits the space H^a_x.

The above implies the ill-posedness for $s \le 0 < a$ as well because we can choose $s' \in (0,a)$ and initial data in $\mathcal{H}^{a,s'} \subset \mathcal{H}^{a,s}$ such that u instantly exits H^a; hence the solution is not in $\mathcal{H}^{a,s}$ either.

5.1.2. DKG for $s > \max(a+1, 1/2)$

In this case, the regularity of u and v is too low for ϕ. We may restrict the region to $a + 2 > s > a + 1 > 1/2$ by the same reasoning as in Section 5.1.1.

We caution that this case is not as simple as the one in Section 5.1.1 because it is not so easy to isolate the leading term in the sense of regularity. Indeed, the leading term is heuristically obvious ($u_F I_\alpha v_F$ or $v_F I_\alpha u_F$), but the previous arguments give only the same regularity for the remainder terms.

To overcome this difficulty, we exploit the following two peculiar properties of singularity at zero (of a continuous function):

(1) *square smoothing*: the square is more regular than that of a nonzero singularity, or that given by the product estimate;

(2) *robustness*: the singularity is not removed by multiplication with a nonzero continuous function, even if the latter has the same or less regularity. (Nonzero singularity, by contrast, can be canceled by multiplication with an irregular function.)

We need only some special cases. More precisely, we use the following.

LEMMA 5.1

Let $p > -1/2$, and assume that $f : \mathbb{R} \to \mathbb{C}$ satisfies

$$f(x) = \begin{cases} |x|^p & (0 \le x < 1), \\ 0 & (x < 0, x > 2), \\ \textit{smooth} & (|x| > \frac{1}{2}) \end{cases} \tag{5.5}$$

and $g : \mathbb{R} \to \mathbb{C}$ satisfies $\inf_{0<x<\varepsilon} |g(x)| > 0$ for some $\varepsilon > 0$. Then for any $s < p + 1/2$, we have $f \in H^s(\mathbb{R})$ but $fg \notin H^{p+1/2}(\mathbb{R})$.

Proof
Let l be the maximal integer less than $p+1/2$, and let $l<s<p+1/2$. We estimate f by the difference norm

$$\|f\|_{H^s}^2 \sim \|f\|_{L^2}^2 + \int_0^\infty t^{-2(s-l)} \|\delta_t \partial_x^l f(x)\|_{L_x^2(\mathbb{R})}^2 \frac{dt}{t}, \tag{5.6}$$

where the difference operator δ_t is defined by $\delta_t f(x) = f(x+t) - f(x)$. Since $\partial_x^l f(x) \sim \pm \max(0,x)^{p-l}$ around $x=0$ and $p-l>-1/2$, we have $f, \partial_x^l f \in L^2$. Hence it suffices to bound the integral for $0<t<1$ in (5.6). Then the L_x^2-norm is bounded by

$$\begin{aligned} &\|\delta_t \partial_x^l f(x)\|_{L_x^2(|x|\lesssim t)} + \|\delta_t \partial_x^l f(x)\|_{L_x^2(t\ll|x|<3)} \\ &\lesssim \||x|^{p-l}\|_{L_x^2(|x|\lesssim t)} + \||x|^{p-l-1} t\|_{L_x^2(t\ll|x|<3)} \lesssim t^{p-l+1/2}(1+|\log t|), \end{aligned} \tag{5.7}$$

and so the integral in (5.6) is bounded because $s<p+1/2$.

Next, we estimate fg by the difference norm. Let $p+1/2 \le m \in \mathbb{N}$. Then

$$\begin{aligned} \|fg\|_{H^{p+1/2}}^2 &\gtrsim \int_0^\infty t^{-2(p+1/2)} \|\delta_t^{m+1}(fg)(x)\|_{L^2}^2 \frac{dt}{t} \\ &\gtrsim \int_0^\varepsilon t^{-2p-1} \|\delta_t^{m+1}(fg)(x)\|_{L^2}^2 \frac{dt}{t} =: \int_0^\varepsilon t^{-2p-1} I(t) \frac{dt}{t}. \end{aligned} \tag{5.8}$$

For $0<t<\varepsilon$, we have

$$\begin{aligned} I(t) &\gtrsim \int_{-(m+1)t}^{-mt} \big|(fg)\big(x+(m+1)t\big) - (m+1)(fg)(x+mt) + \cdots\big|^2 \, dx \\ &= \int_{-(m+1)t}^{-mt} \big|(fg)\big(x+(m+1)t\big)\big|^2 \, dx \\ &= \int_0^t |(fg)(x)|^2 \, dx \gtrsim \mu^2 t^{2p+1}, \end{aligned} \tag{5.9}$$

where $\mu := \inf_{0<x<\varepsilon} |g(x)| > 0$. Hence we conclude that

$$\|fg\|_{H^{p+1/2}}^2 \gtrsim \mu^2 \int_0^\varepsilon \frac{dt}{t} = \infty. \tag{5.10}$$

□

Now we start the proof for $a+2>s>a+1>1/2$. By the previous argument for the well-posedness in $\mathcal{H}^{a,a+1} \supset \mathcal{H}^{a,s} \ni \mathfrak{u}(0)$, we have

$$u_1, v_1, \phi \in H_\alpha^{a+1} H_\beta^{a+1} \subset L_{t,x}^\infty. \tag{5.11}$$

By choosing $v(0) \in H_x^{a+1}$, we may assume in addition that $v \in H_\alpha^{a+1} H_\beta^{a+1}$. Then we get $u_1 v \in H_\alpha^{a+1} H_\beta^{a+1}$, and so $I_{\alpha,\beta}(u_1 v) \in H_\alpha^{a+2} H_\beta^{a+2} \subset C_t(H^s)$. Hence it suffices to show that $I_{\alpha,\beta}(u_F v) \notin H^s$ for any small $t>0$. We frequently use the commutators

$$[I_\alpha, f(\beta)] = [I_\beta, f(\alpha)] = 0, \qquad [\partial_\beta, I_\alpha] = R_\beta. \tag{5.12}$$

We expand it by partial integration and the equation of v,

$$
\begin{aligned}
I_{\alpha,\beta}(u_F v) &= I_\beta[u_F V] = wV - I_\beta[w\partial_\beta V] \\
&= wV - I_\beta\big[w\big(I_\alpha(\phi' u) + R_\beta v\big)\big] \\
&= wV - I_\beta[w u_F \Phi] - I_\beta[w(I_\alpha \phi' u_1 + R_\beta v)],
\end{aligned} \tag{5.13}
$$

where $V := I_\alpha v$, $\phi' = c^3 + c^4\phi$, $\Phi = I_\alpha \phi'$, and $w := I_\beta u_F = \int_{-\alpha}^{\beta} u(0,-\delta)\,d\delta$. We expect that wV is the leading term, and we can dispose of the last term by appropriate choice of the initial data. However, the previous arguments do not give any better regularity to the other term $I_\beta[w u_F \Phi]$ than the whole expression $I_\beta[u_F V]$ since we know only that $\Phi \in H_\alpha^{a+2} H_\beta^{a+1}$.

Here we use $w u_F = \partial_\beta(w^2/2)$ and "square smoothing": we choose $p \in \mathbb{R}$ such that

$$
2p + \frac{5}{2} > s > p + \frac{3}{2} > a + 1 > \frac{1}{2}, \tag{5.14}
$$

and we define the initial data of u by

$$
u(0,x) := U_0'(x), \qquad U_0(x) := \chi(x)|\min(x,0)|^{p+1}. \tag{5.15}
$$

Then $u_F = U_0'(-\beta)$, $w = U_0(\alpha) - U_0(-\beta)$, and from the condition of a, s, p,

$$
u(0) \in H^a, \qquad U_0(-\beta) \notin H_x^s, \qquad U_0(-\beta)U_0'(-\beta) \in H_\alpha^\infty H_\beta^{s-1}, \tag{5.16}
$$

for all $t \in \mathbb{R}$. Hence

$$
I_\beta[w u_F \Phi] = U_0(\alpha) I_\beta[u_F \Phi] + I_\beta[U_0(-\beta)U_0'(-\beta)\Phi], \tag{5.17}
$$

and the last term is in $H_\alpha^s H_\beta^s$. This "square smoothing" does not work for the other term, but it is supported on $\{\alpha \le 0\}$, so that we can neglect it by restricting the region to $x, t > 0$. Similarly, the last term in (5.13) becomes

$$
\begin{aligned}
&I_\beta[w(I_\alpha \phi' u_1 + R_\beta v)] \\
&\quad = U_0(\alpha) I_\beta[I_\alpha \phi' u_1 + R_\beta v] + I_\beta[U_0(-\beta)(I_\alpha \phi' u_1 + R_\beta v)],
\end{aligned} \tag{5.18}
$$

where the last term is bounded in $H_\alpha^{a+2} H_\beta^{a+2}$, while the other term can be neglected by restricting to $x, t > 0$. In short, we have obtained

$$
\phi_1 = U_0(-\beta)V + (\alpha \le 0) + (s \cdot s), \tag{5.19}
$$

where $(\alpha \le 0)$ denotes any function supported on $\alpha \le 0$, and $(s \cdot s)$ denotes any function in $H_\alpha^s H_\beta^s$.

Now we claim that ϕ_1 is as rough as $U_0(-\beta)$ in the region $\alpha = t + x > 0$. Since we know only that $V \in H_\alpha^{a+2} H_\beta^{a+1}$, we cannot simply divide ϕ_1 by V as in the previous case. Instead*, we use the "robustness" of the zero singularity of $U_0(-\beta)$. By the same argument as in the previous case, we can make $|V(t,x)| >$

*Alternatively, we can estimate $U_0(-\beta)V$ by an expansion and a partial integration for V similar to those for ϕ_1, where singularity of V is diminished when multiplied with $U_0(-\beta)$ by the square smoothing.

$t/2$ for small $t > 0$. Then, since $U_0(-\beta) = \max(t - x, 0)^p$ around $x = t$, Lemma 5.1 implies that $\phi_1 \notin H^s_x$ for small $t > 0$.

The above ill-posedness is immediately extended to the region $s > \max(a + 1, 1/2)$ because we can choose $a' \ge a$ satisfying $1/2 < a$ and $s + 1 < a' < s + 2$, and then initial data in $\mathcal{H}^{a',s} \subset \mathcal{H}^{a,s}$, such that ϕ instantly exits H^s; hence the solution is not in $\mathcal{H}^{a,s}$ either. Thus we conclude the proof of Theorem 1.2.

5.2. Irregular flow map for DKG

Next, we consider the remaining region for DKG, where the solution map is not twice differentiable. First, we recall that the second derivative at zero of the solution map is given by the second iterate.

LEMMA 5.2

Let $B_1 \subset B_2 \subset B_3$ be Banach spaces with dense embeddings, and let $L : B_1 \to B_3$ be a bounded linear map such that e^{tL} is a C^0-semigroup on each B_j. Let N be twice differentiable at zero from B_2 to B_3, and let $\|N(\varphi)\|_{B_2} = o(\|\varphi\|_{B_1})$ as $\varphi \to 0$. Suppose that the equation $u_t = Lu + N(u)$ is "locally well posed in B_1"; that is, for some $T > 0$ and for any small $\varphi \in B_1$ there exists $u \in C([0,T]; B_1)$ satisfying $u(0) = \varphi$, the above equation in B_3 for $0 < t < T$, and $\|u\|_{L^\infty(0,T;B_1)} = O(\|\varphi\|_{B_1})$ as $\varphi \to 0$.

Then the map $\mathcal{U} : \varphi \mapsto u$ is twice differentiable at zero from B_1 to $C([0,T]; B_3)$, and

$$\mathcal{U}_0'(\varphi)(t) = e^{tL}\varphi, \qquad \mathcal{U}_0''(\varphi,\varphi)(t) = \int_0^t e^{(t-s)L} N_0''(e^{sL}\varphi, e^{sL}\varphi)\, ds. \tag{5.20}$$

Here one should think of sufficiently regular spaces B_1 and B_2 embedded into the space B_3, where we want to investigate the second derivative, for example, $B = H^{s_j}$ with $s_1 \gg s_2 \gg |s_3| + 1$.

Proof

Integrating the equation, we have

$$u(t) = e^{tL}\varphi + \int_0^t e^{(t-s)L} N\big(u(s)\big)\, ds \quad \text{in } B_3. \tag{5.21}$$

Since $\|N(u(s))\|_{B_2} = o(\|u(s)\|_{B_1}) = o(\|\varphi\|_{B_1})$ and e^{tL} is bounded in B_2, we have

$$u(t) = e^{tL}\varphi + o(\|\varphi\|_{B_1}) \quad \text{in } B_2. \tag{5.22}$$

Similarly, the second term in (5.21) is expanded in B_3 by using the derivatives of N:

$$\begin{aligned} u(t) &= e^{tL}\varphi + \frac{1}{2}\int_0^t e^{(t-s)L} N_0''\big(u(s), u(s)\big)\, ds + o(\|u\|^2_{L^\infty(0,T;B_2)}) \\ &= e^{tL}\varphi + \frac{1}{2}\int_0^t e^{(t-s)L} N_0''(e^{sL}\varphi, e^{sL}\varphi)\, ds + o(\|\varphi\|^2_{B_1}) \quad \text{in } B_3, \end{aligned} \tag{5.23}$$

where in the second step we used (5.22). □

Next, we show that the mass terms (and more generally bounded terms) can be neglected in investigating the second derivative of the flow maps.

LEMMA 5.3

In addition to the assumption of Lemma 5.2, let M be a linear operator bounded on B_2 and B_3. Suppose that the equation

$$v_t = Lv + Mv + N(v) \tag{5.24}$$

is also locally well posed in B_1, and let $\mathcal{V}$ denote its flow map. Let $T > 0$ be such that both equations have solutions on $[0,T]$.

Then $\mathcal{V}_0'' : B_3^2 \to L^\infty(0,T;B_3)$ bounded if and only if $\mathcal{U}_0'' : B_3^2 \to L^\infty(0,T;B_3)$ bounded.

Proof

By the symmetry, it suffices to show the "if" part. By Lemma 5.2, $\mathcal{V}$ is twice differentiable at zero from B_1 to $C([0,T];B_3)$. We define u^0, u^1, v^0, v^1, v^2 by

$$\begin{aligned} &u^0 = \mathcal{U}_0'(\varphi), \qquad u^1 = \mathcal{U}_0''(\varphi,\varphi), \qquad v^0 = \mathcal{V}_0'(\varphi), \qquad v^1 = \mathcal{V}_0''(\varphi,\varphi), \\ &v^2(t) = \int_0^t e^{(t-s)L} N_0''\big(v^0(s), v^0(s)\big)\, ds. \end{aligned} \tag{5.25}$$

Then by Lemma 5.2 together with the Duhamel formula, we have

$$\begin{aligned} v^0 &= u^0 + \int_0^t e^{(t-s)L} M v^0(s)\, ds, \\ v^1 &= v^2 + \int_0^t e^{(t-s)(L+M)} M v^2(s)\, ds. \end{aligned} \tag{5.26}$$

Hence it suffices to bound v^2, which we expand by inserting the formula for v^0:

$$\begin{aligned} v^2 = u^1 &+ 2\int_0^t \int_0^s e^{(t-s)L} N_0''\big(e^{(s-r)L} M v^0(r), u^0(s)\big)\, dr\, ds \\ &+ \int_0^t \int_0^s \int_0^s e^{(t-s)L} N_0''\big(e^{(s-r_1)L} M v^0(r_1), e^{(s-r_2)L} M v^0(r_2)\big)\, dr_1\, dr_2\, ds. \end{aligned} \tag{5.27}$$

The second term on the right-hand side equals, by change of variable $s \mapsto s + r$,

$$\begin{aligned} 2&\int_0^t \int_0^{t-r} e^{(t-s-r)L} N_0''\big(e^{sL} M v^0(r), e^{(s+r)L}\varphi\big)\, ds\, dr \\ &= 2\int_0^t \mathcal{U}_0''\big(M v^0(r), u^0(r)\big)(t-r)\, dr, \end{aligned} \tag{5.28}$$

and the last term of (5.27) equals, by a similar change of variable,

$$\begin{aligned} 2&\int_0^t \int_0^{r_1} \int_0^{t-r_1} e^{(t-s-r_1)L} N_0''\big(e^{sL} M v^0(r_1), e^{(s+r_1-r_2)L} M v^0(r_2)\big)\, ds\, dr_2\, dr_1 \\ &= 2\int_0^t \int_0^{r_1} \mathcal{U}_0''\big(M v^0(r_1), e^{(r_1-r_2)L} M v^0(r_2)\big)(t-r_1)\, dr_2\, dr_1. \end{aligned} \tag{5.29}$$

Hence v^2 is bounded in B_3 if $\mathcal{U}_0'': B_3^2 \to B_3$ bounded. $\square$

Proof of Theorem 1.3
Thanks to Lemmas 5.2 and 5.3, it suffices to give a bounded sequence of initial data for which the second iterate is unbounded in the massless case. The second iterate is given by using the free solutions

$$u^{(1)} = u_F + c_2 I_\alpha(\phi v), \qquad v^{(1)} = v_F + c_4 I_\beta(\phi u),$$
$$\phi^{(1)} = \phi_F + c_6 I_{\alpha,\beta}(uv), \tag{5.30}$$

where $c_2, c_4, c_6 \in \mathbb{C}$ are the same constants as in (2.8).

First, in the case $a+s<0$, we choose initial data with a parameter $N \to \infty$ such that the free parts take the forms

$$u_F = 0, \qquad v_F = v_0(x+t), \qquad \phi_F = \phi_0(x+t), \tag{5.31}$$

and $u^{(1)}(t)$ is unbounded for $N \to \infty$ at any small $t>0$. The Fourier transform of $u^{(1)}$ is given by

$$\begin{aligned}\widehat{u^{(1)}}(t,\xi) &= c_2 \int_0^t e^{-i\xi(t-s)} \widehat{(\phi_0 v_0)(x+s)}\, ds \\ &= \frac{c_2}{2\pi} \int_0^t e^{-it\xi+2is\xi}\, ds\, \widehat{\phi_0} * \widehat{v_0} \\ &= \frac{c_2 \sin(2t\xi)}{2\pi\xi} \widehat{\phi_0} * \widehat{v_0}.\end{aligned} \tag{5.32}$$

We put

$$\widehat{\phi_0}(\xi) = \langle\xi\rangle^{-s}\chi_1(\xi+N), \qquad \widehat{v_0}(\xi) = \langle\xi\rangle^{-a}\chi_1(\xi-N). \tag{5.33}$$

Then we have $\|\phi_0\|_{H^s} + \|v_0\|_{H^a} \lesssim 1$, and by (5.32),

$$\begin{aligned}\|u^{(1)}(t)\|_{H^a_x} &\gtrsim \|\widehat{u^{(1)}}(t)\|_{L^1_\xi(|\xi|\le 1)} \\ &\gtrsim \int_{-1}^1 t\, d\xi \int_{N-1}^{N+1} d\eta\ \langle\xi-\eta\rangle^{-s}\langle\eta\rangle^{-a} \sim tN^{-a-s}\end{aligned} \tag{5.34}$$

for $0<t\ll 1\ll N$, which is unbounded as $N\to\infty$, as desired.

We next consider the case $(a,s)=(-1/2,1/2)$. For any small $t_0>0$, we choose initial data such that the free parts take the forms

$$u_F = u_0(x-t), \qquad v_F = v_0(x+t), \qquad \phi_F = 0, \tag{5.35}$$

and $\phi^{(1)}(t_0)$ is unbounded as $N\to\infty$. The Fourier transform of $\phi^{(1)}$ is

$$\begin{aligned}\widehat{\phi^{(1)}}(t,\xi) &= \int_0^t \frac{\sin(t-s)\xi}{\xi} \widehat{u_0(x-s)v_0(x+s)}\, ds \\ &= \int_0^t \int \frac{\sin(t-s)\xi}{2\pi\xi} \widehat{u_0}(\xi-\eta)e^{-is(\xi-\eta)}\widehat{v_0}(\eta)e^{is\eta}\, d\eta\, ds\end{aligned} \tag{5.36}$$

$$= \int \int_0^t \frac{e^{it\xi} e^{is(2\eta - 2\xi)} - e^{-it\xi} e^{is(2\eta)}}{4i\pi\xi} \, ds \, \widehat{u_0}(\xi - \eta) \widehat{v_0}(\eta) \, d\eta$$
$$= \int \frac{1}{8\pi\xi} \Big[e^{it\xi} \frac{e^{2it(\eta - \xi)} - 1}{\xi - \eta} + e^{-it\xi} \frac{e^{2it\eta} - 1}{\eta} \Big] \widehat{u_0}(\xi - \eta) \widehat{v_0}(\eta) \, d\eta.$$

We put

$$\begin{aligned} &\widehat{u}_0(\xi) = \chi_N(\xi - N^2) N^{1/2}, \\ &\widehat{v}_0(\xi) = \sum_{j=1}^{N} \chi_{\pi/4}\Big(t_0\xi - (2j-1)\pi\Big)(\log N)^{-1/2}. \end{aligned} \tag{5.37}$$

Then for $0 < t_0 \ll 1 \ll N$, we have

$$\begin{aligned} &\|u_0\|_{H^{-1/2}} \lesssim N^{-1+1/2} \|\chi_N\|_{L^2} \sim 1, \\ &\|v_0\|_{H^{-1/2}}^2 \lesssim (\log N)^{-1} \int_{\pi/(4t_0)}^{2N\pi/t_0} \frac{d\xi}{\xi} \sim 1, \end{aligned} \tag{5.38}$$

and

$$\begin{aligned} \|\phi^{(1)}(t_0)\|_{H^{1/2}} &\gtrsim N^{1/2} \|\widehat{\phi^{(1)}}(t_0)\|_{L^1_\xi(|\xi - N^2| < N)} \\ &\gtrsim N(\log N)^{-1/2} \int_{N^2-N}^{N^2+N} \frac{d\xi}{\xi} \sum_{j=1}^{N} \int \frac{d\eta}{\eta} \chi_{\pi/4}\big(t_0\eta - (2j-1)\pi\big) \\ &\sim (\log N)^{1/2} \to \infty. \end{aligned} \tag{5.39}$$

□

5.3. Instant exit for QD and WM

Finally, we prove the ill-posedness part of Theorems 1.5 and 1.6 by instant exit for QD and WM in the special cases of coefficients. This is due to some algebraic structure of these equations and is essentially known, at least for the wave maps (see [18], [17]). Here we give a full proof for the following massless QD for $u = (u_+, u_-)$:

$$(\partial_t \pm \partial_x) u_\pm = u_+ u_-. \tag{5.40}$$

For any free wave solution w, $u_\pm := (w_t \mp w_x)/(1 \mp w)$ solves equation (5.40) in the region $w \neq 1$. If w is in the form $w = \varphi(x+t) - \varphi(x-t)$, then we have

$$u_\pm(t,x) = \frac{2\varphi'(x \mp t)}{1 \mp \varphi(x+t) \pm \varphi(x-t)}, \quad u_\pm(0,x) = 2\varphi'(x). \tag{5.41}$$

It suffices to give a φ satisfying $\varphi' \in H^{-1/2}$ and $u \notin H^{-1/2}$ for any $t > 0$. We set

$$\varphi(x) = -\chi(x) \log\big|\log|x|\big| \tag{5.42}$$

for some $\chi \in C_0^\infty(\mathbb{R})$ satisfying $\chi(x) = 1$ for $|x| < e^{-2}$ and $\chi(x) = 0$ for $|x| > e^{-1}$.

PROPOSITION 5.4

The function φ of (5.42) is in $H^{1/2}$.

REMARK 5.5

One may wonder if $\Phi = \chi(x)\log|x|$ belongs to $H^{1/2}$ or not. The answer is No, since the derivative contains the singularity

$$\chi(x)\frac{1}{x} \notin H^{-1/2}, \tag{5.43}$$

which is clear by the Fourier transform. This fact is used again in the proof of Proposition 5.6.

Proof of Proposition 5.4

Since $\varphi \in L^2$ is obvious, it suffices to bound the following part of formula (5.6):

$$\int_0^{e^{-2}} t^{-1}\|\delta_t\varphi\|_{L_x^2}^2 \frac{dt}{t}. \tag{5.44}$$

If the difference operator δ_t hits χ, the estimate it easy. So we investigate only the term $\chi\delta_t \log|\log|x||$. Since $|\delta_t f(-x-t)| = |\delta_t f(x)|$ if $f(x) = f(|x|)$, we may restrict the L_x^2-norm to the region $x > -t/2$. In the region $x > t/2$, we bound the difference by the derivative

$$\big\|\delta_t \log\big|\log|x|\big|\big\|_{L^2(x>t/2)}^2 \lesssim \int_{t/2}^{\infty} \frac{t^2\,dx}{x^2(\log|x|)^2} \lesssim t|\log t|^{-2}. \tag{5.45}$$

In the region $|x| < t/2$, we have $|x| < |x+t| < 1$, and so $\log|x| < \log|x+t| < 0$. By using that $\log|1+\alpha| \le \alpha$ for $\alpha > 0$, we estimate

$$\begin{aligned} \big|\delta_t \log|\log|x||\big| &= \Big|\log\frac{\log|x|}{\log|x+t|}\Big| = \Big|\log\Big[1 + \frac{\log|x/(x+t)|}{\log|x+t|}\Big]\Big| \\ &\le \Big|\frac{\log|x/(x+t)|}{\log|x+t|}\Big| \lesssim \Big|\frac{\log|x/t|}{\log t}\Big|. \end{aligned} \tag{5.46}$$

Hence we have

$$\big\|\delta_t \log\big|\log|x|\big|\big\|_{L^2(|x|<t/2)}^2 \lesssim \int_{|x|<t/2} \frac{|\log|x/t||^2}{|\log t|^2}\,dx \lesssim t|\log t|^{-2}. \tag{5.47}$$

Thus (5.44) is finite. □

PROPOSITION 5.6

The functions $u_\pm$ defined by (5.41) and (5.42) are not in $H_x^{-1/2}$ for any small $t > 0$.

Proof

Since $u_-(t,x) = -u_+(t,-x)$, it suffices to check u_+. Fix $0 < t < e^{-2}/2$. We investigate the denominator of (5.41) in the region $-e^{-2} < x-t < x+t < e^{-2}$,

$$g(x) := 1 - \log\big|\log|x+t|\big| + \log\big|\log|x-t|\big|.$$

Since $g(-t) = -\infty$ and $g(0) = 1 > 0$, by continuity there is $x_0 \in (-t, 0)$ satisfying $g(x_0) = 0$. Moreover, $g'(x) = 0$ only at $x = 0$; hence $C := g'(x_0) \neq 0$. The Taylor expansion implies that near $x = x_0$,

$$\frac{1}{g(x)} = \frac{1}{C(x - x_0) + O((x - x_0)^2)} = \frac{1}{C(x - x_0)} + O(1). \tag{5.48}$$

Since $\pi(x)(x - x_0)^{-1} \notin H^{-1/2}$ for any smooth cutoff π satisfying $\pi(x_0) \neq 0$, and $\varphi'(x - t)$ is nonzero and continuous around $x = x_0$, we deduce that $u_+(t, x) = \varphi'(x - t)/g(x) \notin H^{-1/2}$. □

References

[1] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations, II: The KdV-equation*, Geom. Funct. Anal. **3** (1993), 209–262.

[2] N. Bournaveas, *A new proof of global existence for the Dirac-Klein-Gordon equations in one space dimension*, J. Funct. Anal. **173** (2000), 203–213.

[3] ———, *Local well-posedness for a nonlinear Dirac equation in spaces of almost critical dimension*, Discrete Contin. Dyn. Syst. **20** (2008), 605–616.

[4] N. Bournaveas and D. Gibbeson, *Global charge class solutions of the Dirac-Klein-Gordon equations in one space dimension*, Differential Integral Equations **19** (2006), 1001–1018.

[5] ———, *Low regularity global solutions of the Dirac-Klein-Gordon equations in one space dimension*, Differential Integral Equations **19** (2006), 211–222.

[6] J. M. Chadam, *Global solutions of the Cauchy problem for the (classical) coupled Maxwell-Dirac equations in one space dimension*, J. Funct. Anal. **13** (1973), 173–184.

[7] P. D'Ancona, D. Foschi, and S. Selberg, *Local well-posedness below the charge norm for the Dirac-Klein-Gordon system in two space dimensions*, J. Hyperbolic Differ. Equ. **4** (2007), 295–330.

[8] ———, *Null structure and almost optimal local regularity for the Dirac-Klein-Gordon system*, J. Eur. Math. Soc. (*JEMS*) **9** (2007), 877–899.

[9] Y.-F. Fang, "A direct proof of global existence for the Dirac-Klein-Gordon equations in one space dimension" in *Proceedings of the Third East Asia Partial Differential Equation Conference*, Taiwanese J. Math. **8** Math. Soc. Repub. China (Taiwan), Kaohsiung, 2004, 33–41.

[10] ———, *On the Dirac-Klein-Gordon equations in one space dimension*, Differential Integral Equations **17** (2004), 1321–1346.

[11] Y.-F. Fang and M. G. Grillakis, *On the Dirac-Klein-Gordon equations in three space dimensions*, Comm. Partial Differential Equations **30** (2005), 783–812.

[12] ———, *On the Dirac-Klein-Gordon system in* $2+1$ *dimensions*, preprint.

[13] Y.-F. Fang and H.-C. Huang, *A critical case of the Dirac-Klein-Gordon equations in one space dimension*, Taiwanese J. Math. **12** (2008), 1045–1059.

[14] J. Ginibre, Y. Tsutsumi, and G. Velo, *On the Cauchy problem for the Zakharov system*, J. Funct. Anal. **151** (1997), 384–436.

[15] A. Gruenrock and H. Pecher, *Global solutions for the Dirac-Klein-Gordon system in two space dimensions*, preprint, arXiv:0903.3189v1 [math.AP]

[16] J. Holmer, *Local ill-posedness of the* 1*D Zakharov system*, Electron. J. Differential Equations **2007**, no. 24.

[17] M. Keel and T. Tao, *Local and global well-posedness of wave maps on* $\mathbf{R}^{1+1}$ *for rough data*, Internat. Math. Res. Notices **1998**, no. 21, 1117–1156.

[18] S. Klainerman and M. Machedon, *Space-time estimates for null forms and the local existence theorem*, Comm. Pure Appl. Math. **46** (1993), 1221–1268.

[19] S. Machihara, *One dimensional Dirac equation with quadratic nonlinearities*, Discrete Contin. Dyn. Syst. **13** (2005), 277–290.

[20] ———, *Dirac equation with certain quadratic nonlinearities in one space dimension*, Commun. Contemp. Math. **9** (2007), 421–435.

[21] ———, *The Cauchy problem for the* 1*-D Dirac-Klein-Gordon equation*, NoDEA Nonlinear Differential Equations Appl. **14** (2007), 625–641.

[22] F. Melnyk, *Local Cauchy problem for the nonlinear Dirac and Dirac-Klein-Gordon equations on Kerr space-time*, J. Math. Phys. **47** (2006), no. 052503.

[23] H. Pecher, *Low regularity well-posedness for the one-dimensional Dirac-Klein-Gordon system*, Electron. J. Differential Equations **2006**, no. 150.

[24] T. Roy, *Adapted linear-nonlinear decomposition and global well-posedness for solutions to the defocusing cubic wave equation on* $\mathbb{R}^3$, Discrete Contin. Dyn. Syst. **24** (2009), 1307–1323.

[25] S. Selberg, *Global well-posedness below the charge norm for the Dirac-Klein-Gordon system in one space dimension*, Int. Math. Res. Not. IMRN **2007**, no. 17, art. ID rnm058.

[26] S. Selberg and A. Tesfahun, *Low regularity well-posedness of the Dirac-Klein-Gordon equations in one space dimension*, Commun. Contemp. Math. **10** (2008), 181–194.

Machihara: Department of Mathematics, Faculty of Education, Saitama University, Saitama 338-8570, Japan

Nakanishi: Department of Mathematics, Kyoto University, Kyoto 606-8502, Japan

Tsugawa: Graduate School of Mathematics, Nagoya University, Nagoya 464-8602, Japan