Twisted Poincaré lemma and twisted Čech–de Rham isomorphism in case dimension $=1$

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Abstract For a compact Riemann surface, $(n+1)$-tuple $x := (x_0, \dots, x_n)$ of points on it, and a holomorphic vector bundle with an integrable connection on the open Riemann surface X_x deprived of $(n+1)$ points $x_0, \dots, x_n$, let $\mathcal{L}$ be the local system of horizontal sections of the connection. In this article, we give a suitable covering of X_x to calculate the Čech cohomology and describe the isomorphism between the cohomology and the *twisted* de Rham cohomology, which is the cohomology of the complex with the differentials given by the connection. This isomorphism is given by the integrations over Aomoto's *regularized* paths, the so-called *Euler type integrals*.

For the family $\{X_x\}_x$ parametrized by x, we give a variant of the isomorphism.

1. Introduction

For a compact Riemann surface $\overline{X}$ of genus g and an $(n+1)$-tuple $x = (x_0, x_1, \dots, x_n)$ of points on $\overline{X}$, let X_x be the punctured Riemann surface: $X_x = \overline{X} \setminus \{x_0, x_1, \dots, x_n\}$. We consider a local system $\mathcal{L}$ defined by horizontal sections of a holomorphic connection ∇ on a (not necessarily trivial) vector bundle $\mathcal{V}$ over X_x:

$$\mathcal{L} := \operatorname{Ker}(\nabla : \mathcal{V} \longrightarrow \mathcal{V} \otimes \Omega^1_{X_x}).$$

In this article, we give an explicit description of the isomorphism between the Čech cohomology with its coefficients in $\mathcal{L}$ (called *twisted* Čech cohomology) and the *twisted* de Rham cohomology, that is, the cohomology of the de Rham complex whose differential is given by ∇. In the formula describing the isomorphism, the *Euler*-type integral, that is, the integration over an $\mathcal{L}^\vee$-valued cycle (called a *twisted* cycle), appears.

Our approach to getting such an explicit description is to refine Poincaré lemma, that is, to describe explicitly the solutions of the equation $\nabla u = \eta$. This

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type of equation is locally reduced to a system of inhomogeneous linear differential equations

$$(d-A)\begin{bmatrix} g_1 \\ \vdots \\ g_N \end{bmatrix} = \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_N \end{bmatrix}$$

for n 1-forms $\eta_1, \ldots, \eta_N$ and an $(N \times N)$-matrix A whose entries are 1-forms. As is well known in elementary calculus, it can be solved by the *method of variation of constants*, which is summarized as follows. The following diagram is commutative:

$$\begin{tikzcd} \mathcal{O}^{\oplus N} \arrow[r, "\Phi", "\cong"'] \arrow[d, "d-A"] & \mathcal{L} \otimes_{\mathbb{C}} \mathcal{O} \arrow[d, "1\otimes d"] \\ \mathcal{O}^{\oplus N} \otimes_{\mathcal{O}} \Omega^1 \arrow[r, "\Phi", "\cong"'] & \mathcal{L} \otimes_{\mathbb{C}} \Omega^1. \end{tikzcd}$$

Thus, we have $d - A = \Phi^{-1} \circ (1 \otimes d) \circ \Phi$, and a solution is given by

$$\begin{bmatrix} g_1 \\ \vdots \\ g_N \end{bmatrix} = \Phi^{-1} \circ \int \circ \Phi \left(\begin{bmatrix} \eta_1 \\ \vdots \\ \eta_N \end{bmatrix} \right). \tag{1.1}$$

The isomorphism Φ is *locally* given by $\Phi(\varsigma_1 f_1 + \cdots + \varsigma_N f_N) = \varsigma_1 \otimes f_1 + \cdots + \varsigma_N \otimes f_N$, where $\{\varsigma_1, \ldots, \varsigma_N\}$ is a (local) basis of $\mathcal{L}$. The right-hand side of (1.1) actually makes sense, especially when it is considered on a domain homotopic to a punctured disk and each ς_i is *not* single valued. In such a situation, one can calculate the solution (1.1), which turns out to be *single* valued, using a carefully chosen integration path (the so-called *regularized paths* by Aomoto [1]). These are the keystones of our desired description of twisted Čech–de Rham isomorphism.

By the above-mentioned Poincaré lemma, it is sufficient, in order for the Čech cohomology to be calculated, that we take a covering $\{U_\mu\}$ such that each U_μ is homotopic to a punctured disk. We give such a covering and a basis of the Čech cohomology for this covering.

A variation in a relative case is also treated. The punctured Riemann surfaces of the form X_x are parametrized by x. We fix such an x_0 once and for all. Then x runs through the configuration space S of n-points on $\overline{X}$. So the collection $\{X_x\}_{x\in S}$ gives rise to an analytic family $\pi : \mathcal{X} \longrightarrow S$, where $\mathcal{X} = \{(t,x) \in \overline{X} \times S \mid t \neq x_0, \ldots, x_n\}$. We consider a rank N vector bundle $\mathcal{V}_{\mathcal{X}}$ with an integrable connection $\nabla_{\mathcal{X}}$ over $\mathcal{X}$:

$$\nabla_{\mathcal{X}} : \mathcal{V}_{\mathcal{X}} \longrightarrow \mathcal{V}_{\mathcal{X}} \otimes \Omega^1_{\mathcal{X}}.$$

It induces a vector bundle $\mathcal{H}^1$ with a natural integrable connection $\nabla^{(\mathrm{GM})}$ (called *Gauss-Manin connection*) over S, each of whose fibers is the twisted de Rham cohomology on X_x. On the other hand, the Čech cohomology forms another analytic vector bundle $\check{\mathcal{H}}^1$. We give N horizontal sections of $\check{\mathcal{H}}^1$ in terms of Čech cocycles for a covering similar to the above-mentioned one and an explicit description of the isomorphism between $\mathcal{H}^1$ and $\check{\mathcal{H}}^1$ in terms of an Euler-type integral.

2. Twisted Poincaré lemma

For a compact Riemann surface $\overline{X}$ and an $(n+1)$-tuple $x = (x_0, x_1, \ldots, x_n)$ of points on $\overline{X}$, let X_x be the punctured Riemann surface $X_x = \overline{X} \setminus \{x_0, x_1, \ldots, x_n\}$. We consider a rank N vector bundle $\mathcal{V}$ with an (integrable) connection ∇ over X_x:

$$\nabla : \mathcal{V} \longrightarrow \mathcal{V} \otimes \Omega^1_{X_x}.$$

Let $\mathcal{L}$ be the kernel of ∇, which is a local system, for ∇ is integrable. As is well known, the integrability says that $\mathcal{V}$ (resp., $\mathcal{V} \otimes \Omega^1_{X_x}$) is isomorphic to $\mathcal{L} \otimes \mathcal{O}_{X_x}$ (resp., $\mathcal{L} \otimes \Omega^1_{X_x}$) and that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow[\cong]{\Phi} & \mathcal{L} \otimes_{\mathbb{C}} \mathcal{O}_{X_x} \\ \downarrow{\scriptstyle \nabla} & & \downarrow{\scriptstyle 1 \otimes d} \\ \mathcal{V} \otimes_{\mathcal{O}_{X_x}} \Omega^1_{X_x} & \xrightarrow[\cong]{\Phi} & \mathcal{L} \otimes_{\mathbb{C}} \Omega^1_{X_x} \end{array} \tag{2.1}$$

where $\Phi^{-1}(s \otimes h) = sh$. Using this diagram (of the *method of variation of constants*), we prove the *twisted Poincaré lemma*.

THEOREM 2.1 (TWISTED POINCARÉ LEMMA)

Let U be an open set in X_x, and let o be a base point on U. Put $[o,p]$ a path in U connecting two points o and p.

(1) *If U is simply connected, then the following holds. For $\eta \in \Gamma(U, \mathcal{V} \otimes \Omega^1_{X_x})$, there exists $u \in \Gamma(U, \mathcal{V})$ such that $\nabla u = \eta$. Moreover, this u is given by the following. We can take linearly independent N-sections $\{s_1, \ldots, s_N\}$ of $\mathcal{L}$ over U and its dual basis $\{s_1^\vee, \ldots, s_N^\vee\}$; that is, $s_i^\vee(s_j) = \delta_{ij}$, where $s_i^\vee \in \Gamma(U, \mathcal{L}^\vee)$. Then we have*

$$u(p) = \sum_i s_i(p) \int_{[o,p]} (s_i^\vee \otimes 1)\big(\Phi(\eta)\big), \tag{2.2}$$

and u is independent of the choices of s_i and $[o,p]$.

(2) *If $\pi_1(U, o)$ is isomorphic to the free group $\langle \sigma \rangle$ generated by one element corresponding to a closed loop σ and the eigenvalues of its monodromy action M_σ on the stalk $\mathcal{L}_o^\vee$ do not contain 1, then the following holds. For $\eta \in \Gamma(U, \mathcal{V} \otimes \Omega^1_{X_x})$, there exists the unique section $u \in \Gamma(U, \mathcal{V})$ such that $\nabla u = \eta$. Moreover, this u is given by the following. We can take linearly independent N-germs $\{s_{1,o}, \ldots, s_{N,o}\}$ of $\mathcal{L}$ over o and its dual basis $\{s_{1,o}^\vee, \ldots, s_{N,o}^\vee\}$. For a path γ in U with its initial*

point at o, we denote $s_{i,\gamma}$ (resp., $s^\vee_{i,\gamma}$) the analytic continuation of $s_{i,o}$ (resp., $s^\vee_{i,o}$) along γ. Then we have

$$u(p) = \sum_i s_{i,[o,p]}(p)\Big(\int_{[o,p]}(s^\vee_{i,[o,p]}\otimes 1)\big(\Phi(\eta)\big) \\ + \int_\sigma \big((M_\sigma - \mathrm{id})^{-1}s^\vee_{i,\sigma}\otimes 1\big)\big(\Phi(\eta)\big)\Big), \tag{2.3}$$

where $(M_\sigma - \mathrm{id})^{-1}s^\vee_{i,\sigma}$ is the analytic continuation of the germ $(M_\sigma - \mathrm{id})^{-1}s^\vee_{i,o}$ along σ, and u is independent of the choices of $s_{i,o}$, $[o,p]$, and o.

Proof
The diagram (2.1) tells us that (2.2) or (2.3), if it is well defined, satisfies $\nabla u = \eta$. In the case when U is simply connected, the integral (2.2) is well defined. Thus, we have the assertion (1). To prove assertion (2), we prove that u is well defined, that is, that $u(p)$ is determined independently of a choice of paths $[o,p]$. We take another path $[o,p]'$. It is sufficient to prove that

$$\sum_i s_{i,[o,p]'}\Big(\int_{[o,p]'}(s^\vee_{i,[o,p]'}\otimes 1)\big(\Phi(\eta)\big) \\ + \int_\sigma \big((M_\sigma - \mathrm{id})^{-1}s^\vee_{i,\sigma}\otimes 1\big)\big(\Phi(\eta)\big)\Big) \\ - \sum_i s_{i,[o,p]}\Big(\int_{[o,p]}(s^\vee_{i,[o,p]}\otimes 1)\big(\Phi(\eta)\big) \\ + \int_\sigma \big((M_\sigma - \mathrm{id})^{-1}s^\vee_{i,\sigma}\otimes 1\big)\big(\Phi(\eta)\big)\Big) = 0$$

in the case when $[o,p]^{-1}\circ[o,p]'$ is homotopic in U to σ. Applying $s^\vee_{j,[o,p]'}$ each side of this formula, we prove

$$\Big(\int_{[o,p]'}(s^\vee_{j,[o,p]'}\otimes 1)\big(\Phi(\eta)\big) + \int_\sigma \big((M_\sigma - \mathrm{id})^{-1}s^\vee_{j,\sigma}\otimes 1\big)\big(\Phi(\eta)\big)\Big) \\ - \sum_i s^\vee_{j,[o,p]'}(s_{i,[o,p]})\Big(\int_{[o,p]}(s^\vee_{i,[o,p]}\otimes 1)\big(\Phi(\eta)\big) \\ + \int_\sigma \big((M_\sigma - \mathrm{id})^{-1}s^\vee_{i,\sigma}\otimes 1\big)\big(\Phi(\eta)\big)\Big) = 0. \tag{2.4}$$

Note that $s^\vee_{i,[o,p]'} = M_\sigma s^\vee_{i,[o,p]}$, $\sum_i M_\sigma s^\vee_{j,\gamma}(s_{i,\gamma})s^\vee_{i,\gamma} = M_\sigma s^\vee_{j,\gamma}$, and $M_\sigma s^\vee_{j,\gamma}(s_{i,\gamma})$ does not depend on a path γ but on the germs $s_{j,o}$ and $s^\vee_{i,o}$. Then the left-hand side of (2.4) equals

$$\Big(\int_{[o,p]}(M_\sigma s^\vee_{j,[o,p]}\otimes 1)\big(\Phi(\eta)\big) \\ + \int_\sigma (s^\vee_{j,\sigma}\otimes 1)\big(\Phi(\eta)\big) + \int_\sigma \big((M_\sigma - \mathrm{id})^{-1}s^\vee_{j,\sigma}\otimes 1\big)\big(\Phi(\eta)\big)\Big)$$

$$
\begin{aligned}
&- \Big(\int_{[o,p]} (M_\sigma s^\vee_{j,[o,p]} \otimes 1)\big(\Phi(\eta)\big) \\
&\qquad + \int_\sigma \big((M_\sigma - \mathrm{id})^{-1} M_\sigma s^\vee_{j,\sigma} \otimes 1\big)\big(\Phi(\eta)\big)\Big) = 0.
\end{aligned}
$$

The uniqueness of u follows from the fact $\Gamma(U,\mathcal{L}) = 0$. We have thus proved the theorem. □

REMARK 1

The above theorem implies that we have the following exact sequence:

$$0 \longrightarrow j_*\mathcal{L} \longrightarrow j_*\mathcal{V} \xrightarrow{\nabla} j_*(\mathcal{V} \otimes \Omega^1_{X_x}) \longrightarrow 0,$$

where $j : X_x \hookrightarrow \overline{X}$ is the inclusion map.

3. Integrations over regularized paths

The integration (2.3) in Section 2 can be regarded as an integration over a *regularized* path, which is formulated by Aomoto [1] in the case $\operatorname{rank}\mathcal{L} = 1$. We generalize it to the higher-rank case. (The special case of higher-rank local systems appears in the work of Mimachi, Ohara, and Yoshida [3].)

DEFINITION 3.1 (TWISTED CHAIN)

A twisted 1-chain is a 1-chain with its coefficients in $\mathcal{L}^\vee$, that is, a linear combination of $\{\gamma \otimes s^\vee_\gamma\}_\gamma$, where γ is a singular 1-simplex (i.e., a path) and $s^\vee_\gamma$ is a local section of $\mathcal{L}^\vee$ on γ.

DEFINITION 3.2 (REGULARIZATION)

Let o, U, σ, M_σ be as in Theorem 2.1(2). Let γ be a path on U whose initial point is o, and let $s^\vee_\gamma$ be a section of $\mathcal{L}^\vee$ over γ. The regularization of $\gamma \otimes s^\vee_\gamma$ is defined by

$$\mathrm{reg}_U\, \gamma \otimes s^\vee_\gamma := \gamma \otimes s^\vee_\gamma + \sigma \otimes (M_\sigma - \mathrm{id})^{-1} s^\vee_\sigma,$$

where $s^\vee_{\sigma,o} = s^\vee_{\gamma,o}$.

DEFINITION 3.3 (INTEGRATION OVER TWISTED 1-SIMPLEX)

Let γ be a path on an open set U of X_x. For $\eta \in \Gamma(U, \mathcal{V} \otimes \Omega^1_{X_x})$, the integration over $\gamma \otimes s^\vee_\gamma$ is defined by

$$\int_{\gamma \otimes s^\vee_\gamma} s^\vee(\eta) := \int_\gamma (s^\vee_\gamma \otimes 1)\big(\Phi(\eta)\big),$$

where Φ is defined in (2.1).

Using this formulation, we have the following expression of the integration (2.3):

$$u(p) = \sum_i s_{i,[o,p]}(p) \int_{\mathrm{reg}_U [o,p] \otimes s^\vee_{i,[o,p]}} s^\vee_i(\eta). \tag{3.1}$$

4. Twisted Čech–de Rham isomorphism

Associated to a covering $\mathfrak{U}=\{U_\mu\}$, the Čech complex with its coefficients in $\mathcal{L}$ is given by

$$\begin{aligned}(4.1)\qquad & 0\longrightarrow \bigoplus_{\mu}\Gamma(U_\mu,\mathcal{L})\xrightarrow{\partial^0}\bigoplus_{\mu<\nu}\Gamma(U_\mu\cap U_\nu,\mathcal{L})\\ &\xrightarrow{\partial^1}\bigoplus_{\mu<\nu<\lambda}\Gamma(U_\mu\cap U_\nu\cap U_\lambda,\mathcal{L})\longrightarrow\cdots,\end{aligned}$$

where

$$\begin{aligned}\bigl(\partial^0(s_\mu)_\mu\bigr)_{\mu\nu}&=s_\nu|_{U_\mu\cap U_\nu}-s_\mu|_{U_\mu\cap U_\nu},\\ \bigl(\partial^1(s_{\mu\nu})_{\mu\nu}\bigr)_{\mu\nu\lambda}&=s_{\nu\lambda}|_{U_\mu\cap U_\nu\cap U_\lambda}-s_{\mu\lambda}|_{U_\mu\cap U_\nu\cap U_\lambda}+s_{\mu\nu}|_{U_\mu\cap U_\nu\cap U_\lambda}.\end{aligned}$$

We assume that the covering $\mathfrak{U}$ satisfies the following conditions.

ASSUMPTION 4.1

(1) $o\in\bigcap_\mu U_\mu$, where $o\in X_x$ is a base point.

(2) $\pi_1(U_\mu,o)$ is isomorphic to the free group $\langle\sigma_\mu\rangle$ generated by a single element corresponding to a closed loop σ_μ.

(3) $U_\mu\cap U_\nu$ is connected.

On the other hand, the *twisted de Rham complex* is defined by the following:

$$(4.2)\qquad 0\longrightarrow\Gamma(X_x,\mathcal{V})\xrightarrow{\nabla}\Gamma(X_x,\mathcal{V}\otimes\Omega^1_{X_x})\longrightarrow 0.$$

The twisted Poincaré lemma (Theorem 2.1) tells us that the first twisted de Rham cohomology $H^1_\nabla(X_x)$ (defined by the complex (4.2)) is isomorphic to the first Čech cohomology $H^1(\mathfrak{U},\mathcal{L})$ (defined by the complex (4.1)).

THEOREM 4.1

We assume that the eigenvalues of the monodromy action M_{σ_μ} on the stalk $\mathcal{L}^\vee_o$ do not contain 1*. Let $\{s_{1,o},\dots,s_{N,o}\}$ be linearly independent N-germs of $\mathcal{L}$ over o, and let $\{s^\vee_{1,o},\dots,s^\vee_{N,o}\}$ be its dual basis. For a path γ with its initial point at o, let $s^\vee_{i,\gamma}$ be the analytic continuation of $s^\vee_{i,o}$ along γ. We denote by $s_{i,\mu\nu}$ the section of $\mathcal{L}$ over $U_\mu\cap U_\nu$ whose germ coincides with $s_{i,o}$. (In the case $\pi_1(U_\mu\cap U_\nu,o)\neq\{1\}$, $s_{i,\mu\nu}$ indicates zero.) Then, the morphism $\Psi: H^1_\nabla(X_x)\longrightarrow H^1(\mathfrak{U},\mathcal{L})$ given by*

$$\Psi(\eta)=\Bigl(-\sum_i s_{i,\mu\nu}\int_{\mathrm{reg}_{\mu\nu}s^\vee_i}s^\vee_i(\eta)\Bigr)_{\mu\nu}$$

is well defined and an isomorphism, where

$$\begin{aligned}\mathrm{reg}_{\mu\nu}s^\vee_i&=\sigma_\mu\otimes(M_{\sigma_\mu}-\mathrm{id})^{-1}s^\vee_{i,\sigma_\mu}\\ &\quad-\sigma_\nu\otimes(M_{\sigma_\nu}-\mathrm{id})^{-1}s^\vee_{i,\sigma_\nu}.\end{aligned}$$

Proof
We have the following commutative diagram:

$$\begin{tikzcd}
 & & 0 & 0 & \\
0 \arrow[r] & \Gamma(X_x, \mathcal{V}\otimes\Omega^1_{X_x}) \arrow[r] & \bigoplus_\mu \Gamma(U_\mu, \mathcal{V}\otimes\Omega^1_{X_x}) \arrow[r, "\partial^0_1"] \arrow[u] & Z^1(\mathfrak{U}, \mathcal{V}\otimes\Omega^1_{X_x}) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \Gamma(X_x, \mathcal{V}) \arrow[r, "\iota"] \arrow[u] & \bigoplus_\mu \Gamma(U_\mu, \mathcal{V}) \arrow[r, "\partial^0_0"] \arrow[u, "\nabla"] & Z^1(\mathfrak{U}, \mathcal{V}) \arrow[r] \arrow[u, "\nabla"] & 0 \\
 & & \bigoplus_\mu \Gamma(U_\mu, \mathcal{L}) \arrow[r] \arrow[u, "\iota'"] & Z^1(\mathfrak{U}, \mathcal{L}) \arrow[u] & \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

In this diagram, both of the two vertical sequences are exact due to the twisted Poincaré lemma (Theorem 2.1), and both of the two horizontal sequences are exact because X_x and $U_{\mu_0}\cap U_{\mu_1}\cap\cdots\cap U_{\mu_k}$ are Stein (Cartan's theorem B). Here we have

$$H^1_\nabla(X_x) \cong \operatorname{Ker}\partial^0_1/\operatorname{Im}\nabla\circ\iota,$$
$$H^1(\mathfrak{U},\mathcal{L}) \cong \operatorname{Ker}\nabla/\operatorname{Im}\partial^0_0\circ\iota',$$

and Ψ should be defined by $\partial^0_0\circ\nabla^{-1}$. A standard argument by diagram chasing tells us that Ψ is well defined and an isomorphism. For a $\mathcal{V}$-valued 1-form η, we calculate $\Psi(\eta)$ explicitly by using formula (3.1) in the proof of the twisted Poincaré lemma:

$$\begin{aligned}
\Psi(\eta) &= \partial^0_0\Big(\sum_i s_{i,[o,p]}(p)\int_{\operatorname{reg}_{U_\mu}[o,p]\otimes s^\vee_{i,[o,p]}} s^\vee_i(\eta)\Big)_\mu \\
&= \Big(\sum_i s_{i,[o,p]}(p)\int_{\operatorname{reg}_{U_\nu}[o,p]\otimes s^\vee_{i,[o,p]}} s^\vee_i(\eta) \\
&\qquad - \sum_i s_{i,[o,p]}(p)\int_{\operatorname{reg}_{U_\mu}[o,p]\otimes s^\vee_{i,[o,p]}} s^\vee_i(\eta)\Big)_{\mu\nu}.
\end{aligned}$$

Note that $[o,p]$ is on $U_\mu\cap U_\nu$. Thus $s_{i,[o,p]}(p)=s_{i,\mu\nu}(p)$, and we have

$$\operatorname{reg}_{U_\nu}[o,p]\otimes s^\vee_{i,[o,p]} - \operatorname{reg}_{U_\mu}[o,p]\otimes s^\vee_{i,[o,p]} = -\operatorname{reg}_{\mu\nu} s^\vee_i.$$

(In the case $\pi_1(U_\mu\cap U_\nu, o)\neq\{1\}$, the restriction of $\sum_i s_{i,[o,p]}(p)\int s^\vee_i(\eta)$ to $U_\mu\cap U_\nu$ vanishes.) We have thus proved the theorem. □

REMARK 2

The integral $\int_{\mathrm{reg}_{\mu\nu} s_i^\vee} s_i^\vee(\eta)$ is a so-called *Euler-type integral*, that is, a pairing between $H^1_\nabla(X_x)$ and $H_1(X_x, \mathcal{L})$ because $\mathrm{reg}_{\mu\nu} s_i^\vee$ can be thought of as a representative of an element of $H_1(X_x, \mathcal{L}^\vee)$.

5. Explicit description of twisted Čech–de Rham isomorphism

In this section, we give a covering satisfying Assumption 4.1 explicitly, and we take integration paths (twisted *cycles*) accordingly. Using integrations over these paths, we describe the twisted Čech–de Rham isomorphism.

Let g be the genus of the compact Riemann surface $\overline{X}$, and let $\gamma_1, \dots, \gamma_{2g}$ be the (ordinary) cycles on $\overline{X}$ whose ends are at the point x_0, that is, the generators of $\pi_1(\overline{X}, x_0)$. We assume that the complement of the γ_i's is a simply connected region Δ and contains $x_1, \dots, x_n$; Δ is identified with the interior of a convex $4g$-sided polygon D, and each side of it is identified with some γ_i. Fix a vertex $\widetilde{x_0}$ of D. Let $[\widetilde{x_0}, x_i]$ be a segment connecting two points $\widetilde{x_0}$, x_i. We assume that each two of $\gamma_1, \dots, \gamma_{2g}$, $[\widetilde{x_0}, x_1], \dots, [\widetilde{x_0}, x_n]$ intersect in $\overline{X}$ only at x_0. Now we have the open covering $\mathfrak{U} = \{U_\mu\}_{\mu \in \{\gamma_1, \dots, \gamma_{2g}, x_1, \dots, x_n\}}$ of X_x satisfying Assumption 4.1:

$$U_{\gamma_i} = \overline{X} \setminus \Big(\bigcup_{j \neq i} \gamma_j \cup \bigcup_{k=1}^{n} [\widetilde{x_0}, x_k] \Big),$$

$$U_{x_k} = \overline{X} \setminus \Big(\bigcup_{i=1}^{2g} \gamma_i \cup \bigcup_{j \neq k} [\widetilde{x_0}, x_j] \Big).$$

REMARK 3

Let U be the open set of X_x deprived of all γ_i's and all $[\widetilde{x_0}, x_k]$'s from $\overline{X}$:

$$U = \overline{X} \setminus \Big(\bigcup_{j=1}^{2g} \gamma_j \cup \bigcup_{k=1}^{n} [\widetilde{x_0}, x_k] \Big).$$

Each two of $\mathfrak{U}$ intersect on $U : U_\mu \cap U_\nu = U$. We can take linearly independent N sections $s_1, \dots, s_N$ of $\mathcal{L}$ over U because it is simply connected.

We take a point $o \in U$ and generators σ_μ of $\pi_1(U_\mu, o)$: In the case $\mu = \gamma_i$, σ_μ is a loop transverse to γ_i; and in the case $\mu = x_k$, σ_μ is a loop surrounding the point x_k.

PROPOSITION 5.1

We assume that the eigenvalues of the monodromy action of σ_μ on the stalk $\mathcal{L}_o$ do not contain 1. *The first cohomology $H^1(\mathfrak{U}, \mathcal{L})$ of the Čech complex has a basis formed by the $N(n+2g-1)$-cocycles $e_\mu \otimes s_k$ ($\mu \in \{\gamma_1, \dots, \gamma_{2g}, x_1, \dots, x_{n-1}\}$, $k = 1, \dots, N$) defined by*

$$e_\mu \otimes s_k := (e^{(\mu)}_{\nu\lambda} s_k)_{\nu\lambda}, \quad e^{(\mu)}_{x_n\lambda} = -\delta_{\mu\lambda}, e^{(\mu)}_{\nu\lambda} = e^{(\mu)}_{x_n\lambda} - e^{(\mu)}_{x_n\nu}.$$

Proof
By Remark 3, an arbitrary cochain $(s_{\nu\lambda})_{\nu\lambda}$ can be expressed by such a form as $s_{\nu\lambda} = \sum_k a^k_{\nu\lambda} s_k$, where $a^k_{\nu\lambda} \in \mathbb{C}$. Note that $\Gamma(U_\mu, \mathcal{L}) = 0$ because the eigenvalues of the monodromy action of σ_μ do not contain 1 by the assumption. So $H^1(\mathfrak{U}, \mathcal{L})$ coincides with cocycles $\operatorname{Ker} \partial^1$. The cocycle condition is equivalent to $a^k_{\nu\lambda} - a^k_{\mu\lambda} + a^k_{\mu\nu} = 0$. Hence $a^k_{\nu\lambda} = a^k_{x_n\lambda} - a^k_{x_n\nu}$, and $(a^k_{\nu\lambda})_{\nu\lambda}$ can be expressed by a linear combination of $\{e^{(\mu)}_{\nu\lambda}\}_\mu$, whence the assertion. □

Applying Theorem 4.1 for the covering given above, we obtain the following.

COROLLARY 5.1
We assume that the eigenvalues of the monodromy action M_{σ_μ} on the stalk $\mathcal{L}^\vee_o$ do not contain 1. Let $\{s_{1,o}, \ldots, s_{N,o}\}$ be the germs of $\{s_1, \ldots, s_N\}$ at o, and let $\{s^\vee_{1,o}, \ldots, s^\vee_{N,o}\}$ be its dual basis. For a path γ with its initial point at o, let $s^\vee_{i,\gamma}$ be the analytic continuation of $s^\vee_{i,o}$ along γ. Then, the isomorphism $\Psi : H^1_\nabla(X_x) \longrightarrow H^1(\mathfrak{U}, \mathcal{L})$ is given by

$$\Psi(\eta) = \sum_{\mu,k} \Big(\int_{\mathrm{reg}_{x_n\mu}\, s^\vee_k} s^\vee_k(\eta) \Big) e_\mu \otimes s_k.$$

REMARK 4
This implies that $\{\mathrm{reg}_{x_n\mu}\, s^\vee_k\}_{\mu,k}$ forms a basis of $H_1(X_x, \mathcal{L}^\vee)$.

6. Relative twisted Čech–de Rham isomorphism

The punctured Riemann surfaces of the form X_x are parametrized by x. We shall fix x_0. Then x runs through the configuration space S of n-points on $\overline{X}$: $S := \overline{X}^n \setminus \bigcup_{i\neq j}\{x_i = x_j\}$. So the collection $\{X_x\}_{x\in S}$ forms an analytic family $\pi : \mathcal{X} \longrightarrow S$, where $\mathcal{X} = \{(t, x) \in \overline{X} \times S \mid t \neq x_0, \ldots, x_n\}$. We consider a rank N vector bundle $\mathcal{V}_{\mathcal{X}}$ with an integrable connection $\nabla_{\mathcal{X}}$ over $\mathcal{X}$:

$$\nabla_{\mathcal{X}} : \mathcal{V}_{\mathcal{X}} \longrightarrow \mathcal{V}_{\mathcal{X}} \otimes \Omega^1_{\mathcal{X}}.$$

Let $\mathcal{L}_{\mathcal{X}}$ be the kernel of $\nabla_{\mathcal{X}}$, which is a local system because $\nabla_{\mathcal{X}}$ is integrable. It induces a vector bundle $\mathcal{H}^1$ over S, each of whose fibers is isomorphic to $H^1_\nabla(X_x)$. Let $DR^\bullet_{\nabla_{\mathcal{X}/S}}$ be the relative de Rham complex with the differential $\nabla_{\mathcal{X}/S}$ induced from the above connection $\nabla_{\mathcal{X}}$:

$$0 \longrightarrow \mathcal{V}_{\mathcal{X}} \xrightarrow{\nabla_{\mathcal{X}/S}} \mathcal{V}_{\mathcal{X}} \otimes \Omega^1_{\mathcal{X}/S} \longrightarrow 0.$$

The vector bundle $\mathcal{H}^1$ is the first cohomology of $\mathbb{R}\pi_* DR^\bullet_{\nabla_{\mathcal{X}/S}}$. Because $\pi : \mathcal{X} \longrightarrow S$ is Stein, we have the identification $\mathcal{H}^1 \cong \pi_*(\mathcal{V}_{\mathcal{X}} \otimes \Omega^1_{\mathcal{X}/S})/\nabla_{\mathcal{X}/S}(\pi_*\mathcal{V}_{\mathcal{X}})$, whose sections are represented by $\mathcal{V}_{\mathcal{X}}$-valued relative 1-forms.

The vector bundle $\mathcal{H}^1$ has a natural connection ∇^{GM} (*Gauss-Manin connection*): for $[\eta] \in \mathcal{H}^1$ represented by a 1-form on $\mathcal{X}$ and a vector field v over S, $\nabla^{\mathrm{GM}}_v[\eta] := [\nabla_{\widetilde{v}}\eta]$, where $\widetilde{v}$ is a lift of v to $\mathcal{X}$ and $[\bullet]$ indicates the element of $\mathcal{H}^1$ represented by a 1-form $\bullet$ on $\mathcal{X}$.

We have another vector bundle $\check{\mathcal{H}}^1$ corresponding to Čech cohomology. Let $\mathcal{L}_{\mathcal{X}/S}$ be the kernel of $\nabla_{\mathcal{X}/S} : \mathcal{V}_{\mathcal{X}} \longrightarrow \mathcal{V}_{\mathcal{X}} \otimes \Omega^1_{\mathcal{X}/S}$. The vector bundle $\check{\mathcal{H}}^1$ should be defined by $R^1\pi_*\mathcal{L}_{\mathcal{X}/S}$. By the projection formula, $\check{\mathcal{H}}^1$ is isomorphic to $R^1\pi_*\mathcal{L}_{\mathcal{X}} \otimes_{\mathbb{C}_S} \mathcal{O}_S$ because $\mathcal{L}_{\mathcal{X}/S}$ is isomorphic to $\mathcal{L}_{\mathcal{X}} \otimes_{\mathbb{C}_S} \pi^{-1}\mathcal{O}_S$. We construct and compute $R^1\pi_*\mathcal{L}_{\mathcal{X}}$ by means of Čech resolution.

We use the symbols $\widetilde{x_0}$, γ_i, Δ, and D, the same as in Section 5. Fix an identification of Δ with the interior of D. For $x_i, x_j \in D$, we denote by $\theta(x_i, x_j)$ the angle contained in D whose sides are $[\widetilde{x_0}, x_i]$ and $[\widetilde{x_0}, x_j]$. For $I = (i_1, \dots, i_n)$, we take an open set $V_I := \{\theta(x_{i_1}, x_{i_n}) > \theta(x_{i_2}, x_{i_n}) > \cdots > \theta(x_{i_{n-1}}, x_{i_n})\} \subset S$ and compute $\Gamma(V_I, R^1\pi_*\mathcal{L}_{\mathcal{X}})$. We have the following Čech resolution:

$$0 \longrightarrow \Gamma(V_I, \pi_*\mathcal{L}_{\mathcal{X}}) \longrightarrow \bigoplus_{\mu} \Gamma(U^I_\mu, \mathcal{L}_{\mathcal{X}}) \xrightarrow{\partial^0} \bigoplus_{\mu \prec \nu} \Gamma(U^I_\mu \cap U^I_\nu, \mathcal{L}_{\mathcal{X}}) \longrightarrow \cdots, \tag{6.1}$$

where μ, ν belong to an ordered set $\{\gamma_1 \prec \cdots \prec \gamma_{2g} \prec x_{i_1} \prec \cdots \prec x_{i_n}\}$ and

$$U^I_{\gamma_i} = \Big\{(t, x) \in \overline{X} \times V_I \,\Big|\, t \notin \Big(\bigcup_{j \neq i} \gamma_j \cup \bigcup_{k=1}^{n} [\widetilde{x_0}, x_k]\Big)\Big\},$$

$$U^I_{x_k} = \Big\{(t, x) \in \overline{X} \times V_I \,\Big|\, t \notin \Big(\bigcup_{i=1}^{2g} \gamma_i \cup \bigcup_{j \neq k} [\widetilde{x_0}, x_j]\Big)\Big\}.$$

Let U^I be an open set given by

$$\Big\{(t, x) \in \overline{X} \times V^I \,\Big|\, t \notin \Big(\bigcup_{i=1}^{2g} \gamma_j \cup \bigcup_{k=1}^{n} [\widetilde{x_0}, x_k]\Big)\Big\}.$$

Note that $U^I_\mu \cap U^I_\nu$ coincides with U^I.

LEMMA 6.1

Let o be a point in U^I. The fundamental group $\pi_1(U^I_\mu, o)$ is a free group generated by one element σ_μ, and U^I is contractible.

Proof

We have the chain of smooth surjective morphisms

$$V_I^{(1)} \longleftarrow V_I^{(2)} \longleftarrow \cdots \longleftarrow V_I^{(n)} = V_I \longleftarrow U^I_\mu,$$

where $V_I^{(r)}$ is the image of V_I under the projection $\overline{X}^n \longrightarrow \overline{X}^r$. Each fiber of the above surjective morphisms is contractible except for the rightmost one. The fiber of $V_I \longleftarrow U^I_\mu$ is homeomorphic to U_μ in Section 5. Thus $\pi_1(U^I_\mu, o)$ is isomorphic to the fundamental group of U_μ, which is a free group generated by one element. The fiber of $V_I \longleftarrow U^I$, the restriction of $V_I \longleftarrow U^I_\mu$ to U^I, is also contractible. Thus U^I is contractible. □

Due to Lemma 6.1, we can take N (single-valued) sections $s^I_1, \dots, s^I_N$ of $\mathcal{L}_{\mathcal{X}}$ over U^I. Thus $\Gamma(U^I_\mu \cap U^I_\nu, \mathcal{L}_{\mathcal{X}})$ is generated by $s^I_1, \dots, s^I_N$. And we take the generators σ_μ of $\pi_1(U^I_\mu, o)$. Now we have the following.

PROPOSITION 6.1

We assume that the eigenvalues of the monodromy action of $\sigma_\mu \in \pi_1(U_\mu^I, o)$ on the stalk $\mathcal{L}_{\mathcal{X},o}$ do not contain 1. The first cohomology of the Čech complex corresponding to (6.1) has a basis consisting of the $N(n+2g-1)$-cocycles $e_\mu^I \otimes s_k^I$ ($\mu \in \{\gamma_1, \ldots, \gamma_{2g}, x_{i_1}, \ldots, x_{i_{n-1}}\}$, $k = 1, \ldots, N$) defined by

$$e_\mu^I \otimes s_k^I := (e_{\nu\lambda}^{(\mu)} s_k^I)_{\nu\lambda}, \quad e_{\lambda x_{i_n}}^{I,(\mu)} = \delta_{\mu\lambda}, e_{\nu\lambda}^{I,(\mu)} = -e_{\lambda x_{i_n}}^{I,(\mu)} + e_{\nu x_{i_n}}^{I,(\mu)}.$$

Note that we have the following commutative diagram:

$$\begin{tikzcd} \mathcal{V}_{\mathcal{X}} \arrow[r, "\Phi_{\mathcal{X}}", "\cong"'] \arrow[d, "\nabla"] & \mathcal{L}_{\mathcal{X}} \otimes_{\mathbb{C}_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}} \arrow[d, "1\otimes d"] \\ \mathcal{V}_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{X}}} \Omega^1_{\mathcal{X}/S} \arrow[r, "\Phi_{\mathcal{X}}", "\cong"'] & \mathcal{L}_{\mathcal{X}} \otimes_{\mathbb{C}_{\mathcal{X}}} \Omega^1_{\mathcal{X}/S} \end{tikzcd}$$

where $\Phi_{\mathcal{X}}^{-1}(s \otimes h) = sh$. Put $\mathcal{L}_{\mathcal{X}}^\vee := \mathcal{H}om_{\mathbb{C}_{\mathcal{X}}}(\mathcal{L}_{\mathcal{X}}, \mathbb{C}_{\mathcal{X}})$.

COROLLARY 6.1

We assume that the eigenvalues of the monodromy action M_{σ_μ} on the stalk $\mathcal{L}_{\mathcal{X},o}^\vee$ do not contain 1. Let $\{s_{1,o}^I, \ldots, s_{N,o}^I\}$ be the germs of $\{s_1^I, \ldots, s_N^I\}$ at o, and let $\{s_{1,o}^{I\vee}, \ldots, s_{N,o}^{I\vee}\}$ be its dual basis. For a path γ with its initial point at o, let $s_{i,\gamma}^{I\vee}$ be the analytic continuation of $s_{i,o}^{I\vee}$ along γ. Then, the isomorphism $\Psi_{\mathcal{X}} : \mathcal{H}^1 \longrightarrow \check{\mathcal{H}}^1$ is given by

$$\Psi_{\mathcal{X}}(\eta) = \sum_{\mu,k} \Big(\int_{\mathrm{reg}_{x_{i_n}\mu}\, s_k^{I\vee}} (s_k^{I\vee} \otimes 1)\big(\Phi_{\mathcal{X}}(\eta)\big) \Big) e_\mu^I \otimes s_k^I.$$

REMARK 5

This fact implies that solutions of the differential equations corresponding to a *Gauss-Manin* connection, that is, the induced connection on relative de Rham cohomology, have integral representations of Euler type. This conforms with the following fact: De Rham cohomology classes are represented by differential forms, which should be integrated; there is a nondegenerate paring between $\mathcal{H}^1$ and $\mathcal{H}_1 := \bigcup_{x \in S} H_1(X_x, \mathcal{L}_{\mathcal{X}}|_{X_x})$ whose basis is given by the regularizations of paths.

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