Comment

Clive W. J. Granger

There is a great deal to be admired in the extensive work on chaos that has appeared in recent years, including some startling but simple theorems, and also the best art work produced by mathematics. However, in my opinion, it is often surrounded by an unnecessary amount of hype, considerable zeal and possibly some illogical arguments and confusion.

To simplify this discussion, I will consider only series that are "white chaos" and compare them with iid series. A process will be called "white" if its (estimated) autocorrelations are all zero and thus the (estimated) spectrum is flat, with estimates based on a long realization of the process. White chaos is a deterministic process with these white properties. As an example, I will consider the process generated by

$$x_{t+1} = 4x_t(1 - x_t) \tag{1}$$

with starting value $x_0 = s$, s being the "seed" value. I will also assume that truly stochastic processes exist—an assumption that I think most scientists will accept with probability one. Thus, an iid series y_t exists, and such a series is also obviously white.

Let G_1, G_2 be a pair of generating mechanisms, producing series x_{1t}, x_{2t}; then, it is obviously possible that the two series will have some properties in common, such as zero means and identical (estimated) spectra. Many generating mechanisms can produce series having the white properties, as pointed out in Granger (1983). An example is the bilinear process generated by

$$y_t = \alpha y_{t-1}\varepsilon_{t-2} + \varepsilon_t, \tag{2}$$

where ε_t is zero mean iid. It is clearly possible for a (deterministic) white chaos to have many properties of an iid process. Statisticians are familiar with pseudo-random numbers (prn) generated on computers by a somewhat complex deterministic model. These numbers are chaos of "high dimension," as defined in the papers being discussed, or "space-filling." It is generally agreed that it would take an enormous amount of data—a sample size of billions—to distinguish prn from a true iid. The only questions then worth considering is how to distinguish between a low-dimensional white chaos and an iid series, and thus whether or not white chaos occurs in reality rather than in computer simulations or physics laboratory experiments.

The papers emphasize the similarities between white chaos and iid series, such as the similar appearance of their plots through time or the values taken by statistics such as autocorrelations. The fact that white chaos can look like iid, which can be restated as an iid series that looks like white chaos, has no implication. If two generating mechanisms produce series, each of which has some properties, P, it does not mean that the mechanisms are identical or similar. There is a danger of falling into the famous logical fallacy that says, "If A then B, observe B therefore A." An example would be, "If chaos (A) then positive Lyapunov exponent (B), if data has a positive Lyapunov exponent then it must be chaos," which is seen occasionally in chaos literature but is, of course, false because some stochastic processes, such as an AR(1) with the coefficient larger than one, also have positive Lyapunov exponents. It follows that this exponent cannot be used as a "popular measure of chaos" (Berliner, Section 3) without the added assumption that the process *is* chaos.

A similar problem arises with the interpretation of ergodicity. Let the proportion of time that a series lies in some region R asymptotically tend to a constant, for every R. This asymptotic proportion could be called the likelihood that the series is (eventually) in R. The fact that chaotic series have such likelihood is interesting but not especially surprising. If the series are also assumed, or known, to be stochastic, then these likelihoods can be called probabilities and interpreted in the usual frequency count manner. There is no philosophical problem in doing this. Without the assumption of stochasticity, the likelihood need not be called a probability and then no unnecessary confusion occurs. The likelihood can be put together for different sets R to derive a marginal "distribution" for the series. However, of much greater interest is the joint distribution of a set of adjacent values of the series, which can be derived in a similar manner. Consider a pair of random variables X, Y, with a joint distribution $f(x, y)$. They can be called "singular" if there exists some combination $X - g(Y)$,

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say, which is nonrandom, so that prob$(X - g(Y) = 0) = 1$. If one considers the joint distribution of x_{t+1}, x_t for x_t generated by (1), then their joint distribution will be singular. This is an enormously different property of a white chaos series compared with an iid series that will not be singular. Other chaotic processes will also be singular, possibly using several lags.

It is interesting to ask how white chaos and iid series differ. The similarities have concentrated on statistics that measure the *linear* properties of series, such as autocorrelations and spectra, so that the extent to which x_{t+1} can be explained linearly from past x_t is considered. Of course, much more could be learned from statistics that measure nonlinearity, such as the bispectrum, the maximum autocorrelation [the correlation between $g(x_{t+h})$, $h(x_t)$, where the functions g, h are chosen to maximize the correlation], rank autocorrelation or the "shadow autocorrelation"

$$r(x_{t+h}, x_t) = \left[1 - \exp\left(-2\sigma(x_{t+h}, x_t)^{1/2}\right)\right]$$

where σ is the relative entropy

$$\sigma(x, y) = \iint f(x, y)\log\left[\frac{f(x, y)}{f_1(x)f_2(y)}\right] dx\,dy,$$

$f(x, y)$ is the joint distribution and f_1, f_2 are the marginal distributions. This measure is discussed by Granger and Lin (1991) and has the property of being invariant to instantaneous transformations of x_{t+h}, x_t separately. A white chaos may not have similar properties as an iid process using such statistics. For example, corr$(x_{t+1}^2, x_t^2) = -0.221$, with x_t generated by (1), as shown in Liu, Granger and Heller (1991) by simulation.

Despite the comments in Chatterjee and Yilmaz, there are big differences in the forecasting of white chaos and iid processes. Neither are forecastable linearly, but white chaos is (virtually) perfectly forecastable nonlinearly in the short run, as seen in (1), whereas iid is never forecastable.

To achieve perfect forecastability, the generating map needs to be known, but given sufficient data from a noise-free, low-dimensional white chaos neural network techniques can often provide an excellent estimator of the map. Of course, a chaotic process using seed s is perfectly forecastable from a process previously generated using the same seed and generating mechanism. This would not be true for an iid process.

The question of forecastability in the log run is a little more complex. Because of round-off errors in computing, a white chaos is not actually generated by a map such as (1) but rather by

$$x_{t+1} = 4x_t^{(T)}\left(1 - x_t^{(T)}\right), \tag{3}$$

where $x_t^{(T)}$ is a truncated x_t. The difference between the actual and truncated processes gives a long-run divergence between future values and values predicted by (3), the Lyapunov exponent measuring this rate of divergence. Thus, in a sense, a white chaos is not forecastable, even nonlinearity, in the long run. However, it is known that eventually all future x values will lie on the (strange) attractor A, which will be of limited (and possibly fractional) dimension. Thus, likelihood (x_{t+h} on $A \mid x_t) = 1$ for h large, and this is a forecast with substantial implications that will not hold for a true iid process.

With the introduction of noise, the situation becomes very different. Consider a generating mechanism

$$x_{t+1} = f(x_t) + e_{t+1}, \tag{4}$$

$$y_t = x_t + \varepsilon_t, \tag{5}$$

where e_t, ε_t are independent zero-mean iid series. The noise e_t in (4) is inherent and will generally continue to affect future x's. In decision sciences, such as economics, e_t will affect current decisions and so be embedded in future values of economic variables. The noise in (5) can be thought of as measurement noise and does not affect current or future x_t. Suppose that $x_{t+1} = f(x_t)$ generates white chaos. If σ_e^2 = variance (e_t) is zero, then the observed series, y_t, is a mixture of chaos and iid. Techniques for distinguishing chaos from stochastic, such as those based on estimates of the correlation coefficient, as in Liu, Granger and Heller (1991), are likely to "see" the high-dimensional ε_t rather than the low-dimensional white chaos, unless $\sigma_\varepsilon^2 = \operatorname{var}\varepsilon$ is very small indeed. If σ_e^2 is not zero, one has a nonlinear AR(1) and not chaos, and in this case all the traditional nonlinear statistical techniques are appropriate, as discussed by Tong (1990) for the univariate case, and by Granger and Teräsvirta (1992) for multivariate situations. If $f(\ )$ were linear, then x_t could be decomposed into the solution of the deterministic equation and a stochastic component consisting of e_{t+1-j}, $j \geq 0$. Such a decomposition is not usually available for nonlinear models, although Lord (1979) has some results that may be relevant. If σ_e^2 is not zero or extremely small, it is unclear why chaos should even be considered. A further difficulty is that the series may be unstable if $\sigma_e^2 > 0$. For example, if an iid series is added to the right-hand side of (1) the resulting quadratic AR(1) will be explosive and will have none of the properties of chaos, other than positive Lyapunov exponent! Of course, the noise e_{t+1} could be embedded into the model other than linearly, but then one ends up with a completely different model. I think that intrinsic noise with $\sigma_e^2 > 0$ leads us immediately to a stochastic world, and if $\sigma_e^2 = 0$ but $\sigma_\varepsilon^2 \neq 0$ and not small, as is often the case in economics, distinguishing between iid and low-dimensional white chaos will be extremely difficult.

This leads to the question of whether the real world, such as an actual economy, contains chaos. Chatterjee and Yilmaz take the position that it is ubiquitous, finding examples in "such diverse fields as physiology, geology, ..., economics ..." and "theoretical models of population biology." There are also theoretical models in economics that produce chaos, but that does not imply that it occurs in practice. I would prefer to suggest the opposite view that there is *no* evidence of chaos outside of laboratories. My reason is that there exists no statistical test, that I know of, that has chaos as its null hypothesis. There are plenty of tests that have as a null H_0:iid (or linear) and are designed to have power against chaos. However, as is well known by all statisticians, if one rejects the null a specific alternative hypothesis cannot be accepted. If a null of linearity or iid is rejected, one cannot accept (white) chaos, as nonlinear stochastic models are also appropriate. For example, the test (based on the correlation dimension) by Brock, Dechert and Scheinkman (1987) (the BDS test) that was applied in Brock and Sayers (1988) often finds evidence of nonlinearity but not of chaos. Until a property P can be found that holds *only* for chaos and not for stochastic series, and a test is based on P with chaos as the null, can there be a suggestion that chaos is found in the real world.

Finally, I would suggest that bifurcation and fractional integrated models are irrelevant for the main topic discussed in the articles, but space limitations prevent me from expanding on this point.

In conclusion, I think that scientists working on the area of chaos are doing a disservice to this important area of research by overselling its relevance, by trying to equate it with randomness and by using concepts (such as probability) that are unnecessary and only lead to confusion. The techniques being developed for analysis of chaotic processes, such as the BDS test or estimates of the Lyapunov exponent, or methods of forecasting using $\sigma_e^2 = 0$, are potentially powerful and useful when applied to truly stochastic, real-world series. There is a need for statistical methods to investigate the properties of these techniques in this case, and this, in my opinion, is the natural link between chaos and statistics.

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Comment: Randomness in Complex Systems

David Griffeath

1. WHAT IS RANDOM?

Professors Berliner, Chatterjee and Yilmaz are to be commended for their thoughtful overviews of the recent explosion in experimental and theoretical research on chaos. They identify a host of challenging statistical questions fundamental to the subject and make timely appeals for the readership of *Statistical Science* to join the fray. Over the past decade, I have tried to track the major currents of chaos, studying many of the articles and books mentioned in the authors' fine reference list. I strongly urge others to peruse those sources and seek out a few.

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Berliner and Chatterjee and Yilmaz note that the term "chaos" is not used in a consistent manner by the scientific community; for example, there is no universally accepted mathematical definition. In my experience, the word means so many different things to different people that it threatens to become scientifically dangerous. Apparently, Bernoulli shift, the most basic stochastic process, is deemed chaotic. But how is it distinguished from those strange attractors, delicately perched on the boundary between order and randomness, that have dramatically captured the imagination of both scientists and the general public? The phenomenology of chaos is leaving its mark across a broad spectrum of contemporary culture: from physics to philosophy to recreational computing to textile design. At the hairdresser, I discovered an article in a summer issue of *Gentleman's Quarterly* linking mathematical chaos, Silicon Valley nerds and late-