Comment

Evarist Giné and Joel Zinn

Since Dudley's influential paper of 1978 the theory of empirical processes has undergone a vigorous development. David Pollard and his collaborators, among others, have applied some of these developments in asymptotic statistics. However, probably due to the technical character of this theory, applications are slow in coming. The present article will certainly help to make the subject better known to potential users.

We have no criticism to offer on this interesting paper. Instead, we take it as a basis for a digression both on points of view and aspects of empirical process theory that we have found useful in our work.

In the present article, Pollard describes how to obtain maximal inequalities for Gaussian and related processes using the "chaining method" associated to metric entropy. It is important to highlight this subject as Pollard has done, because, directly or indirectly, it is at the core of most of the progress made on empirical processes since 1978. Closely connected to this subject is Talagrand's (1987a) landmark work characterizing sample boundedness and continuity of Gaussian processes by means of properties of their covariances. These properties are the so called majorizing measure conditions which, like metric entropy, are conditions on the size of the index set for the Gaussian pseudo-distance. Actually these are the minimal conditions under which a chaining proof quite similar to the one here can still be carried out (see, e.g., Remark 2.6 in Andersen, Giné, Ossiander and Zinn, 1988). Rhee and Talagrand (1988) show how this more refined chaining method can be of practical interest. They obtain a precise maximal inequality for an empirical process in a concrete situation with implications in bin packing by constructing the appropriate majorizing measure. More applications of majorizing measures to empirical processes, in connection with bracketing, can be found in Andersen, Giné, Ossiander and Zinn (1988).

Given the wealth of results available for Gaussian processes, notably deviation and concentration inequalities, integrability, comparison theorems, and, of course, the already mentioned maximal inequalities, direct and converse (for references see, e.g., Pisier, 1986; Giné and Zinn, 1986), it is sometimes effective to hypothesize Gaussian properties for $\mathscr{F}$ either instead of, or in conjunction with, more analytical conditions such as entropy. As a very naive instance, here is a Gaussian definition of manageable class weaker than Definition 4.3 and which does essentially the same job. Let $\mathscr{F}$ be a class of functions with envelope F. For $Q = \sum \alpha_i \delta_{s_i} \in \mathscr{P}_f(S)$, the set of probability measures on S with finite support, define the Gaussian process $W_Q(f) = \sum \alpha_i^{1/2} g_i f(s_i)/(QF^2)^{1/2}$, where g_i are i.i.d. $N(0, 1)$, and let $W_Q(f, g) = [E(W_Q(f) - W_Q(g))^2]^{1/2}$. Then say that $\mathscr{F}$ is manageable if

(i) $\sup_{Q \in \mathscr{P}_f(S)} E \| W_Q \|_{\mathscr{F}} < \infty$ and
(ii) $\lim_{\delta \to 0} \sup_{Q \in \mathscr{P}_f(S)} E \| W_Q \|_{\mathscr{F}'(\delta, W_Q)} = 0$

where $\mathscr{F}'(\delta, W_Q) = \{f - g: f, g \in \mathscr{F}, W_Q(f, g) \le \delta\}$. Stability properties and results similar to Theorems 4.5 and 4.7 still hold for such classes. For instance, a proof of (a weaker form of) Corollary 4.6 goes as follows: using property (i) together with symmetrization and Jensen's inequality as in the text, we have

$$E \| \nu_n \|_{\mathscr{F}} \le 2E \left\| n^{-1/2} \sum_{i=1}^n \sigma_i f(X_i) \right\|_{\mathscr{F}}$$
$$\le \sqrt{2\pi}\, E \left\| n^{-1/2} \sum_{i=1}^n g_i f(X_i) \right\|_{\mathscr{F}}$$
$$= \sqrt{2\pi}\, E[(P_n F^2)^{1/2} E_g \| W_{P_n} \|_{\mathscr{F}}] \le C(PF^2)^{1/2}.$$

The proof of Theorem 4.7 would use (ii) and a comparison theorem of Fernique (1985). If $\mathscr{F} = \{f_n: \|f_n\|_\infty = o(1/(\log n)^{1/2})\}$ then $\mathscr{F}$ is manageable in this weaker sense but not necessarily in the sense of Definition 4.3. For more details on these classes of functions, see Giné and Zinn (1989).

In the applications presented by Pollard, error terms in Taylor expansions are controlled by the size of $\| \nu_n \|_{\mathscr{F}}$, the sup of the empirical process over a class of functions $\mathscr{F}$, and therefore probability inequalities for $\| \nu_n \|_{\mathscr{F}}$, i.e., maximal inequalities, yield the desired results. Other types of possible applications of empirical processes would relate to the construction of asymptotic confidence regions and tests of hypotheses based on the statistics $\| P_n - P \|_{\mathscr{F}} = n^{-1/2} \| \nu_n \|_{\mathscr{F}}$. These would require knowledge of the limiting distribution of $\| \nu_n \|_{\mathscr{F}}$ or, in general, of the limiting law of

Evarist Giné is Professor of Mathematics at Texas A & M University. He is presently on leave at The College of Staten Island, City University of New York. His address is Department of Mathematics, The College of Staten Island, 130 Stuyvesant Place, Staten Island, New York 10301. Joel Zinn is Professor, Department of Mathematics, Texas A & M University, College Station, Texas 77843-3368.

the processes $\{\nu_n(f)): f \in \mathscr{F}\}$ regarded as random elements with values in the space $\ell^\infty(\mathscr{F})$ of bounded functions on $\mathscr{F}$. As indicated in Section 6 of Pollard's article, much effort has been devoted to proving limit theorems for ν_n. There are, however, some stumbling blocks for this type of application to materialize which are already encountered in the multidimensional Kolmogorov-Smirnov case; particularly, that the limiting Gaussian process G_P depends on the law P of the data which, after all, is unknown. The bootstrap is a way around this difficulty; see, e.g., Beran and Millar (1986) for confidence regions based on $\mathscr{F} = \{$the half spaces of $\mathbb{R}^d\}$. Let X^ω_{nj}, $j = 1, \cdots, n$, $n \in \mathbb{N}$, be iid with the law of the empirical measure P^ω_n, and set $\nu^\omega_n(f) = n^{-1/2} \sum_{j=1}^n [f(X^\omega_{nj}) - P^\omega_n(f)]$. It can be shown that, under measurability conditions, the processes $\{\nu^\omega_n\}$ converge weakly to G_P for almost every ω if and only if ν_n converges weakly to G_P and $PF^2 < \infty$ (actually $PF^2 < \infty$ is not required if convergence of ν^ω_n takes place not ω-a.s., but in ω-probability) (Giné and Zinn, 1988). Then the distribution of $\| G_P \|_{\mathscr{F}}$ is approximated by that of $\| \nu^\omega_n \|_{\mathscr{F}}$ ω-a.s. and the latter can be computed up to any degree of approximation by Monte Carlo methods. We should point out that in the proof of the bootstrap central limit theorem for empirical processes we use several results and techniques from Probability in Banach spaces other than chaining. Among the most useful ones are a lemma on Poissonization by Le Cam (1970) and an inequality by Hoffman-Jørgensen (1974) that is basic for integrability of sums of independent random vectors. For a larger list of useful results see the introduction in our paper.

Pollard (1981) introduced Rademacher randomization (i.e., considering $n^{-1} \sum_{i=1}^n \sigma_i \delta_{X_i}$ instead of $P_n - P$) in the context of empirical processes and since then symmetrization has played an important role in this theory. Jain and Marcus (1975), as well as Hoffmann-Jørgensen and Pisier (1976), motivated by Kahane's (1968) book, used it in the related subject of the CLT in Banach spaces. However Gaussian randomization (i.e., considering $n^{-1} \sum_{i=1}^n g_i \delta_{X_i}$ instead of $P_n - P$) took a little more time to come into the subject essentially because (a) it does not add to randomization with ± 1's for proving (the sufficiency part of) limit theorems since chaining works for sub-Gaussian processes equally well and (b) although it is obvious that $\| \sum_{i=1}^n g_i \delta_{X_i}/n^{1/2} \|_{\mathscr{F}}$ dominates $\| \sum_{i=1}^n \sigma_i \delta_{X_i}/n^{1/2} \|_{\mathscr{F}}$, the "almost" converse, which is due to Fernique and Pisier, is less obvious and was not published until 1984 (Giné and Zinn). It is in this converse direction that Gaussian randomization is essential: used in conjunction with strictly Gaussian theory (e.g., Sudakov's inequality), it allows one to prove that certain sufficient conditions for P to satisfy the CLT or the LLN uniformly in $\mathscr{F}$ are also necessary. A previous case in point is the proof in Marcus and Pisier (1981) that certain entropy conditions are necessary and sufficient for a.s. uniform convergence of random Fourier series, with Rademacher series used for sufficiency and Gaussian series for necessity.

Regarding history of recent research on empirical processes directly related to chaining, we would like to mention the works of Talagrand (1987b) and Ledoux and Talagrand (1989) where P-Donsker classes of functions $\mathscr{F}$ are characterized (up to measurability) in a random-geometric way: their decomposition of $\mathscr{F}$ into a Gaussian or L_2 part and a part where sign cancellation plays no role, which is based on chaining, captures in our view the essence of P-Donsker classes.

ADDITIONAL REFERENCES

BERAN, R. and MILLAR, P. W. (1986). Confidence sets for a multivariate distribution. *Ann. Statist.* **14** 431–443.

FERNIQUE, X. (1985). Sur la convergence étroite des mésures gaussiennes. *Z. Wahrsch. verw. Gebiete* **68** 331–336.

GINÉ, E. and ZINN, J. (1986). Lectures on the central limit theorem for empirical processes. *Probability and Banach Spaces. Lecture Notes in Math.* **1221** 50–113. Springer, New York.

GINÉ, E. and ZINN, J. (1988). Bootstrapping general empirical measures. *Ann. Probab.* To appear.

GINÉ, E. and ZINN, J. (1989). Gaussian characterization of uniform Donsker classes of functions. Preprint.

HOFFMANN-JØRGENSEN, J. (1974). Sums of independent Banach space valued random variables. *Studia Math.* **52** 159–186.

HOFFMANN-JØRGENSEN, J. and PISIER, G. (1976). The law of large numbers and the central limit theorem in Banach spaces. *Ann. Probab.* **4** 587–599.

KAHANE, J. P. (1968). *Some Random Series of Functions.* Heath, Lexington, Mass.

LE CAM, L. (1970). Remarques sur le théorème limite central dans les espaces localement convexes. In *Les Probabilités sur les Structures, Algebriques* 233–249. CNRS, Paris.

LEDOUX, M. and TALAGRAND, M. (1989). Comparison theorems, random geometry and some limit theorems for empirical processes. *Ann. Probab.* **17** 596–631.

PISIER, G. (1986). Probabilistic methods in the geometry of Banach spaces. *Probability and Analysis. Lecture Notes in Math.* **1206** 167–241. Springer, New York.

POLLARD, D. (1981). Limit theorems for empirical processes. *Z. Wahrsch. verw. Gebiete* **57** 181–195.

RHEE, W. T. and TALAGRAND, M. (1988). Exact bounds for the stochastic upward matching problem. *Trans. Amer. Math. Soc.* **307** 109–125.

TALAGRAND, M. (1987a). Regularity of Gaussian processses. *Acta Math.* **159** 99–149.

TALAGRAND, M. (1987b). Donsker classes and random geometry. *Ann. Probab.* **15** 1327–1338.