The number of 1-codimensional cycles on projective varieties

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Abstract In this article, we investigate the converging radius of the "generalized zeta function," which is, roughly speaking, the generating function of the number of effective cycles. In Section 3, we give the explicit value of the converging radius when the codimension of the cycles is 1. In Section 4, we deal with 1-dimensional cycles on a projective space and give a lower bound of the convergent radius.

1. Introduction

Let X be a projective scheme defined over a finite base field $\mathbb{F}_q$. The power series

$$\zeta(X;t) := \exp\Bigl(\sum_{r=1}^{\infty} \#X(\mathbb{F}_{q^r})\frac{t^r}{r}\Bigr)$$

is the well-known zeta function of X. Many beautiful properties of the zeta function are known, such as the analogue of the Riemann hypothesis, functional equations, relations with the Betti numbers, and so on.

We can look at the zeta function in a slightly different way: It can be regarded as a generating function of the numbers of zero-dimensional cycles: Let $N_d(X)$ be the number of effective zero-dimensional cycles Y defined over $\mathbb{F}_q$ with $\deg Y = d$.

An easy calculation gives the next equation:

$$\zeta(X;t) = \sum_{d=0}^{\infty} N_d(X)t^d.$$

It might as well be natural to consider the generating function of the number of higher-dimensional cycles.

DEFINITION 1.1

Let (X,H) be a $\mathbb{Q}$-polarized quasi-projective scheme (i.e., H is a $\mathbb{Q}$-ample line bundle on X) of dimension n over a finite field $\mathbb{F}_q$. Let us fix a compactification $(\bar{X},\bar{H})$ of (X,H).

Fix an integer l with $0 \le l \le n-1$. For any positive integer $d \in \mathbb{Z}_{>0}$, define $G_d(X,H,l;\mathbb{F}_q)$ as the set of all effective l-dimensional cycles Y on X of degree d,

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defined over $\mathbb{F}_q$; that is, Y is an effective l-dimensional cycle on $\bar{X}$ such that $\deg_{\bar{H}} Y = d$ and every irreducible component of Y is not contained in $\bar{X} \setminus X$.

Also, define $N_d(X,H,l;\mathbb{F}_q)$ as the cardinal of $G_d(X,H,l;\mathbb{F}_q)$. Note that $N_d(X,H,l;\mathbb{F}_q)$ is finite since the cycles are defined over the finite field, with the degree bounded.

DEFINITION 1.2

Let (X,H) and l be as above. Set

$$Z(X,H,l;t) := \sum_{d=0}^{\infty} N_d(X,H,l;\mathbb{F}_q)t^{d^{l+1}} \in \mathbb{Z}[[t]].$$

This is a generating function of $N_d(X,H,l;\mathbb{F}_q)$.

Define $C(X,H,l;\mathbb{F}_q)$ as

$$C(X,H,l;\mathbb{F}_q) := \limsup_{d\to\infty} \frac{\log_q N_d(X,H,l;\mathbb{F}_q)}{d^{l+1}}.$$

THEOREM 1.3 (MORIWAKI [5])

With the notation above, we have

$$0 < C(X,H,l;\mathbb{F}_q) < \infty.$$

From Theorem 1.3, we see that $Z(X,H,l;t)$ converges (in the complex analytic sense) at the origin $t=0$, with the convergence radius $q^{-C(X,H,l;\mathbb{F}_q)}$.

When $l=0$, we can easily see that $Z(X,H,0;t)$ coincides with Weil's zeta function:

$$Z(X,H,0;t) = \zeta(X;t).$$

Thus, if X is a smooth projective variety, we have $C(X,H,0;\mathbb{F}_q) = n$ via Weil conjecture.

EXAMPLE 1.4

Let us calculate $N_d(X,H,1;\mathbb{F}_q)$ and $Z(X,H,1;t)$ for $(X,H) = (\mathbb{P}^2, \mathcal{O}(1))$. Since a 1-dimensional cycle is a Cartier divisor on $\mathbb{P}^2$, $\operatorname{Pic}\mathbb{P}^2 = \mathbb{Z}\mathcal{O}(1)$, and $h^0(\mathbb{P}^2, \mathcal{O}(d)) = (1/2)(d+1)(d+2)$, we obtain

$$N_d\big(\mathbb{P}^2, \mathcal{O}(1), 1; \mathbb{F}_q\big) = \frac{q^{(1/2)(d+1)(d+2)} - 1}{q-1}.$$

(For a further explanation of this calculation, see Section 3.) So we have

$$Z\big(\mathbb{P}^2, \mathcal{O}(1), 1; t\big) = \frac{1}{q-1} \sum_{d=0}^{\infty} (q^{(1/2)(d+1)(d+2)} - 1)t^{d^2}.$$

From this formula, we can easily see that $C(\mathbb{P}^2, \mathcal{O}(1), 1; \mathbb{F}_q) = 1/2$.

Although it is difficult to analyze $Z(X,H,l;t)$ in general, we can calculate $C(X, H, \dim X - 1; \mathbb{F}_q)$ using asymptotic theories.

Here is the main result of this article.

THEOREM 1.5 (CF. THEOREM 3.5)
Let (X, H) be a $\mathbb{Q}$-polarized n-dimensional smooth projective variety over $\mathbb{F}_q$. Then

$$C(X, H, n-1; \mathbb{F}_q) = \frac{1}{n!(H^n)^{n-1}}.$$

REMARK 1.6
When $l \leq \dim X - 2$, we do not have any good calculating method for $N_d(X, H, l; \mathbb{F}_q)$. In fact, we cannot hope for a simple formula for $C(X, H, l; \mathbb{F}_q)$ like that in the main theorem (see Remark 2.5).

Here we explain the strategy of the proof of the main theorem.

For simplicity, assume that X is smooth. Since 1-codimensional cycles (i.e., Weil divisors) are parametrized by complete linear systems of (effective) line bundles, the problem is reduced to the evaluation of the maximum value of the dimension of $H^0(X, L)$ when we increase the degree of the line bundle L. A rough estimate is sufficient for our purpose.

In [7] we verified that the asymptotic behavior of the dimension of global sections of a line bundle can be approximated by the volume function. The volume function is a continuous, positive real-valued function on $\mathrm{N}^1(X)_{\mathbb{R}}$, naively defined as

$$\mathrm{vol}_X(L) := \limsup_{m \to \infty} \frac{h^0(X, mL)}{m^n/n!}.$$

We also verified that Fujita's approximation theorem also holds in positive characteristics, which gives us the upper bound of the dimension of linear systems.

In this way, we can calculate the explicit value of $C(X, H, n-1; \mathbb{F}_q)$, namely, the growth of the number of 1-codimensional cycles. However, it becomes extremely difficult to calculate $C(X, H, l; \mathbb{F}_q)$ if the codimension is bigger than 1. It is difficult even in the simplest case, that is, $X = \mathbb{P}^3$ and $l = 1$. In the rest of this article we struggled to find the upper bound of $C(\mathbb{P}^3, \mathcal{O}(1), 1; \mathbb{F}_q)$. Yet it is still unknown whether the result obtained in this article is the best possible. The interesting thing is that, when the codimension is bigger than 1, the map $H \mapsto C(X, H, l; \mathbb{F}_q)$ (regarded as a function on $\mathrm{N}^1(X)_{\mathbb{R}}$) seems to be a piecewise analytic function on the ample cone, dividing the ample cone into several smaller cones. Compare this with the case when the codimension is 1, in which case, $C(X, H, n-1; \mathbb{F}_q)$ is determined only by the self-intersection of H.

This article consists of four sections.

In Section 2, we discuss general properties of $C(X, H, l; \mathbb{F}_q)$. The results in this chapter are used later, often without being noted. In Section 3, we prove the main theorem explained above. In Section 4, we try to give a good upper bound of $C(\mathbb{P}^3, \mathcal{O}(1), 1; \mathbb{F}_q)$.

Notation and conventions

In this article, any scheme is separated and of finite type over its base field. A variety is a geometrically integral scheme. We use Snapper's definition of the intersection theory (see [2]).

Since we work mainly on projective varieties, we use the notion of line bundles and Cartier divisors interchangeably.

For a projective scheme X, we denote the Picard group of X by $\mathrm{Pic}(X)$, the group of line bundles on X numerically equivalent to zero by $\mathrm{Num}(X)$, and the Néron-Severi group of X by $\mathrm{N}^1(X)$: $\mathrm{N}^1(X) = \mathrm{Pic}(X)/\mathrm{Num}(X)$. The $\mathbb{Q}$-Néron-Severi group $\mathrm{N}^1(X)_{\mathbb{Q}}$ (resp., $\mathrm{N}^1(X)_{\mathbb{R}}$) is defined by $\mathrm{N}^1(X)_{\mathbb{Q}} = \mathrm{N}^1(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ (resp., $\mathrm{N}^1(X)_{\mathbb{R}} = \mathrm{N}^1(X) \otimes_{\mathbb{Z}} \mathbb{R}$), and $\rho(X) := \operatorname{rank} \mathrm{N}^1(X)$ is the Picard number of X, which is finite (see [3]). Similarly, we call the linear combination of 1-codimensional subvarieties with rational (resp., real) coefficients "$\mathbb{Q}$-divisor" (resp., "$\mathbb{R}$-divisor"). We sometimes use the term "$\mathbb{Z}$-divisor" for the usual Cartier divisor (i.e., with integral coefficients). $\overline{\mathrm{Eff}}(X) \subset \mathrm{N}^1(X)_{\mathbb{R}}$ is the closure of the convex cone spanned by the classes of effective $\mathbb{Q}$-divisors. The elements in $\overline{\mathrm{Eff}}(X)$ are called "pseudoeffective" (see [4]).

We use the word "l-cycle" for the abbreviation of "l-(equi)dimensional cycle." If X is integral, then we denote the function field (or the constant sheaf associated to the function field) by $\mathrm{Rat}(X)$.

2. Properties of $C(X,H,l;\mathbb{F}_q)$

Throughout this section, (X,H) is a $\mathbb{Q}$-polarized n-dimensional projective variety over $\mathbb{F}_q$.

DEFINITION 2.1

Define a positive integer e as

$$\begin{aligned} e &= e(X,H,l;\mathbb{F}_q) \\ &:= \min\{a \in \mathbb{Z}_{>0} \mid N_{ad}(X,H,l;\mathbb{F}_q) > 0 \quad (\forall d \gg 0)\}. \end{aligned}$$

LEMMA 2.2

Let e be as above. There exists a positive integer $c > 0$ such that

$$N_{ed}(X,H,l;\mathbb{F}_q) \le N_{e(d+r)}(X,H,l;\mathbb{F}_q)$$

for all $r \ge c$.

Proof

There exists c such that $N_{er}(X,H,l;\mathbb{F}_q) > 0$ for all $r \ge c$. Choose $Y_r \in G_{er}(X,H,l;\mathbb{F}_q)$ for $r \ge c$. There is an injective map

$$\varphi_r : G_{ed}(X,H,l;\mathbb{F}_q) \hookrightarrow G_{e(d+r)}(X,H,l;\mathbb{F}_q)$$

given by $Y \mapsto Y + Y_r$. □

PROPOSITION 2.3

For any positive integer $s \in \mathbb{Z}_{>0}$,

$$\limsup_{d\to\infty} \frac{\log_q N_{sd}(X,H,l;\mathbb{F}_q)}{(sd)^{l+1}} = \limsup_{d\to\infty} \frac{\log_q N_{d}(X,H,l;\mathbb{F}_q)}{d^{l+1}}.$$

Proof

It is clear that the right-hand side is not smaller than the left-hand side. Hence it suffices to show

$$\limsup_{d\to\infty} \frac{\log_q N_{sd}(X,H,l;\mathbb{F}_q)}{(sd)^{l+1}} \geq \limsup_{d\to\infty} \frac{\log_q N_{d}(X,H,l;\mathbb{F}_q)}{d^{l+1}}.$$

For arbitrary small $\epsilon > 0$, there exists a strictly increasing sequence $d_1, d_2, \ldots \to \infty$ such that

$$\frac{\log_q N_{ed_i}(X,H,l;\mathbb{F}_q)}{(ed_i)^{l+1}} \geq C(X,H,l;\mathbb{F}_q) - \epsilon.$$

Fix c that satisfies the condition in Lemma 2.2. For each i, there exists a positive integer $c \leq \exists r_i \leq c+s$ such that $e(d_i + r_i)$ is a multiple of s. Then, by Lemma 2.2,

$$\begin{aligned}\frac{\log_q N_{e(d_i+r_i)}(X,H,l;\mathbb{F}_q)}{(e(d_i+r_i))^{l+1}} &\geq \frac{\log_q N_{ed_i}(X,H,l;\mathbb{F}_q)}{(ed_i)^{l+1}} \cdot \frac{(ed_i)^{l+1}}{(e(d_i+r_i))^{l+1}} \\ &\geq \big(C(X,H,l;\mathbb{F}_q) - \epsilon\big) \cdot \frac{(ed_i)^{l+1}}{(e(d_i+r_i))^{l+1}}.\end{aligned}$$

Taking $d_i \to \infty$, the last value converges to $C(X,H,l;\mathbb{F}_q) - \epsilon$. Taking $\epsilon \to 0$, we obtain the result. □

COROLLARY 2.4

For any positive rational number $a \in \mathbb{Q}_{>0}$, we have

$$C(X,aH,l;\mathbb{F}_q) = \frac{1}{a^{l(l+1)}} C(X,H,l;\mathbb{F}_q).$$

Proof

We may assume that a is an integer. Using Proposition 2.3, we have

$$\begin{aligned}C(X,aH,l;\mathbb{F}_q) &= \limsup_{d\to\infty} \frac{\log_q N_{d}(X,aH,l;\mathbb{F}_q)}{d^{l+1}} \\ &= \limsup_{d\to\infty} \frac{\log_q N_{a^l d}(X,aH,l;\mathbb{F}_q)}{(a^l d)^{l+1}} \\ &= \frac{1}{a^{l(l+1)}} \limsup_{d\to\infty} \frac{\log_q N_{d}(X,H,l;\mathbb{F}_q)}{d^{l+1}} \\ &= \frac{1}{a^{l(l+1)}} C(X,H,l;\mathbb{F}_q).\end{aligned}$$

□

REMARK 2.5

Corollary 2.4 says that $C(X,H,l;\mathbb{F}_q)$ is a homogeneous function on $\mathrm{N}^1(X)_{\mathbb{Q}}$. In [5], the finiteness of $C(X,H,l;\mathbb{F}_q)$ is proven only for very ample H, but the proposition says that this holds for any $\mathbb{Q}$-ample H.

Also, note that Proposition 2.3 implies that $C(X,H,l;\mathbb{F}_q)$ is *not* a rational function with respect to (H^n) if $1 \le l \le n-2$.

Now, we prepare some lemmas, which enable us to calculate $C(X,H,l;\mathbb{F}_q)$ somewhat more easily.

LEMMA 2.6

Fix a positive real number l. Let $\{\alpha_d\}_{d=1,2,\dots}$ be a sequence of positive real numbers, and assume that

$$\alpha := \limsup_{d\to\infty} \frac{\log \alpha_d}{d^l}$$

is finite. Then

$$\limsup_{d\to\infty} \frac{\log(\sum_{r\le d} \alpha_r)}{d^l} = \alpha. \tag{2.1}$$

Proof

It is sufficient to prove that the left-hand side of (2.1) is not bigger than α. For any small real number $\epsilon > 0$, there exists a positive integer N such that $\log \alpha_d \le (\alpha+\epsilon)d^l$ for all $d \ge N$. Set $C := \sum_{r<N} \alpha_r$. Then

$$\begin{aligned}
\limsup_{d\to\infty} \frac{\log \sum_{r\le d} \alpha_r}{d^l} &\le \limsup_{d\to\infty} \frac{\log(C + \sum_{N\le r\le d} e^{(\alpha+\epsilon)d^l})}{d^l} \\
&= \limsup_{d\to\infty} \frac{\log(C + (d-N)e^{(\alpha+\epsilon)d^l})}{d^l} \\
&= \limsup_{d\to\infty} \frac{\log(d-N) + (\alpha+\epsilon)d^l}{d^l}.
\end{aligned}$$

Since l is positive, $(\log(d-N))/d^l \to 0$ as d increases to infinity. Thus we obtain the result. $\square$

LEMMA 2.7

Fix a positive real number $l > 0$. Let $\{\alpha_d\}_{d=1,2,\dots}$ and $\{\beta_d\}_{d=1,2,\dots}$ be two sequences of positive real numbers, and assume that

$$\alpha = \limsup_{d\to\infty} \frac{\log \alpha_d}{d^l}, \qquad \beta = \limsup_{d\to\infty} \frac{\log \beta_d}{d^l}$$

are both finite. Then

$$\limsup_{d\to\infty} \frac{\log(\alpha_d + \beta_d)}{d^l} = \max\{\alpha, \beta\}. \tag{2.2}$$

Proof
We may assume that $\alpha \le \beta$. It is sufficient to prove that the left-hand side of (2.2) is not bigger than β. For any positive $\epsilon > 0$,

$$\frac{\log \alpha_d}{d^l} < \alpha + \epsilon, \qquad \frac{\log \beta_d}{d^l} < \beta + \epsilon$$

for sufficiently large d. Then

$$\begin{aligned}
&\limsup_{d\to\infty} \frac{\log(\alpha_d + \beta_d)}{d^l} \\
&\quad \le \limsup_{d\to\infty} \frac{\log(e^{d^l(\alpha+\epsilon)} + e^{d^l(\beta+\epsilon)})}{d^l} \\
&\quad \le \limsup_{d\to\infty} \frac{\log 2e^{d^l(\beta+\epsilon)}}{d^l} \\
&\quad = \limsup_{d\to\infty} \Big(\frac{\log 2}{d^l} + \frac{d^l(\beta+\epsilon)}{d^l} \Big) \\
&\quad = \beta + \epsilon.
\end{aligned}$$

Taking $\epsilon \to 0$, we obtain the result. □

A similar argument shows that multiplication by a sequence of polynomial order does not change the value of $\limsup_{d\to\infty} (\log \alpha_d)/d^l$. More precisely, we have the following.

LEMMA 2.8
Fix a positive number $l > 0$. Let $\{\alpha_d\}_{d=1,2,\ldots}$ and $\{\beta_d\}_{d=1,2,\ldots}$ be two sequences of positive real numbers, and assume that $\beta_d = O(d^n)$ for some $n \in \mathbb{Z}$. Then

$$\limsup_{d\to\infty} \frac{\log \alpha_d}{d^l} = \limsup_{d\to\infty} \frac{\log \beta_d \alpha_d}{d^l}. \tag{2.3}$$

LEMMA 2.9
Let S be a set, and let $\deg : S \to \mathbb{Z}_{>0}$ be a map from S to the set of natural numbers, such that $S_d := \deg^{-1}(d)$ is a finite set for all $d \in \mathbb{Z}_{>0}$. Let M be the free commutative monoid generated by S. Then we can extend the degree map to a monoid homomorphism

$$\deg : M \to (\mathbb{Z}_{\ge 0}, +)$$

in a natural way. Set $M_d := \deg^{-1}(d)$ for $d \in \mathbb{Z}_{\ge 0}$. (This is a finite set.) If

$$\alpha := \limsup_{d\to\infty} \frac{\log \# S_d}{d^l}$$

is finite for a fixed constant $l > 1$, then

$$\limsup_{d\to\infty} \frac{\log \# M_d}{d^l} = \alpha. \tag{2.4}$$

Proof
It is sufficient to prove that the left-hand side of (2.4) is not bigger than α. For any $\epsilon > 0$, there exists a constant C such that

$$\log \#S_d < (\alpha+\epsilon)d^l + C$$

for all d.

Since any element of M is a linear combination of elements of S, we have an upper bound of the cardinal of M_d:

$$\#M_d \leq \sum_{(a_1,\ldots,a_r)\in\sigma(d)} \prod_{i=1}^{r} \#S_{a_i}.$$

Here $\sigma(d)$ is the set of partitions of d; that is, $(a_1,\ldots,a_r)\in\sigma(d)$ means $1\leq a_1\leq \cdots \leq a_r$ and $\sum_{i=1}^r a_r = d$. Since the number of partition is less than 2^d,

$$\#M_d \leq 2^d \max_{(a_1,\ldots,a_r)\in\sigma(d)} \prod_{i=1}^{r} \#S_{a_i}.$$

Hence,

$$\begin{aligned}
\limsup_{d\to\infty} \frac{\log \#M_d}{d^l} &\leq \limsup_d \frac{d\log 2 + \log \max_{(a_1,\ldots,a_r)\in\sigma(d)} \prod_{i=1}^r \#S_{a_i}}{d^l} \\
&= \limsup_d \frac{\max_{(a_1,\ldots,a_r)} \sum_{i=1}^r \log \#S_{a_i}}{d^l} \\
&\leq \limsup_d \frac{\max_{(a_1,\ldots,a_r)}((\alpha+\epsilon)\sum_i d^l + rC)}{d^l} \\
&\leq \alpha+\epsilon.
\end{aligned}$$

Taking $\epsilon \to 0$, we obtain the result. □

REMARK 2.10
If we define $\tilde{N}_d(X,H,l;\mathbb{F}_q)$ as the number of effective l-cycles Y on X defined over $\mathbb{F}_q$, with $\deg_H Y \leq d$, and

$$\tilde{C}(X,H,l;\mathbb{F}_q) := \limsup_{d\to\infty} \frac{\log_q \tilde{N}_d(X,H,l;\mathbb{F}_q)}{d^{l+1}},$$

then we have $\tilde{C}(X,H,l;\mathbb{F}_q) = C(X,H,l;\mathbb{F}_q)$. This follows immediately from Lemma 2.6.

PROPOSITION 2.11
Define $G_d^{\mathrm{int}}(X,H,l;\mathbb{F}_q)$ as the set of l-dimensional integral subschemes Y of X, defined over $\mathbb{F}_q$, with $\deg_H Y = d$, and $N_d^{\mathrm{int}}(X,H,l;\mathbb{F}_q)$ as the cardinal of $G_d^{\mathrm{int}}(X, H,l;\mathbb{F}_q)$.

Also, set

$$C^{\mathrm{int}}(X,H,l;\mathbb{F}_q) := \limsup_{d\to\infty} \frac{\log_q N_d^{\mathrm{int}}(X,H,l;\mathbb{F}_q)}{d^{l+1}}.$$

Then for $l \geq 1$, we have $C(X,H,l;\mathbb{F}_q) = C^{\mathrm{int}}(X,H,l;\mathbb{F}_q)$.

Proof
This follows immediately from Lemma 2.9. □

PROPOSITION 2.12
Let $Z \subset X$ be a closed subscheme of X, defined over $\mathbb{F}_q$. Set $U := X \setminus Z$. We fix the compactification (X,H) of $(U,H|_U)$. Then we have

$$C(X,H,l;\mathbb{F}_q) = \max\bigl\{C(U,H|_U,l;\mathbb{F}_q), C(Z,H|_Z,l;\mathbb{F}_q)\bigr\}.$$

Proof
Since $C(X,H,l;\mathbb{F}_q) = C^{\mathrm{int}}(X,H,l;\mathbb{F}_q)$ and $G_d^{\mathrm{int}}(X,H,l;\mathbb{F}_q) = G_d^{\mathrm{int}}(U,H|_U,l;\mathbb{F}_q) \cup G_d^{\mathrm{int}}(Z,H|_Z,l;\mathbb{F}_q)$, the result follows from Lemma 2.7. □

3. The value of $C(X,H,n-1;\mathbb{F}_q)$

Throughout this section, X is a projective variety, unless otherwise stated. In this section, we give the precise value of $C(X,H,n-1;\mathbb{F}_q)$. First, we give the lower bound.

LEMMA 3.1
Let D be a Cartier divisor on X. Then, the number of effective divisors linearly equivalent to D defined over $\mathbb{F}_q$ is

$$\frac{q^{h^0(X,\mathcal{O}(D))} - 1}{q-1}.$$

This holds also for any Weil divisor D if X is normal.

Proof
$|D|$ has a natural structure of the set of closed points of the projective space $\mathbb{P} := \mathbb{P}\bigl(H^0(X,\mathcal{O}(D))\bigr)$, and divisors defined over $\mathbb{F}_q$ correspond to $\mathbb{F}_q$-rational points of $\mathbb{P}$. Thus, the number of divisors linearly equivalent to D (these are all Cartier) is

$$\#\mathbb{P}(\mathbb{F}_q) = \frac{q^{h^0(X,\mathcal{O}(D))} - 1}{q-1}.$$

The second statement is also proven by a similar argument (see [7]). □

PROPOSITION 3.2
We have the following lower bound of $C(X,H,n-1;\mathbb{F}_q)$:

$$C(X,H,n-1;\mathbb{F}_q) \geq \frac{1}{n!(H^n)^{n-1}}.$$

Proof
By multiplying H by a sufficiently big positive integer and applying Corollary 2.4, we may assume that H is an ample line bundle defined over $\mathbb{F}_q$. Then

$$G_{d(H^n)}(X,H,n-1;\mathbb{F}_q) \supset |dH|(\mathbb{F}_q),$$

where $|dH|(\mathbb{F}_q)$ is the set of divisors linearly equivalent to dH, defined over $\mathbb{F}_q$. Using Lemma 3.1, we see that

$$\lim_{d\to\infty} \frac{\log_q \#|dH|(\mathbb{F}_q)}{h^0(X,dH)} = 1.$$

Combining all these, we have

$$\begin{aligned} C(X,H,n-1;\mathbb{F}_q) &= \limsup_{d\to\infty} \frac{\log_q N_d(X,H,n-1;\mathbb{F}_q)}{d^n} \\ &= \limsup_{d\to\infty} \frac{\log_q N_{d(H^n)}(X,H,n-1;\mathbb{F}_q)}{(d(H^n))^n} \\ &\geq \limsup_{d\to\infty} \frac{\log_q \#|dH|(\mathbb{F}_q)}{(d(H^n))^n} \\ &= \limsup_{d\to\infty} \frac{h^0(X,dH)}{(d(H^n))^n} \\ &= \frac{\mathrm{vol}_X(H)}{n!(H^n)^n} = \frac{1}{n!(H^n)^{n-1}}. \end{aligned}$$ □

Next, we show that the lower bound of $C(X,H,n-1;\mathbb{F}_q)$ given above is, in fact, also the upper bound.

LEMMA 3.3

We have that

$$\#\{\delta \in \mathrm{N}^1(X) \cap \overline{\mathrm{Eff}}(X) \mid \deg_H \delta = d\}$$

is at most polynomial order with respect to d.

Proof

Induce an ordinary topology and a measure on the finite-dimensional vector space $\mathrm{N}^1(X)_{\mathbb{R}}$. Kleiman's criterion of ampleness implies that $\overline{\mathrm{Eff}}(X) \cap \{\deg_H = 1\}$ is compact. So the measure (as a hyperplane) of $\overline{\mathrm{Eff}}(X) \cap \{\deg_H = d\}$ increases at most polynomial order with respect to d. Since $\mathrm{N}^1(X)_{\mathbb{Z}} \subset \mathrm{N}^1(X)_{\mathbb{R}}$ is a discrete lattice, the result follows. □

LEMMA 3.4

The number of numerically trivial line bundles on X, defined over $\mathbb{F}_q$, is finite.

Proof

The set $\mathrm{Num}(X)$ of numerically trivial line bundles is a bounded family; that is, there exists an algebraic scheme T of finite type over $\mathbb{F}_q$ and a line bundle L on $X \times T$ such that for any numerically trivial line bundle $M \in \mathrm{Num}(X)$, there exists a geometric point $t \in T$ such that $M \simeq L|_{X\times\{t\}}$. Moreover, if M is defined over $\mathbb{F}_q$, t can be taken as an $\mathbb{F}_q$-rational point of T.

T is of finite type; hence the number of its $\mathbb{F}_q$-rational points is finite. Therefore, the number of numerically trivial line bundles defined over $\mathbb{F}_q$ is finite (see [6] for further explanations). □

THEOREM 3.5 (MAIN THEOREM)
If (X,H) is a $\mathbb{Q}$-polarized smooth projective variety defined over $\mathbb{F}_q$ of dimension n, we have

$$C(X,H,n-1;\mathbb{F}_q) = \frac{1}{n!(H^n)^{n-1}}.$$

Proof
It is already proven in Proposition 3.2 that the left-hand side is not smaller than the right-hand side. Hence, it is sufficient to show

$$C(X,H,n-1;\mathbb{F}_q) \le \frac{1}{n!(H^n)^{n-1}}.$$

From Lemma 3.1, we have the following:

$$N_d(X,H,n-1;\mathbb{F}_q) = \sum_{\substack{\delta\in \mathrm{N}^1(X)\\ \deg_H \delta = d}} \sum_{L\in[\delta]} \frac{q^{h^0(X,L)}-1}{q-1}.$$

Here L runs over all the line bundles in the class δ. By Lemmas 2.8, 3.3, and 3.4, this yields

$$C(X,H,n-1;\mathbb{F}_q) \le \limsup_{d\to\infty} \max_{\substack{\delta\in \mathrm{N}^1(X)\cap\overline{\mathrm{Eff}}(X)\\ \deg_H \delta = d}} \frac{h^0(\delta)}{d^n},$$

where

$$h^0(\delta) := \max\bigl\{h^0(X,L) \mid L\in[\delta]\bigr\}.$$

Note that this value is finite via the semicontinuity theorem since $\mathrm{Num}(X)$ is a bounded family.

Further, [7, Theorem 3.2] shows that

$$\limsup_{d\to\infty} \max_{\substack{\delta\in \mathrm{N}^1(X)\cap\overline{\mathrm{Eff}}(X)\\ \deg_H \delta = d}} \frac{h^0(\delta)}{d^n} \le \max_{\substack{\delta\in \mathrm{N}^1(X)_{\mathbb{R}}\\ \deg_H \delta = 1}} \frac{\mathrm{vol}_X(\delta)}{n!}.$$

Using [7, Corollary 2.19], we have

$$\max_{\substack{\delta\in \mathrm{N}^1(X)_{\mathbb{R}}\\ \deg_H \delta = 1}} \frac{\mathrm{vol}_X(\delta)}{n!} \le \frac{1}{n!(H^n)^{n-1}}.$$

Combining all of these, we obtain the result. □

Note that $C(X,H,n-1;\mathbb{F}_q)$ depends only on (H^n), and independently of q, the order of the base field. In other cases, such as $l \le n-2$, this is not assured, but it is meaningful to consider the next value.

Define

$$\bar{C}(X,H,l) := \limsup_{r\to\infty} C(X,H,l;\mathbb{F}_{q^r}).$$

Note that this invariant is not assured to be finite for $l \le n-2$ at the present.

PROPOSITION 3.6

Let (X,H) be a $\mathbb{Q}$-polarized normal n-dimensional projective variety defined over $\mathbb{F}_q$. Suppose that there exists resolution of singularities of X; that is, suppose that there exists a projective birational morphism $\pi : X' \to X$ from a smooth variety. Then we have

$$\bar{C}(X,H,n-1) = \frac{1}{n!(H^n)^{n-1}}.$$

Here it is convenient to use $\bar{C}$ because it allows us to change the base field.

Proof

Since π^*H is nef and big, there is an effective divisor D such that $H_\epsilon := \pi^*H - \epsilon D \in \mathrm{N}^1(X')_{\mathbb{R}}$ is ample for any sufficiently small number ϵ. We have $(H_\epsilon^{n-1} \cdot \pi^*Y) \le (H^{n-1} \cdot Y)$ for any effective Weil divisor Y on X. Here π^*Y is defined merely by the pullback of ideals corresponding to the irreducible components of Y. The closed subscheme π^*Y may have some components of codimension bigger than 1, but we may ignore them since it does not affect the intersection number.

So we have the injection $\tilde{G}_d(X',H,n-1;\mathbb{F}_q) \hookrightarrow \tilde{G}_d(X,H_\epsilon,n-1;\mathbb{F}_q)$ via $Y \mapsto \pi^*Y$, which shows that

$$\tilde{N}_d(X,H,n-1;\mathbb{F}_q) \le \tilde{N}_d(X',H_\epsilon,n-1;\mathbb{F}_q).$$

This implies

$$C(X,H,n-1;\mathbb{F}_q) \le C(X',H_\epsilon,n-1;\mathbb{F}_q) \le \frac{1}{n!(H_\epsilon^n)^{n-1}}.$$

The right-hand side converges to $1/(n!(H^n)^{n-1})$ as we take $\epsilon \to 0$.

Note that in the above proof, we did not care whether the resolution π is defined over $\mathbb{F}_q$ or not; We may change the base field and resolve this problem if necessary. □

COROLLARY 3.7

If (X,H) is a $\mathbb{Q}$-polarized projective surface (or a 3-fold with characteristic not less than 5) defined over $\mathbb{F}_q$, then we have

$$\bar{C}(X,H,n-1) = \frac{1}{n!(H^n)^{n-1}}.$$

Proof

The desingularization theorem exists for surfaces (and 3-folds, when the characteristic is not less than 5; see [1]). □

COROLLARY 3.8

Let (X, H) be a $\mathbb{Q}$-polarized projective scheme of dimension n, defined over $\mathbb{F}_q$. For any closed smooth $(l+1)$-dimensional subvariety Z defined over $\mathbb{F}_q$, we have

$$C(X, H, l; \mathbb{F}_q) \geq \frac{1}{(l+1)!(\deg_H Z)^l}.$$

Proof

This follows from the fact that $G_d(Z, H|_Z, l; \mathbb{F}_q) \subset G_d(X, H, l; \mathbb{F}_q)$. □

COROLLARY 3.9

Let (X, H) be a $\mathbb{Q}$-polarized smooth projective variety of dimension $n > 0$, defined over $\mathbb{F}_q$. Then $\bar{C}(X, H, l) > 0$ for any $0 \leq l \leq n-1$.

Proof

We can obtain a smooth $(l+1)$-dimensional subvariety of X by applying Bertini's theorem inductively. Since we have a positive lower bound of $C(X, H, l; \mathbb{F}_{q^r})$ which is independent of r, the result follows. □

4. The Value of $C(\mathbb{P}^3, \mathcal{O}(1), 1; \mathbb{F}_q)$

The aim of this section is to obtain the upper bound of $C(\mathbb{P}^3, \mathcal{O}(1), 1; \mathbb{F}_q)$. Here we explain the outline of the proof. We cannot calculate the value directly, so we replace $\mathbb{P}^3$ by $\mathbb{P}^2 \times \mathbb{P}^1$ and consider the upper bound of $C(\mathbb{P}^2 \times \mathbb{P}^1, H, 1; \mathbb{F}_q)$ instead. Then we compare the number of cycles on the above two varieties by a fixed birational map.

LEMMA 4.1

Let X and Y be a projective variety over an algebraically closed field k, and let J be the Picard scheme of Y. Then we have a (noncanonical) isomorphism

$$\operatorname{Pic}(X \times Y) \simeq \operatorname{Pic}(X) \oplus \operatorname{Pic}(Y) \oplus \operatorname{Hom}(X, J).$$

In particular,

$$\operatorname{Pic}(X \times \mathbb{P}^n) = \operatorname{Pic}(X) \oplus \mathbb{Z}\mathcal{O}_{\mathbb{P}^n}(1).$$

Proof

Fix closed points $x_0 \in X$ and $y_0 \in Y$. Let p and q be the first and the second projection of $X \times Y$, respectively. Define a map

$$F: \operatorname{Pic}(X \times Y) \to \operatorname{Pic}(X) \oplus \operatorname{Pic}(Y) \oplus \operatorname{Hom}(X, J)$$

as follows. Let $L_1 := L|_{X \times \{y_0\}}$ and $L_2 := L|_{\{x_0\} \times Y}$. Regard L_1 (resp., L_2) as a line bundle on X (resp., Y). Set $M := L \otimes p^* L_1^{-1} \otimes q^* L_2^{-1}$. Since $M|_{\{x_0\} \times Y}$ is trivial, $M|_{\{x\} \times Y}$ is algebraically equivalent to zero for all closed points $x \in X$. Thus, from the universal property of the Jacobian variety, we obtain a unique map $\varphi : X \to J$ which satisfies $M \simeq (\varphi \times \mathrm{Id}_Y)^* \mathcal{E}$, where $\mathcal{E} \in \operatorname{Pic}(J \times Y)$ is the universal bundle. Set $F(L) := (L_1, L_2, \varphi)$.

On the other hand, define

$$G:\ \mathrm{Pic}(X)\oplus \mathrm{Pic}(Y)\oplus \mathrm{Hom}(X,J)\to \mathrm{Pic}(X\times Y)$$

by $G(L_1,L_2,\varphi):=p^*L_1\otimes q^*L_2\otimes(\varphi\times \mathrm{Id}_Y)^*\mathcal{E}$. It is easy to see that G gives the inverse of F. □

PROPOSITION 4.2

Let B be a smooth projective surface defined over $\mathbb{F}_q$, and let $X:=B\times\mathbb{P}^1$. Let p and q be the first and the second projection from X, respectively. Let $H:=p^*H_1+aq^*\mathcal{O}(1)$ be an $\mathbb{Q}$-ample line bundle on X, where H_1 is a $\mathbb{Q}$-ample line bundle on B, and $a\in\mathbb{Q}$ is a positive rational number. Then we have

$$C(X,H,1;\mathbb{F}_q)\le \max\left\{\frac{(1+c^+)}{2(H_1^2)},\frac{1}{4a\mu}\right\},$$

where

$$c^+=c^+(\mathbb{F}_q):=2\log_q(\sqrt{q}+1)$$

and $\mu=\mu(H_1;\mathbb{F}_q)$ is the minimum value of the degree (with respect to H_1) of a 1-dimensional integral subscheme of B defined over $\mathbb{F}_q$.

Proof

It is sufficient to show that $C^{\mathrm{int}}(X,H,1;\mathbb{F}_q)$ is not bigger than the right-hand side.

We need some notation. For a 1-dimensional integral subscheme C of B defined over $\mathbb{F}_q$, let $S_C:=C\times\mathbb{P}^1$ be the inverse image of C via p. Set

$$G_d(C):=\left\{D\in G_d(X,H,1;\mathbb{F}_q)\;\middle|\;\mathrm{Supp}(D)\subset S_C\right\},$$
$$G_d^{\mathrm{int}}(C):=\left\{D\in G_d^{\mathrm{int}}(X,H,1;\mathbb{F}_q)\;\middle|\;\mathrm{Supp}(D)\subset S_C\right\}.$$

Then we see that

$$G_d^{\mathrm{int}}(X,H,1;\mathbb{F}_q)\subset\bigcup_C G_d(C).$$

Moreover, if $D\in G_d(C)$ and $p(\mathrm{Supp}(D))=C$, then it is obvious that $\deg_H D\ge \deg_{H_1} C$. Hence,

$$G_d^{\mathrm{int}}(X,H,1;\mathbb{F}_q)=\bigcup_{C;\deg_{H_1}C\le d} G_d(C).$$

In order to evaluate $\#G_d(X,H,1;\mathbb{F}_q)$, we must

(i) count the number of C's the degree of which is not greater than d,
(ii) evaluate $\#G_d(C)$ for each C.

CLAIM 4.3

Define T_e as the set of 1-dimensional integral subschemes C of B, defined over $\mathbb{F}_q$ with $\deg_{H_1} C=e$. Then for any small $\epsilon>0$, there exists a constant c_0 such that, *for any e,*

$$\log_q \#T_e \leq e^2\Big(\frac{1}{2(H_1^2)} + \epsilon\Big) + c_0.$$

This follows from the fact that

$$C(B, H_1, 1; \mathbb{F}_q) = \frac{1}{2(H_1^2)}.$$

(See Main Theorem 3.5.)

CLAIM 4.4

Let $C \in T_e$. Then for any small $\epsilon > 0$,

$$\log_q \#G_d(C) \leq \max_{e \leq d} f(e, d),$$

where

$$f(e, d) = e^2\Big(\frac{c^+}{2(H_1^2)} + \epsilon\Big) + \frac{d^2}{4ae} + O(d).$$

Proof

First, we prove the claim when C is geometrically integral and nonsingular. (This is the essential case.) Then $S = S_C = C \times \mathbb{P}^1$ is also nonsingular; hence any element of $G_d(C)$ is a Cartier divisor of S.

Since $\mathrm{Pic}(S) = \mathrm{Pic}(C) \oplus \mathrm{Pic}(\mathbb{P}^1)$, any line bundle L on S can be described as $L = p_1^* M + p_2^* N$, where M and N are line bundles on C, $\mathbb{P}^1$, respectively. Let x and y be the degree of M, N, respectively. Suppose that $\deg_H L = d$. Then we have $d = ax + ey$, and Künneth's formula implies

$$\begin{aligned} h^0(S, L) &= h^0(C, M) h^0(\mathbb{P}^1, N) \\ &\leq (x+1)(y+1) \\ &= xy + x + y + 1 \qquad (4.1) \\ &\leq \frac{1}{ae}(aexy) + ax + ey + 1 \\ &\leq \frac{1}{4ae} d^2 + d + 1. \end{aligned}$$

Next, we show that

$$\begin{aligned} &\log_q \#\big\{L \in \mathrm{Pic}(S) \;\big|\; L \text{ is defined over } \mathbb{F}_q \text{ and } \deg_H L = d\big\} \qquad (4.2) \\ &\leq \frac{1}{2} c^+ e^2 + O(e). \end{aligned}$$

Since the number of effective classes $\xi \in \mathrm{N}^1(S)$ which satisfy $\deg_H \xi = d$ is at most linear order with respect to d, we may ignore it.

We need to know the upper bound of the number of line bundles numerically equivalent to zero. $\mathrm{Num}(S)$ is equal to the Jacobian variety $\mathrm{Jac}(C)$ of C, and the analogue of the Riemann hypothesis shows that

$$\#\,\mathrm{Jac}(C)(\mathbb{F}_q) = \prod_{i=1}^{2g} (\omega_i - 1),$$

where $g = g(C)$ is the genus of C, and $|\omega_i| = \sqrt{q}$ for all i. Let K be the canonical divisor of B. There exists $r > 0$ such that $rH_1 - K$ is ample. The adjunction formula yields

$$\begin{aligned} 2g &= 2 + (C + K) \cdot C \\ &\leq 2 + \frac{e^2}{(H_1^2)} + (K \cdot C) \\ &\leq 2 + \frac{e^2}{(H_1^2)} + (rH_1 \cdot C) \\ &= \frac{e^2}{(H_1^2)} + O(e). \end{aligned}$$

Hence,

$$\log_q \#\,\mathrm{Jac}(C)(\mathbb{F}_q) \leq \frac{c^+}{2(H_1^2)} e^2 + O(e),$$

from which (4.2) follows. Combining with (4.1), we obtain

$$\#G_d^{\mathrm{int}}(X, H, 1; \mathbb{F}_q) \leq \sum_{e \leq d} \sum_{\substack{C:\text{integral} \\ \deg_{H_1} C = e}} \#G_d(C) \leq \sum_{e \leq d} \#T_e \cdot \max_{C; \deg_{H_1} = e} \#G_d(C).$$

Since

$$\log_q \#G_d(C) \leq \frac{d^2}{4ae} + \frac{c^+}{2(H_1^2)} e^2 + O(d),$$

we obtain the result.

Second, we prove the claim when C is geometrically integral but singular.

Let $Z \subset S$ be the fiber of the singular locus of C, and let $\pi : \tilde{C} \to C$ be the normalization of C. Note that the genus of $\tilde{C}$ is less than that of C. Set $S_{\tilde{C}} := \tilde{C} \times \mathbb{P}^1$ and $H' := \pi^* H|_S$. Note that since $\tilde{C} \to C$ is finite, H' is also ample. Let $\tilde{G}_d(C)$ be the set of effective 1-cycles on $S_{\tilde{C}}$ defined over $\mathbb{F}_q$ satisfying $\deg_{H'} D = d$.

Also, let $G_d^{\mathrm{gen}}(C)$ be the subset of $G_d(C)$, consisting of prime divisors whose support is not contained in Z. We count only the cardinals of $G_d^{\mathrm{gen}}(C)$ since the number of prime divisors contained in Z is comparatively small.

The natural map $\pi : S_{\tilde{C}} \to S_C$ induces an injective map

$$\pi^* : G_d^{\mathrm{gen}}(C) \to \tilde{G}_d(C)$$

via $D \mapsto \overline{\pi^{-1}(D \cap (S_C \setminus Z))}$. Then, an argument similar to that above shows that

$$\log_q \#\tilde{G}_d(C) \leq \frac{1}{4ae} d^2 + \frac{c^+}{2(H_1^2)} e^2 + O(d).$$

Finally, we see the case when C is not geometrically integral. There exists a sufficiently large positive integer $r \in \mathbb{Z}_{>0}$ such that $C \times_{\mathbb{F}_q} \mathbb{F}_{q^r}$ decomposes into geometrically integral components:

$$C \times_{\mathbb{F}_q} \mathbb{F}_{q^r} = C_1 \cup \cdots \cup C_r.$$

Note that $C \times_{\mathbb{F}_q} \mathbb{F}_{q^r}$ is reduced since a finite field is perfect. The Galois action maps C_i's transitively. Note that $\deg_H C_1 = e/r$. Set $S_1 := C_1 \times \mathbb{P}^1$. Then, there is a one-to-one correspondence

$$G_d(C) \overset{1:1}{\leftrightarrow} G_{d/r}(C_1; \mathbb{F}_{q^r}),$$

where $G_{d/r}(C_1; \mathbb{F}_{q^r})$ is the set of effective 1-cycles D on S_1 defined over $\mathbb{F}_{q^r}$ with $\deg_H D = d/r$. Hence,

$$\begin{aligned}\log_q \#G_d(C) &= r \log_{q^r} \#G_{d/r}(C_1; \mathbb{F}_{q^r}) \\ &\leq r\Big(\frac{c^+}{2}\frac{e^2}{r^2} + \frac{d^2}{4r^2ae} + O\Big(\frac{d}{r}\Big)\Big) \\ &= \frac{1}{r}\Big(\frac{c^+e^2}{2} + \frac{d^2}{4rae} + O\Big(\frac{d}{r}\Big)\Big).\end{aligned}$$

The above claim is thus proved. □

From the above claim, we have

$$C(X, H, 1; \mathbb{F}_q) \leq \limsup_d \frac{1}{d^2} \max_{e \leq d}\Big(e^2\Big(\frac{1 + c^+}{2(H_1^2)} + \epsilon\Big) + \frac{d^2}{4ae}\Big).$$

Note that we ignored the terms of lower degree, since they converge to zero when divided by d^2. It is easy to see that

$$M_{d,e} := e^2\Big(\frac{1 + c^+}{2(H_1^2)} + \epsilon\Big) + \frac{d^2}{4ae}$$

takes the maximum value only when e takes the smallest or the largest value. The largest value of e is d, and $M_{d,d}/d^2 \to \frac{1+c^+}{2(H_1^2)} + \epsilon$ as $d \to \infty$. The smallest value of e is $\mu = \mu(H_1; \mathbb{F}_q)$, and $M_{d,\mu}/d^2 \to \frac{1}{4a\mu}$ as $d \to \infty$. Thus, we have proved the theorem.

We have used Landau's O in the formulas. Note that these values behave properly, so that they will not disturb our argument when taking limits, and so on. □

COROLLARY 4.5

Let (X, H) be as above. Then, $\bar{C}(X, H, 1)$ is finite, and

$$\bar{C}(X, H, 1) \leq \max\Big\{\frac{1}{(H_1^2)}, \frac{1}{4a\mu}\Big\}.$$

In particular, if $(H_1^2) \geq 4a\mu$, then

$$\bar{C}(X, H, 1) = \frac{1}{4a\mu}.$$

Proof
The first statement follows immediately from the fact that

$$\lim_{r\to\infty} c^+(\mathbb{F}_{q^r}) = 1.$$

The second statement follows from Corollary 3.8. □

EXAMPLE 4.6
Let $B := \mathbb{P}^2$ and $H := ap^*\mathcal{O}(1) + bq^*\mathcal{O}(1)$. We have $H_1 = a\mathcal{O}(1)$, so $\mu(H_1) = a$. Hence,

$$C(X, H, 1; \mathbb{F}_q) \le \max\left\{\frac{1+c^+}{2a^2}, \frac{1}{4ab}\right\}$$

and

$$\bar{C}(X, H, 1) \le \max\left\{\frac{1}{a^2}, \frac{1}{4ab}\right\}.$$

EXAMPLE 4.7
Let $X := \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, and let p_i be the ith projection $(i = 1, 2, 3)$. Set $H := ap_1^*\mathcal{O}(1) + bp_2^*\mathcal{O}(1) + cp_3^*\mathcal{O}(1)$. We may assume that $a \ge b \ge c$. Let B be the image of $p_1 \times p_2$. (Hence, $H_1 = ap_1^*\mathcal{O}(1) + bp_2^*\mathcal{O}(1)$.)

We have $\mu(H_1) = b$, so we obtain

$$C(X, H, 1; \mathbb{F}_q) \le \max\left\{\frac{1+c^+}{4ab}, \frac{1}{4bc}\right\}.$$

In particular,

$$\bar{C}(X, H, 1) \le \max\left\{\frac{1}{2ab}, \frac{1}{4bc}\right\}.$$

By symmetry, there are other ways of defining B, but the others give larger upper bounds.

CONJECTURE 4.8
We hope that the inequality of Corollary 4.5 is in fact, equal:

$$\bar{C}(X, H, 1) = \max\left\{\frac{1}{(H_1^2)}, \frac{1}{4a\mu}\right\}.$$

We still have a gap between the upper bound and the lower bound at the present; the upper bound is at most the double of the lower bound. This gap arises essentially from the fact that the Jacobians we must look at are not only one but infinitely many.

Also, we must be careful at several points when calculating the bounds more precisely; for example, we need the next statement to be proven to fill the gap.

CONJECTURE 4.9
Let (B, H) be a $\mathbb{Q}$-polarized smooth projective surface defined over $\mathbb{F}_q$. Let M_e be the number of nonsingular curves on B defined over $\mathbb{F}_q$, which satisfies $\deg_H = e$.

Then

$$\limsup_{e\to\infty} \frac{\log_q M_e}{e^2} = \frac{1}{2(H^2)}.$$

Let A be a very ample divisor on B. According to Bertini's theorem, most of the hypersection is reduced and nonsingular; that is, there is a Zariski open subset $U \subset |A|$ (regarding $|A|$ as a projective space) such that any section in U is a nonsingular curve. The above conjecture is asking whether the number of the $\mathbb{F}_q$-rational points in U grows properly if we increase the degree e. In other words, are most of the elements of $G_d(B, H, 1; \mathbb{F}_q)$ nonsingular?

COROLLARY 4.10

Let $\omega := \frac{1+2\log_q(\sqrt{q}+1)}{2}$. *Then we have*

$$C\big(\mathbb{P}^3, \mathcal{O}(1), 1; \mathbb{F}_q\big) \le \frac{(4\omega+1)^2}{16\omega}.$$

In particular,

$$\bar{C}\big(\mathbb{P}^3, \mathcal{O}(1), 1\big) \le \frac{25}{16}.$$

Proof

Define a rational map

$$\pi : \mathbb{P}^3 \to \mathbb{P}^2 \times \mathbb{P}^1,$$
$$(x:y:z:w) \mapsto \big((x:y:w),(z:w)\big).$$

The indeterminancy locus of this map is

$$\{w=z=0\} \cup \{w=x=y=0\}.$$

For any $Y \in G_d^{\mathrm{int}}(\mathbb{P}^3, \mathcal{O}(1), 1; \mathbb{F}_q)$ which is not contained in the plane $Z = \{w=0\}$ (π is injective outside Z), consider the strict transform $\tilde{Y}$ of Y by π. Obviously, the map $Y \mapsto \tilde{Y}$ is injective. Set

$$a := \frac{4\omega}{4\omega+1}, \qquad b := \frac{1}{4\omega+1},$$

and fix an ample line bundle $H := aA + bB$ on $\mathbb{P}^2 \times \mathbb{P}^1$, $A := p_1^*\mathcal{O}_{\mathbb{P}^2}(1)$, and $B := p_2^*\mathcal{O}_{\mathbb{P}^1}(1)$. Since $a+b=1$, and $A\cdot\tilde{Y} = B\cdot\tilde{Y} = \deg Y$, we may conclude that

$$\begin{aligned} C\big(\mathbb{P}^3, \mathcal{O}(1), 1; \mathbb{F}_q\big) &\le C(\mathbb{P}^2\times\mathbb{P}^1, H, 1; \mathbb{F}_q) \\ &\le \frac{(4\omega+1)^2}{16\omega}. \end{aligned}$$

Note that we ignored the effective 1-cycles which are included in the plane $Z = \{w=0\}$ because $C(Z, \mathcal{O}(1)|_Z, 1; \mathbb{F}_q) = 1/2$ is smaller than the above upper bound. Also, note that the above a, b give the smallest upper bound in this approach. The two constants a and b are not rational numbers, but suitable approximations of a and b by rational numbers give the same result. It is not the essential part of the proof, so we omitted it. □

REMARK 4.11
We can also consider the birational map

$$\mathbb{P}^3 \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$

and calculate $C(\mathbb{P}^3, \mathcal{O}(1), 1; \mathbb{F}_q)$. But this gives a bigger upper bound.

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