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NONLINEAR OPERATORS AND SUFFICIENCY

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Let $(X, \mathscr{A})$ be a measurable space and $\mathscr{P} | \mathscr{A}$ a family of probability measures. Given a sufficient sub-σ-field, we define the sufficiency operator by assigning to each integrable function a conditional expectation which is independent of $P \in \mathscr{P}$. The operator thus defined is $\mathscr{P}$-a.e. linear, monotone, idempotent, and expectation invariant for every $P \in \mathscr{P}$. Not all of these properties are necessary to make such an operator useful in statistical theory: For the most important applications, linearity may be replaced by homogeneity and translation invariance; idempotency may be relinquished. It was therefore suggested (Pfanzagl (1967), page 416) to study homogeneous, translation invariant, monotone, and expectation invariant operators as a possibly useful generalization of sufficiency-operators.

The purpose of this paper is to show that there exists to any such operator a sufficient sub-σ-field whose sufficiency operator effects at least the same reduction. (Hence there is no real gain in introducing the idea of "reduction by homogeneous, translation invariant, monotone, and expectation invariant operators" into statistical theory.)

This result generalizes Proposition 9 of LeCam (1964), page 1435, where, roughly speaking, a similar result was obtained for monotone and expectation invariant operators which are linear (rather than homogeneous and translation invariant).

An ergodic lemma for homogeneous and translation invariant (but not necessarily linear) operators, needed here as a tool for the proof of the main theorem, may be of independent interest.

1. The main theorem. Let $(X, \mathscr{A})$ be a measurable space and $\mathscr{P} | \mathscr{A}$ a family of probability measures.

Let $\mathscr{F} \subset \bigcap_{P \in \mathscr{P}} \mathscr{L}_r(X, \mathscr{A}, P)$, $1 \leqq r \leqq \infty$, be a linear subspace containing 1 which is relatively closed in $\bigcap_{P \in \mathscr{P}} \mathscr{L}_r(X, \mathscr{A}, P)$ under the formation of countable pointwise infima (and therefore also under the formation of countable pointwise suprema).

Let $T: \mathscr{F} \to \mathscr{F}$ be an operator with the following properties:

homogeneous:	$Taf = aTf\ \mathscr{P}$-a.e. for all $a \in \mathbb{R}$, $f \in \mathscr{F}$.
translation invariant:	$T(1 + f) = 1 + Tf\ \mathscr{P}$-a.e. for all $f \in \mathscr{F}$.
monotone:	$f \leqq g\ \mathscr{P}$-a.e. implies $Tf \leqq Tg\ \mathscr{P}$-a.e. for all $f, g \in \mathscr{F}$.
expectation invariant:	$P(Tf) = P(f)$ for all $P \in \mathscr{P}$, $f \in \mathscr{F}$.

For notational convenience we shall drop the phrase "$\mathscr{P}$-a.e." whenever this is possible without danger of confusion.

Given a sub-σ-field $\mathscr{A}_0 \subset \mathscr{A}$ and $f \in \mathscr{L}_1(X, \mathscr{A}, P)$ let $P^{\mathscr{A}_0}f$ denote the P-equivalence class consisting of all conditional expectations of f relative to P,

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given $\mathscr{A}_0$. The sub-σ-field $\mathscr{A}_0$ is sufficient for $\mathscr{P}|\mathscr{A}$ if $\bigcap_{P\in\mathscr{P}} P^{\mathscr{A}_0}f \neq \emptyset$ for every $f \in \bigcap_{P\in\mathscr{P}} \mathscr{L}_1(X, \mathscr{A}, P)$.

For a sufficient sub-σ-field $\mathscr{A}_0$ a sufficiency operator is defined by assigning to each $f \in \bigcap_{P\in\mathscr{P}} \mathscr{L}_1(X, \mathscr{A}, P)$ an element of $\bigcap_{P\in\mathscr{P}} P^{\mathscr{A}_0}f$. Any sufficiency operator is $\mathscr{P}$-a.e. linear, monotone, idempotent, and expectation invariant for every $P \in \mathscr{P}$.

THEOREM. *For every homogeneous, translation invariant, monotone, and expectation invariant operator* $T: \mathscr{F} \to \mathscr{F}$ *there exists a sufficient sub-σ-field* $\mathscr{A}_0 \subset \mathscr{A}$ *such that any sufficiency operator* T_0 *transforms* $\mathscr{F}$ *into itself and fulfills* $T_0T = TT_0 = T_0$.

PROOF. (i) We define a sequence of operators $T_n: \mathscr{F} \to \mathscr{F}$ by

$$T_1 := I, \quad \text{the identity operator on } \mathscr{F}, \tag{1}$$

$$T_{n+1} := (I + nTT_n)/(n+1), \qquad n \in \mathbb{N}.$$

It is straightforward to show for every $n \in \mathbb{N}$ that T_n is homogeneous, translation invariant, monotone, and expectation invariant.

As any homogeneous and translation invariant operator is constant preserving, the sequence $(T_n f)_{n\in\mathbb{N}}$ is bounded for any bounded function f. For this reason we shall restrict ourselves at first to the set of all bounded functions in $\mathscr{F}$, say $\mathscr{F}_0$.

(ii) Now we define an operator $T_0 | \mathscr{F}_0$ by

$$T_0 f := \liminf_{n\in\mathbb{N}} T_n f, \qquad f \in \mathscr{F}_0.$$

It is straightforward to show that T_0 is positive homogeneous (i.e. homogeneous for constants $a \geqq 0$), translation invariant, and monotone.

From (1) we easily obtain

$$TT_n = \frac{n+1}{n} T_{n+1} - \frac{1}{n} I$$

which implies

$$TT_0 = T_0 \quad \text{on} \quad \mathscr{F}_0 \tag{2}$$

by expectation invariance. This relation, in turn, implies $T_nT_0 = T_0$ for every $n \in \mathbb{N}$, whence

$$T_0T_0 = T_0 \quad \text{on} \quad \mathscr{F}_0. \tag{3}$$

(iii) Now we shall prove that T_0 is expectation invariant and homogeneous. We have for $f \in \mathscr{F}_0$ and $P \in \mathscr{P}$

$$P[T_0 f] = P[\liminf_{n\in\mathbb{N}} T_n f] \leqq \liminf_{n\in\mathbb{N}} P[T_n f] = P[f],$$

as T_n is expectation invariant. Similarly, $P[T_0' f] \geqq P[f]$ for $T_0' f := \limsup_{n\in\mathbb{N}} T_n f$. As $T_0 f = T_0' f$ $\mathscr{P}$-a.e. by the ergodic lemma, we have $P[T_0 f] = P[T_0' f]$ and therefore $P[T_0 f] = P[f]$ for every $P \in \mathscr{P}$.

Finally, $T_0(-f) = \liminf_{n\in N} T_n(-f) = -\limsup_{n\in N} T_n f = -T_0' f = -T_0 f\ \mathscr{P}$-a.e. so that T_0 is homogeneous.

(iv) The results of (i)—(iii) together imply that $T_0 \mid \mathscr{F}_0$ is homogeneous, translation invariant, monotone, expectation invariant, and idempotent. By Pfanzagl (1967), Theorem 3, T_0 is the restriction to $\mathscr{F}_0$ of a conditional expectation relative to $P \in \mathscr{P}$, given the σ-field $\mathscr{A}_0 := \{A \in \mathscr{A}: T_0 1_A = 1_A\ \mathscr{P}\text{-a.e.}\}$. (The cited theorem in Pfanzagl (1967) is proved for the particular case $r = 1$. The proof for $1 < r \leqq \infty$ is, however, the same.)

(v) It remains to show that $T_0 T = T_0$ on $\mathscr{F}_0$. As both operators T and T_0 are homogeneous, translation invariant, monotone, and expectation invariant on $\mathscr{F}_0$, so is $T_0 T$. Furthermore, $T_0 T$ is idempotent, for

$$(T_0 T)(T_0 T) = T_0(T T_0)T = T_0 T_0 T = T_0 T .$$

Hence $T_0 T$ is the restriction to $\mathscr{F}_0$ of a conditional expectation relative to $P \in \mathscr{P}$, given the σ-field $\mathscr{A}_* := \{A \in \mathscr{A}: T_0 T 1_A = 1_A\ \mathscr{P}\text{-a.e.}\}$. As $T_0 T 1_A$ is $\mathscr{A}_0$-measurable, we have $\mathscr{A}_* \subset \mathscr{A}_0(\mathscr{P})$ (i.e. for every $A_* \in \mathscr{A}_*$ there exists an $A_0 \in \mathscr{A}_0$ such that $P((A_* \cap \bar{A}_0) \cup (\bar{A}_* \cap A_0)) = 0$ for every $P \in \mathscr{P}$). Conversely, let $A_0 \in \mathscr{A}_0$ be arbitrary. As $T_0 1_{A_0} = 1_{A_0}\ \mathscr{P}$-a.e., we have $T 1_{A_0} = T T_0 1_{A_0} = T_0 1_{A_0} = 1_{A_0}\ \mathscr{P}$-a.e. and therefore $T_0 T 1_{A_0} = T_0 1_{A_0} = 1_{A_0}\ \mathscr{P}$-a.e., so that $\mathscr{A}_0 \subset \mathscr{A}_*$. Hence $\mathscr{A}_0 = \mathscr{A}_*(\mathscr{P})$ and the two operators $T_0 T$ and T_0 are identical on $\mathscr{F}_0$.

(vi) It is a matter of routine to prove that the assertion, so far established on $\mathscr{F}_0$, holds true on $\mathscr{F}$.

The theorem should be compared with LeCam (1964), Proposition 9. If we neglect the fact that LeCam's proposition is formulated in a different conceptual framework, it yields the assertion of our theorem for linear and constant preserving (rather than homogeneous and translation invariant) operators.

We shall close this section by an example of a homogeneous, translation invariant, monotone, and expectation invariant operator which is not additive: Let $X = \{1, 2, 3\}$ and $P(\{1\}) = P(\{2\}) = P(\{3\}) = \frac{1}{3}$. Let $\mathscr{F}$ be the class of all real-valued functions on X. Any element of $\mathscr{F}$ may be represented by a point of $\mathbb{R}^3$. For $(a_1, a_2, a_3) \in \mathbb{R}^3$ we define $T(a_1, a_2, a_3) := (\frac{1}{2}((a_1 \wedge a_2 \wedge a_3) + (a_1 \vee a_2 \vee a_3)), (a_1 \wedge a_2) \vee (a_2 \wedge a_3) \vee (a_3 \wedge a_1), \frac{1}{2}((a_1 \wedge a_2 \wedge a_3) + (a_1 \vee a_2 \vee a_3)))$. It is straightforward to show that T has the asserted properties. T is, however, not additive, since $T(1, 0, 0) + T(0, 1, 0) = (1, 0, 1)$, whereas $T((1, 0, 0) + (0, 1, 0)) = (\frac{1}{2}, 1, \frac{1}{2})$.

2. The ergodic lemma. The following lemma is a generalization of the common ergodic lemma to homogeneous and translation invariant operators T. If the operator T is linear, T_n, $n \in \mathbb{N}$, defined by (1) becomes $T_n = (I + T + T^2 + \cdots + T^{n-1})/n$ and the lemma reduces to the common ergodic theorem. The following proof is modeled after that given by Garsia (1965) for linear operators.

LEMMA. *Let $T: \mathscr{F}_0 \to \mathscr{F}_0$ be homogeneous, translation invariant, monotone, and expectation invariant. Let $(T_n)_{n\in N}$ be the sequence of operators defined by* (1). *Then $(T_n f)_{n\in N}$ converges pointwise* $\mathscr{P}$-a.e. *for $f \in \mathscr{F}_0$.*

PROOF. As before we drop phrases excluding null sets if there is no danger of confusion. (i) At first we shall show that $T1_A = 1_A \mathscr{P}$-a.e. implies $T(g1_A) = (Tg)1_A \mathscr{P}$-a.e. for every nonnegative $g \in \mathscr{F}_0$.

Note that $1_A \in \mathscr{F}_0$ and $g \in \mathscr{F}_0$, $g \geqq 0$, implies $g1_A \in \mathscr{F}_0$, since $g1_A = \bigvee_{n \in \mathbb{N}} (g \wedge (n1_A))$. The relation $0 \leqq g \leqq c$ implies $0 \leqq g1_A \leqq c1_A$ and therefore $0 \leqq T(g1_A) \leqq cT1_A = c1_A \mathscr{P}$-a.e. Similarly, $0 \leqq T(g1_{\bar{A}}) \leqq c1_{\bar{A}} \mathscr{P}$-a.e. As $g = (g1_A) \vee (g1_{\bar{A}})$, we obtain $Tg \geqq T(g1_A) \vee T(g1_{\bar{A}}) = T(g1_A) + T(g1_{\bar{A}}) \mathscr{P}$-a.e. (since $T(g1_A) \wedge T(g1_{\bar{A}}) = 0 \mathscr{P}$-a.e.). Hence expectation invariance implies $Tg = T(g1_A) + T(g1_{\bar{A}}) \mathscr{P}$-a.e. Multiplication by 1_A yields $(Tg)1_A = T(g1_A) \mathscr{P}$-a.e.

(ii) Let $T_0 : \mathscr{F}_0 \rightarrow \mathscr{F}_0$ and $T_0' : \mathscr{F}_0 \rightarrow \mathscr{F}_0$ be defined by $T_0 f := \liminf_{n \in \mathbb{N}} T_n f$ and $T_0' f := \limsup_{n \in \mathbb{N}} T_n f$, respectively, and let for $f \in \mathscr{F}_0$ and $r, s \in \mathbb{R}$, $r < s$

$$M_{r,s}(f) := \{x \in X : T_0 f(x) < r < s < T_0' f(x)\} .$$

At first we shall show that

$$T1_{M_{r,s}(f)} = 1_{M_{r,s}(f)} \mathscr{P}\text{-a.e.} \qquad \text{for all} \quad r, s \in \mathbb{R}, \quad f \in \mathscr{F}_0 . \tag{4}$$

For notational convenience we shall write $M_{r,s}$ instead of $M_{r,s}(f)$.

We have

$$1_{M_{r,s}} = \bigvee_{n \in \mathbb{N}} [0 \vee (1 \wedge n(r - T_0 f))] \wedge \bigvee_{n \in \mathbb{N}} [0 \vee (1 \wedge n(T_0' f - s))] . \tag{5}$$

As $TT_0 = T_0$ on $\mathscr{F}_0$ (see part (ii) of the proof of the main theorem), we have $T(n(r - T_0 f)) = n(r - T_0 f) \mathscr{P}$-a.e. Similarly $T(n(T_0' f - s)) = n(T_0' f - s) \mathscr{P}$-a.e.

As T is monotone, $Tg_n = g_n \mathscr{P}$-a.e. for all $n \in \mathbb{N}$ implies $T(\bigvee_{m \in \mathbb{N}} g_m) \geqq Tg_n = g_n \mathscr{P}$-a.e. for all $n \in \mathbb{N}$ and therefore $T(\bigvee_{m \in \mathbb{N}} g_m) \geqq \bigvee_{m \in \mathbb{N}} g_m \mathscr{P}$-a.e. Together with expectation invariance this implies $T(\bigvee_{m \in \mathbb{N}} g_m) = \bigvee_{m \in \mathbb{N}} g_m \mathscr{P}$-a.e. Similarly, $T(\bigwedge_{m \in \mathbb{N}} g_m) = \bigwedge_{m \in \mathbb{N}} g_m \mathscr{P}$-a.e. Repeated applications of these relations imply that the function on the right side of (5) has property (4).

(iii) Next we shall show that

$$P(f1_{M \cap \{x \in X : \bigvee_{m \in \mathbb{N}} mT_m f(x) > 0\}}) \geqq 0 \tag{6}$$

for every $f \in \mathscr{F}_0$, every $P \in \mathscr{P}$ and every $M \in \mathscr{A}$ with $T1_M = 1_M \mathscr{P}$-a.e.

For $f \in \mathscr{F}_0$ let $g_n := \bigvee_{m=1}^{n} mT_m f$. We have

$$g_n^+ \geqq mT_m f \qquad \text{for every} \quad m \in \{1, \cdots, n\} .$$

Hence

$$f + Tg_n^+ \geqq f + mTT_m f = (m+1)T_{m+1} f \mathscr{P}\text{-a.e.} \qquad \text{for every} \quad m \in \{1, \cdots, n\}$$

and

$$f + Tg_n^+ \geqq f = T_1 f \mathscr{P}\text{-a.e.}$$

so that

$$f + Tg_n^+ \geqq g_n \mathscr{P}\text{-a.e.}$$

Let $G_n := \{x \in X : g_n(x) > 0\}$. We have

$$f1_{G_n} + Tg_n^+ \geqq f1_{G_n} + (Tg_n^+)1_{G_n} \geqq g_n 1_{G_n} = g_n^+ \mathscr{P}\text{-a.e.} \tag{7}$$

Let $M \in \mathscr{A}$ be such that $T1_M = 1_M$. Multiplication of (7) by 1_M and an application of (i) yields

$$f1_{M \cap G_n} + T(g_n{}^+ 1_M) \geqq g_n{}^+ 1_M \, \mathscr{P}\text{-a.e.}$$

As T is expectation invariant, this immediately implies

$$P(f1_{M \cap G_n}) \geqq 0 \qquad \text{for all} \quad P \in \mathscr{P} \quad \text{and} \quad n \in \mathbb{N}\,.$$

(6) now follows for $n \to \infty$.

(iv) $x \in M_{r,s}$ implies $\limsup_{n \in \mathrm{N}} T_n f(x) > s$ and therefore $\bigvee_{m \in \mathrm{N}} m(T_m f(x) - s) > 0$. Hence

$$M_{r,s} \cap \{x \in X: \bigvee_{m \in \mathrm{N}} m(T_m f(x) - s) > 0\} = M_{r,s}$$

and (5) applied for $M_{r,s}$ instead of M and $f - s$ instead of f implies

$$P(f1_{M_{r,s}}) \geqq sP(1_{M_{r,s}}) \qquad \text{for all} \quad P \in \mathscr{P}.$$

The dual argument yields

$$P(f1_{M_{r,s}}) \leqq rP(1_{M_{r,s}}) \qquad \text{for all} \quad P \in \mathscr{P}$$

whence

$$sP(1_{M_{r,s}}) \leqq rP(1_{M_{r,s}}) \qquad \text{for all} \quad P \in \mathscr{P}.$$

As $r < s$ and $P(1_{M_{r,s}}) \geqq 0$, this implies $P(1_{M_{r,s}}) = 0$ for all $P \in \mathscr{P}$.
Hence

$$P(1_{\{x \in X:\, T_0 f(x) < T_0' f(x)\}}) = P(1 \textstyle\bigcup_{r,s \in \mathrm{Q},\, M_{r,s};\, r<s}) \leqq \sum_{r,s \in \mathrm{Q};\, r<s} P(1_{M_{r,s}}) = 0$$

for all $P \in \mathscr{P}$.

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