Comment

James O. Berger

This is a very useful paper. The area of axiomatic approaches to probability appears as something of a maze to the outside observer, and this paper is a highly illuminating guide through the maze. My comments will be directed toward the "axiom reality" interface, in particular toward discussion of what the axiomatic approaches have to say about statistical practice and, conversely, what reality has to say about axiomatics.

The major impact of probability axiomatics on statistical practice is, of course, to reaffirm the central role that probabilistic reasoning should have in dealing with uncertainty. It is somewhat surprising that the centrality of this role is still at issue today, but probabilistic processing of uncertainty is under assault from many quarters. Alternatives, such as "fuzzy set" and logic-based reasoning, are heavily promoted in some circles. The importance of the axiomatics discussed in the paper is that they forcefully indicate that any such nonprobabilistic mode of reasoning must violate some "common sense" principle. It would be nice if any proposer of nonprobabilistic uncertainty reasoning would be required to publically announce which axioms he is rejecting; we would all be saved a good deal of nonsense.

Reality can impact on axiomatics by providing guidance concerning choice among, or limitations of, axioms. This use of reality is often misapplied by arguing, say, that since people do not necessarily intuitively obey the axioms in empirical studies, then these axioms must be wrong (cf. some of the examples reported in Section 3). A succinct refutation of such arguments is in Smith (1984), who states "It is rather like arguing against the continued use of formal logic or arithmetic on the grounds that individuals can be shown to perform badly at deduction or long division in suitable experiments." The point is that, when the axioms appear to clash with real behavior, one's first reaction should be to attempt to change real behavior for the better.

A clash cannot always be resolved in favor of the axioms, however. For instance, any of the axiom systems which lead to a unique probability distribution clash rather glaringly with reality. Suppose we are interested in dealing with the event that it will rain tomorrow. A number of axiom systems would imply that there is a unique (subjective) probability of rain which one should seek to elicit. But as Good (1980) says, "For it would be only a joke if you were to say that the probability of rain tomorrow (however sharply defined) is 0.3057876289." The point is that the infinitely fine comparisons needed to arrive at a unique probability (with infinite precision) are an impossibility in reality. One might argue that the unique probability exists and that we can, through elicitation methods, get arbitrarily close to this probability, but the argument strikes me as fallacious. Even with careful training, probability elicitation is unlikely to ever be more than a very inexact measurement process, and it can be very important to formally recognize this inexactness. Thus, the "real" outcome of the elicitation effort concerning rain might be that the probability of rain is between 0.27 and 0.33.

Tracing back through the axioms, one finds that the essential impact of this understanding is to admit that events may be noncomparable (at least in the appropriate enriched problem needed to argue for uniqueness of probabilities). This is actually comforting, in that there is no need to argue against any of the more intuitively appealing axioms. (Note that, as emphasized in the paper, "noncomparable" is distinct from "comparative equiprobable," and one would not necessarily expect any axioms, such as transitivity, to hold for noncomparable events.)

Methods of dealing with this inexactness in probability elicitation (or noncomparability) include use of upper and lower probability functions, as discussed in the paper. Again, however, I would resist such new constructs because they are not probability functions, and any general system for manipulating them is almost sure to include operations which may violate the desirable probability axioms. The obvious solution is to stick with probabilities, and to simultaneously consider all probabilities (or probability distributions) which are compatible with the elicited probability intervals (or compatible with the event comparisons that can be made). Ensuing probability manipulations are then carried out individually on each element of this collection, leading to a collection of "answers." Hopefully, or course, these answers are all similar enough for an overall conclusion to be reached. If not, one is alerted to the necessity to attempt more careful probability elicitation. The attractive features of this approach are that it involves information combination

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and processing only via probabilistic means (e.g., Bayes theorem), while explicitly recognizing the inexactness of probability elicitation. This approach, long advocated by I. J. Good (cf. Good, 1983), is reviewed and discussed (as the "robust Bayesian" viewpoint) in Berger (1984, 1985). Of particular note, in terms of axiomatics, is that Smith (1961), Good (1962), Giron and Rios (1980), and others show that possible noncomparability, together with a reasonable set of other axioms, essentially yield the robust Bayesian approach.

As a second example of how "reality" might impact on axiomatics, consider the issue of finitely additive versus countably additive probabilities. Axiomatically, additional assumptions must be made to guarantee countably additive probabilities, assumptions which tend to be somewhat obscure and noncompelling. Attempts to work with finitely additive probabilities, however, encounter the difficulty that conditional distributions (or posterior probabilities) are often not well-defined, so that additional assumptions end up being needed anyway. And the nature of these assumptions is perhaps even more obscure than those leading to countable additivity; one might well conclude that the countably additive domain is the least objectionable arena in which to perform.

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Comment

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My remarks focus on the themes of extension, tolerance for limited precision, the restricted applicability of the familiar concept of numerical probability, and the possibilities for other concepts of probability that are suggested by the axiomatic measurement-theoretic approach to comparative probability. Dr. Fishburn provides us with an authoritative survey of several axiom systems for binary relations of comparative (qualitative) probability that have been developed in the context of an interpretation of subjective probability based upon the degrees of belief of an individual. One might hope that a study of such axiom systems for comparative probability would lead us closer to the conceptual issues and roots of probabilistic reasoning and rational beliefs about uncertainty and thereby also enable us to develop such reasoning processes and model such beliefs through a probability-like mathematical structure. A process of axiomatization enables us to decompose a complex issue into a related set of simpler component issues that can then be examined closely on their merits. When properly engaged in, such a study does not prejudge its outcome. By observing the nature and strength of the axioms necessary to insure that the resulting model is a finitely or countably additive numerical probability measure, we can gain insight into the limitations of this familiar and often reliable model. By eliminating those axioms that appear to be objectionable in particular application domains, we can develop alternative concepts of probability useful for fairly representing probabilistic reasoning about either individual beliefs or objective nondeterministic phenomena, as appropriate for the domain. Clearly, the process of axiom selection must be guided by sound interpretations of the probability concept.

Regrettably, but understandably, few of these issues are addressed with sufficient emphasis either in this survey or in much of the related literature cited therein. While the opening quotations might lead us to anticipate an analysis of the link between belief or expectation (on the subjective interpretation) and the mathematical apparatus that is then deployed, this is

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