Comment

D. V. Hinkley and S. Wang

Professor Reid provides a stimulating review of the theory and application of saddlepoint methods in parametric statistical analysis. As is indicated in Sections 6.4 and 6.6, similar approximations can be applied to certain nonparametric statistical calculations. Robinson (1982) applies the saddlepoint technique to obtain approximations to permutation distributions, and more recently Davison and Hinkley (1988) have applied saddlepoint approximations to several bootstrap and randomization problems. Great numerical accuracy is evident in most of these applications. The corresponding theoretical development, which requires some delicacy, is contained in Wang's Ph.D. dissertation for statistics which are sums of random variables. We should like to summarize and illustrate some of the results for a simple bootstrap problem here.

Let $(X_1, \ldots, X_n)$ be independently sampled from the continuous distribution function F whose mean is $\mu = E(X_1)$. Suppose that we wish to calculate the cumulative distribution function (CDF) G of the estimation error $D = \bar{X} - \mu$, where $\bar{X} = n^{-1} \sum X_i$. If F is known, and if the cumulant generating function $K(t) = \log\{\int_{-\infty}^{\infty} e^{t(x-\mu)}\, dF(x)\}$ exists in a neighborhood of $t = 0$ and is calculable, then a saddlepoint formula will give a very accurate approximation to G (see Section 6.3).

But suppose that F is completely unknown. The bootstrap approach (Efron and Tibshirani, 1986) is to calculate G with the empirical CDF $\tilde{F}$ in place of F. That is, one estimates G by $\tilde{G}$, the CDF of $\bar{X}^* - \bar{x}$ when $\bar{X}^*$ is the average of $(X_1^*, \ldots, X_n^*)$ which are sampled randomly with replacement from the fixed, observed set $(x_1, \ldots, x_n)$. A standard implementation of the bootstrap would approximate $\tilde{G}$ by Monte Carlo methods, e.g., by direct simulation of hundreds of samples $(X_1^*, \ldots, X_n^*)$ and calculation of empirical cumulative frequencies for $\bar{X} - \bar{x}$. Saddlepoint methods offer an alternative, efficient approach to approximation to $\tilde{G}$.

In principle some care is needed here because $\tilde{F}$, and hence $\tilde{G}$, are discrete, and slightly different saddlepoint formulas apply in discrete cases. Suppose that the x_i's are given to m decimal places, so that $\bar{X}^*$ is a multiple of $n^{-1}10^{-m}$. Define

$$\text{(A1)} \quad \begin{aligned} \tilde{K}(t) &= \log\left\{\int_{-\infty}^{\infty} e^{t(x-\bar{x})}\, d\tilde{F}(x)\right\} \\ &= \log\left[n^{-1} \sum \exp\{t(x_i - \bar{x})\}\right]. \end{aligned}$$

Then the saddlepoint approximation to $\tilde{G}(d) = \Pr(\bar{X}^* - \bar{x} \le d \mid \tilde{F})$ when d is a multiple of $n^{-1}10^{-m}$ is, corresponding to Reid's equation (28),

$$\text{(A2)} \quad \tilde{G}_s(d) = \begin{cases} \Phi(w) - \phi(w)\,[10^{-m}\{1 - e^{10^{-m}T}\}^{-1} \cdot \{n\tilde{K}''(T)\}^{-1/2} - w^{-1}], & d_1 \neq 0, \\ \tfrac{1}{2} + \tfrac{1}{6}(2\pi n)^{-1/2}\{\tilde{K}''(0)\}^{-3/2}\tilde{K}'''(0) - \tfrac{1}{2}10^{-m}\{2\pi n\tilde{K}''(0)\}^{-1/2}, & d_1 = 0, \end{cases}$$

where $d_1 = d + n^{-1}10^{-m}$, $\tilde{K}'(T) = d_1$ and

$$w = [2n\{Td_1 - \tilde{K}(d_1)\}]^{1/2}\operatorname{sgn}(T).$$

Wang has proved that

$$\text{(A3)} \quad \tilde{G}(d) = \tilde{G}_s(d)\{1 + O_p(n^{-1})\},$$

but that the relative error is *not* strictly uniform in the tails for fixed n. In this latter sense the saddlepoint approximation is not as strong as usual, although in practice this seems unimportant.

Recall that $\tilde{G}$ is itself intended to be an approximation, to the continuous CDF G. For this purpose it may be sensible to modify (A2) with a continuity correction, i.e., to approximate G by

$$\text{(A4)} \quad \tilde{G}_1(d) = \tilde{G}_s(d - \tfrac{1}{2}n^{-1}10^{-m}).$$

Note that $\tilde{G}_s$ is continuous.

A somewhat more casual approach is to ignore the discreteness, and to apply Reid's (28) with $\tilde{K}$ as in (A1) replacing K. We denote the result by $\tilde{G}_2$. In fact, as the following numerical example shows, there will often be negligible differences among $\tilde{G}_s$, $\tilde{G}_1$ and $\tilde{G}_2$.

The numerical example involves the sample of $n = 10$ numbers, with $m = 1$,

9.6 10.4 13.0 15.0 16.6 17.2 17.3 21.8 24.0 33.8.

Approximate percentage points for $\bar{X}^* - \bar{x}$ have been calculated using $\tilde{G}_s$, $\tilde{G}_1$ and $\tilde{G}_2$. Some of the results are

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TABLE A1
Approximations to bootstrap percentage points of $\bar{X} - \mu$ based on a sample of $n = 10$ numbers with 1 d.p.

Probability	"Exact"	Via saddlepoint approximation		Normal approx.
		$\tilde{G}_s$	$\tilde{G}_1$	
0.0001	−6.34	−6.32	−6.3130	−8.46
0.0005	−5.79	−5.79	−5.7842	−7.48
0.001	−5.55	−5.53	−5.4223	−7.03
0.005	−4.81	−4.81	−4.8051	−5.86
0.01	−4.42	−4.44	−4.4331	−5.29
0.05	−3.34	−3.33	−3.3296	−3.74
0.10	−2.69	−2.69	−2.6863	−2.91
0.20	−1.86	−1.86	−1.8556	−1.91
0.80	1.80	1.79	1.7956	1.91
0.90	2.87	2.85	2.8516	2.91
0.95	3.73	3.74	3.7480	3.74
0.99	5.47	5.47	5.4765	5.29
0.995	6.12	6.12	6.1212	5.86
0.999	7.52	7.46	7.4634	7.03
0.9995	8.19	7.98	7.9889	7.48
0.9999	9.33	9.11	9.1145	8.46

compared in Table A1, which includes also the "exact" results obtained by Monte Carlo with 50,000 simulated samples, as well as normal approximation results obtained with the correct mean and variance for D. Calculations using $\tilde{G}_2$ never differ from those using $\tilde{G}_1$ by more than 3×10^{-4}.

The success of the saddlepoint approximation in this example extends to many bootstrap and permutation distributions, so long as we restrict ourselves to problems involving sums as in Daniels's papers. We are unaware of comparable saddlepoint approximations for general nonlinear statistics. For practical purposes the key result would be the analog of Reid's (28) for statistics T_n of the form

$$T_n = \theta + n^{-1} \sum a_j(X_j) + n^{-2} \sum\sum b_{jk}(X_j, X_k),$$

because many statistics are very well approximated by such an expression. The relevant approximation would apply, for example, to the bootstrap distributions of studentized linear estimates.

ADDITIONAL REFERENCES

DAVISON, A. C. and HINKLEY, D. V. (1988). Saddlepoint approximations in resampling methods. *Biometrika.* To appear.

EFRON, B. and TIBSHIRANI, R. (1986). Bootstrap methods for standard errors, confidence intervals and other measures of statistical accuracy (with discussion). *Statist. Sci.* **1** 54–77.

Comment

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Professor Reid's paper is an excellent review of the use of saddlepoint methods in statistics. In this comment I would merely like to expand briefly on Professor Reid's discussion of the relation between saddlepoint approximations for sampling distributions and approximations to posterior moments and marginal densities based on Laplace's method.

As described, for example, in De Bruijn (1970) the basic Laplace method and saddlepoint method both involve approximating a number a_n defined as $a_n = \int f_n(y)\,dy$ for some function f_n when n is large. For Laplace's method the function and its arguments are real, for the saddlepoint method the function is complex and the integral is over a path in the complex plane. In both cases it is assumed that the behavior of the integral is determined by the behavior of f_n in the neighborhood of a particular point y_n. For Laplace's method y_n is a local maximum, for the saddlepoint approximation it is a saddlepoint.

The statistical applications of the saddlepoint approximation discussed by Professor Reid add some new features. Rather than approximate a single number a_n these methods approximate a density function $g_n(x)$ given as $g_n(x) = \int f_n(x, y)\,dy$. As n increases the density $g_n(x)$ becomes concentrated about some point x_n at rate $n^{-1/2}$. The point x_n represents the mean or some other measure of the center of the distribution with density $g_n(x)$. For each value of the argument of the density $g_n(x)$ the saddlepoint approximation involves the determination of the corresponding saddlepoint $y_n = y_n(x)$ of the function $f_n(x, \cdot)$.

This use of the saddlepoint approximation closely resembles the use of Laplace's method for computing approximate marginal posterior densities as described in Leonard (1982) and Tierney and Kadane (1986). In

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