Comment

I. J. Good

For anyone who wishes to delve into the intricacies of the axiomatics of subjective probability, Fishburn's survey would be a fine up-to-date place to start. But in spite of this survey, it remains difficult to obtain an overall view of the extensive literature. It is appropriate that work should be done in so fundamental a part of human reasoning, but my own taste is to adopt as simple a theory as possible to which I can see no serious objection. In doing so it seems necessary unfortunately to concede that the appropriate theory depends on the application. But if one has more than one theory it is advisable that they should supplement rather than contradict one another. My contribution to the discussion will be to explain how this can come about.

Like Fishburn my comments do not require that the reader has much background. So, I first state, somewhat too briefly, the theory of subjective (personal) probability that I adopt (Good, 1950). It is a theory of upper and lower (interval-valued or partially ordered) probabilities, but it begins with a set of axioms of numerical conditional probabilities that appear at first sight to contradict this description. The axioms are

A1 $P(E \mid H)$ is a nonnegative real number.
A2 If $P(E \cdot F \mid H) = 0$, then $P(E \vee F \mid H = P(E \mid H) + P(F \mid H)$.
A3 $P(E \cdot F \mid H) = P(E \mid H) \cdot P(F \mid E.H)$.
A4 If E and F are logically equivalent (i.e., if they imply one another) then $P(E \mid H) = P(F \mid H)$ and $P(H \mid E) = P(H \mid F)$ for any H.
A5 $P(H^* \mid H^*) \neq 0$.
A6 $P(E^* \mid H^*) = 0$ for some proposition E^*.

Here H^* denotes the "usual assumptions of logic and pure mathematics." These axioms are "abstract" in the sense of pure mathematics, that is, *by themselves* they say nothing about degrees of belief. When we wish to talk about comparisons of degrees of belief we can use such notation as $P'(A \mid B) > P'(C \mid D)$, where P' does *not* denote a numerical function. The inequality means that one degree of conviction or belief exceeds another one.

The main rule of application is merely that if $P'(A \mid B) > P'(C \mid D)$ then $P(A \mid B) > P(C \mid D)$ (input to the "black box") and conversely (output from the black box).

We assume that a perfectly rational entity has a body of beliefs which, when combined with the axioms, do not lead to a contradiction.

For the sake of simplicity one can assume in some discussions that all degrees of belief are sharp. Landmarks in the scale can be introduced by imagining perfect packs of cards perfectly shuffled or perfect roulette wheels (Good, 1950, pages 15, 16, and 34). This provides a dense set of numerical probabilities so that any real number between 0 and 1 can be a probability defined by means of a Dedekind section.

In many contexts, the prime on P' can be dropped as an abbreviation, and the ambiguity need cause no confusion.

The partially ordered theory is consistent with the sharp theory and we can choose which to use on a given occasion. The sharp theory is simpler but less realistic, and the advantages of simplicity often outweigh the lack of complete realism.

The theory can be used to produce an axiom set for upper and lower probabilities as in Good (1962).

Judgments for the input to the black box are made more flexible by introducing utilities and embedding the theory in one of rational behavior (for example, Good, 1952). Other forms of judgment are also possible such as those of "weights of evidence" (for example, Good, 1950, Chapter 6; 1985).

The way to apply the theory is summarized in 27 "Priggish Principles" by Good (1971). Judgments of probabilities can be changed without new empirical evidence. Thus, probabilities can be "dynamic" or "evolving"; see, for example, Good (1977). In a sense, therefore, there are acceptable inconsistencies in the application of the theory. But on a given occasion, or rather in a given document, there should be no inconsistency. Dynamic probability requires that A4 be replaced by (A4′): If you have seen that E and F are equivalent then $P(E \mid H) = P(F \mid H)$ and $P(H \mid E) = P(H \mid F)$ (Good, 1950, page 49).

Another (controversial) way to enlarge the area of discourse is to admit that there are "physical" probabilities or "propensities" in addition to subjective probabilities (Poisson, 1837; Carnap, 1950; Good, 1959, 1985). Then we can assume subjective probability distributions for these physical probabilities. This gives more flexibility for enlarging our body of beliefs. Thus there are a few apparently different theories but no real conflict between them.

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Regarding the justification of the usual "sharp" axioms, a convenient reference is Cox (1946), a paper that was unjustly reviewed only by title in *Mathematical Reviews*, and was overlooked by most probabilists until it was essentially reprinted in Cox (1961, pages 1–24).

I'd like to comment concerning Fishburn's discussion of transitivity. It seems intuitively clear to me that if you prefer A to B and B to C, then you *should* rationally prefer A to C. The example concerning Sue's intransitivity seems to me to show that she simply made a mistake, one that, if pointed out to her, should make her reconsider her judgments unless either she is obstinate or, owing to shortage of time, she prefers to live with inconsistency. It may be useful for theoretical psychology, and practical economics, to find axioms that describe actual behavior, but my interest has been in a normative theory.

The example where $A \sim C$, $C \sim B$, and $A > B$ requires more discussion. It is analogous to a situation where A, B, and C are three points on a line, A and C being too close to distinguish, and similarly C and B; but A just far enough from B to be distinguished. The situation is like the one discussed by Good and Tideman (1981). A man in a restaurant can't at first decide between steak and chicken. He then thinks to himself that he would prefer steak to lobster (which wasn't in fact on the menu), but would not be able to *perceive* that chicken is better than lobster. From this he deduces that the utility of chicken lies between those of lobster and steak. Symbolically $U(\text{steak}) > U(\text{chicken}) > U(\text{lobster})$. We described this situation by saying that steak is *discernibly* better than chicken but not *perceptibly* better.

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Comment

Patrick Suppes

Peter Fishburn has provided an excellent, well-documented survey of the substantial literature on the axiomatic foundations of subjective probability. In my comments it is my purpose not to criticize Fishburn's survey but rather to raise some conceptual questions about the literature itself. I hope thereby to stress the importance of some problems that have received little emphasis in the literature, and consequently are scarcely mentioned in Fishburn's survey, but that are fundamental from a foundational standpoint.

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PROBLEM OF UNIQUENESS

In Section 2 of his article, Fishburn brings out the following well-known fact. The known necessary and sufficient conditions for the existence of a probability measure agreeing with the qualitative ordering, given that the algebra of events is finite, do not establish uniqueness of the measure, if no extensions to some additional sort of infinite structure are provided. The point I want to emphasize is what seems to be the fundamental character of the results here. For a given finite algebra and a given ordering, it is of course possible to write down conditions that are necessary and sufficient for existence of a unique measure, but there do not seem to be any very interesting general