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THE MOST POWERFUL SCALE AND LOCATION INVARIANT TEST OF THE NORMAL VERSUS THE DOUBLE EXPONENTIAL

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The most powerful scale and location invariant test of normality against the double exponential alternative is derived by the technique of integrating with respect to the scale and location transformation group. The resultant test is asymptotically equivalent to the likelihood ratio test of this hypothesis and to Geary's test (i.e. mean deviation over standard deviation) for all three test statistics are shown to have the same asymptotic normal distribution when the sampling is from a symmetric, absolutely continuous distribution, whose density is continuous in the neighborhood of its median and whose fourth moment exists.

0. Introduction. The purpose of this paper is threefold: (i) to derive the most powerful scale and location invariant test of the hypothesis

$$H_0: f(x) = (2\pi\sigma^2)^{-\frac{1}{2}} \exp\left[-|x-\mu|^2/2\sigma^2\right]$$

against the alternative

$$H_1: f(x) = (2\sigma)^{-1} \exp\left(-|x-\mu|/\sigma\right),$$

(ii) to find the asymptotic normal distribution of the test statistic and finally, (iii) to show that the most powerful invariant test is asymptotically equivalent to the likelihood ratio test and to Geary's [1] of H_0 versus H_1. Goals (ii) and (iii) are intertwined in the same results and will both be considered in Section 2.

The normality or nonnormality of a distribution is not affected by the values of its scale or location parameters. Thus, it is often desirable that a test of normality be invariant under scale and location transformations. The results of this paper will yield the possibility of making an absolute judgment on the power of the invariant tests of normality in regards to the double exponential alternative.

Throughout this paper $X_1, \cdots, X_n$ will denote a random sample, $x_1, \cdots, x_n$ will denote a collection of reals which may or may not be outcomes of a random sample, $Y_1 \leqq \cdots \leqq Y_n$ will denote the order statistics and $y_1 \leqq \cdots \leqq y_n$ will be the ordered values of $x_1, \cdots, x_n$.

1. Derivation of the test. Note first that the most powerful scale and origin invariant test of composite hypothesis H_0 against the composite alternative H_1 is the same as that of the simple hypothesis

$$H_0^*: f(x) = (2\pi)^{-\frac{1}{2}} \exp(-x^2/2)$$

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against the simple alternative

$$H_1^*: f(x) = 2^{-1}\exp(-|x|),$$

because the distribution, and consequently the power, of the latter test is not affected by a scale and location transformation.

The method of derivation is an application of a technique originated by Stein [5] and further developed by Wijsman [4], Hájek [2] and Koehn [3] for finding the density of the maximal invariant by integrating over the transformation group. Specifically in the normal case the density of the scale and location maximal invariant is

$$p^* = \int_0^\infty \int_{-\infty}^\infty (2\pi)^{-n/2} \exp(-\sum_1^n (\lambda x_i - u)^2/2)\lambda^{n-2}\, du\, d\lambda$$

and for the double exponential it is

$$q^* = \int_0^\infty \int_{-\infty}^\infty 2^{-n} \exp(-\sum_1^n |\lambda x_i - u|)\lambda^{n-2}\, du\, d\lambda .$$

Thus, the test will reject when $q^*/p^* > k$, where k is chosen to provide the desired test size.

The test may be obtained in closed form if p^* and q^* can be obtained in closed form. p^* was evaluated by Hájek ([2] page 49) and thus its evaluation need not be discussed here. The value of

$$p^* = 2^{-1}n^{-\frac{1}{2}}\pi^{-n/2+\frac{1}{2}}\Gamma(n/2 - \tfrac{1}{2})[(n-1)S^2]^{-n/2+\frac{1}{2}}, \tag{1}$$

where $S^2 = (n-1)^{-1}\sum (x_i - \bar{x})^2$ and $\bar{x} = n^{-1}\sum_1^n x_i$.

If $\gamma(t) = \sum_1^n |x_i - t|$, the key to evaluating q^* is in recognizing that γ is a continuous, piecewise linear function, i.e.

$$\begin{aligned}
\gamma(t) &= \textstyle\sum_1^n y_i - nt, && \text{for } t \leqq y_1, \\
&= (2i-n)t + \textstyle\sum_{i+1}^n y_j - \sum_1^i y_j, \\
& && \text{for } y_i \leqq t \leqq y_{i+1} \text{ and } i = 1, \cdots, n-1, \\
&= nt - \textstyle\sum_1^n y_i, && \text{for } y_n \leqq t.
\end{aligned}$$

Thus, integrating first on the scale parameter and letting $t = u/\lambda$, $q^* = 2^{-n}\Gamma(n)\int_{-\infty}^\infty \gamma^{-n}(t)\,dt$. Then $\int_{-\infty}^\infty \gamma^{-n}(t)\,dt$ can be integrated piecewise to yield

$$q^* = 2^{-n+1}\Gamma(n-1)[B_n(x_1, \cdots, x_n)\gamma(M)]^{-n+1}, \tag{2}$$

where

$$M \in [y_{n_1}, y_{n_2}], \qquad n_1 = [(n+1)/2], \qquad n_2 = [n/2] + 1,$$

and

$$\begin{aligned}
&B_n(x_1, \cdots, x_n) \\
&\quad = [\textstyle\sum_1^n \gamma^{n-1}(M)\gamma^{-n+1}(y_i)(2i-n)^{-1}(n+2-2i)^{-1}]^{-1/(n-1)}, && \text{for } n \text{ odd}, \\
&\quad = [(n-1)(y_{n_2} - y_{n_1})2^{-1}\gamma^{-1}(M) + 2^{-1} \\
&\qquad + \textstyle\sum_{i=1, i\neq n_1, i\neq n_2}^n \gamma^{n-1}(M)\gamma^{-n+1}(y_i)(2i-n)^{-1}(n+2-2i)^{-1}]^{-1/(n-1)}, \\
& && \text{for } n \text{ even}.
\end{aligned}$$

A little algebra then yields the following theorem.

THEOREM 1. *Let* $V_n(x_1, \cdots, x_n) = n^{-1}\gamma(M)/S$. *The most powerful scale and location invariant test of* H_0 *against* H_1 *rejects when* $B_n V_n < c$ *where* c *is chosen to provide the required test size.*

2. Asymptotic results. It may be shown that the likelihood ratio test of H_0 versus H_1 rejects when $V_n < c'$. The most powerful scale and location invariant test of the normal against the uniform and against the exponential have previously been found [6] to be the same as the respective likelihood ratio tests. In the case of H_0 versus H_1, however, the two tests are different but are asymptotically equivalent for, as is shown below, V_n and $B_n V_n$ have the same asymptotic normal distribution under H_0 and H_1.

Geary's test of H_0 versus H_1 rejects normality when $G_n = n^{-1}\gamma(\bar{X})[(n-1)/n]^{-\frac{1}{2}}S^{-1} < c''$. G_n shares the common asymptotic distribution of V_n and $B_n V_n$ and thus, Geary's test is asymptotically equivalent to the other two.

THEOREM 2. *If the underlying distribution is symmetric and absolutely continuous, has a finite fourth moment and if its density is continuous in a neighborhood of its median, then* V_n, $B_n V_n$ *and* G_n *have a common asymptotic normal distribution with mean* $\nu_1\nu_2^{-\frac{1}{2}}$ *and variance* $n^{-1}[1 - \nu_1\nu_3\nu_2^{-2} + 4^{-1}\nu_1^2\nu_2^{-1}(\nu_4\nu_2^{-2} - 1)]$, *where* $\nu_i = E[|X-\mu|^i]$ *and* $\mu = E[X] = \text{med } X$.

Before giving proof for Theorem 2, it is first convenient to show that if V_n has an asymptotic distribution then G_n and $B_n V_n$ share it.

LEMMA 1. *Under the hypothesis of Theorem* 2, $n^{\frac{1}{2}}(V_n - G_n') \rightarrow_p 0$ *where* $G_n' = (n/(n-1))^{\frac{1}{2}}G_n$.

PROOF. Let F be the underlying distribution function and let F_n be the empirical distribution function. Furthermore, let $I_{(a,b)}(x)$ be the indicator function of the interval (a, b). For any distribution, if $m = \text{med } X$,

$$
\begin{aligned}
E[|X-c|] &= E[|X-m|] + |c-m|(2F(m)-1) \\
&\qquad + 2E[I_{(m,c)}(X)|c-X|]\,, && \text{if } m < c\,, \\
&= E[|X-m|] + |c-m|(1-2F(m^-)) \\
&\qquad + 2E[I_{(c,m)}(X)|c-X|]\,, && \text{if } m > c\,,
\end{aligned} \tag{3}
$$

where $F(m^-)$ is the limit from the left of F at m. In particular, this is true for the empirical distribution. Calculation then reveals that, if $\gamma'(t) = n^{-1}\gamma(t)$,

$$
n^{\frac{1}{2}}|\gamma'(M) - \gamma'(\bar{X})| \leqq |n^{\frac{1}{2}}(Y_{n_1} - \mu)(2F_n(\bar{X}) - 1)| + |n^{\frac{1}{2}}(\bar{X} - \mu)(2F_n(\bar{X}) - 1|\,. \tag{4}
$$

Note that $|2F_n(\bar{X}) - 1| = 2|F_n(\bar{X}) - F(\mu)|$ and

$$
|F_n(\bar{X}) - F(\mu)| \leqq \max_x |F_n(x) - F(x)| + |F(\bar{X}) - F(\mu)|\,. \tag{5}
$$

By the Glivenko–Cantelli theorem the first term on the right of inequality (5) converges stochastically to zero, and since F is continuous and $\bar{X} \rightarrow_p \mu$, the second term also converges to zero and thus, $2F_n(\bar{X}) - 1 \rightarrow_p 0$. Consequently, since $n^{\frac{1}{2}}(Y_{n_1} - \mu)$ and $n^{\frac{1}{2}}(\bar{X} - \mu)$ are limiting normal, both terms on the right of inequality (4) converge stochastically to zero. Therefore, $n^{\frac{1}{2}}(\gamma'(M) - \gamma'(\bar{X}))$ converges stochastically to zero. Since $S \to_p \nu_2^{\frac{1}{2}}$ and $n^{\frac{1}{2}}(V_n - G_n') = n^{\frac{1}{2}}(\gamma'(M) - \gamma'(\bar{X}))/S$, the result follows. □

The relationship between V_n and $B_n V_n$ is stronger than that between V_n and G_n.

LEMMA 2. *For all sequences $x_1, \cdots, x_n$ and $p < 1$,*

$$V_n(x_1, \cdots, x_n)B_n(x_1, \cdots, x_n) = V_n(x_1, \cdots, x_n) + o(n^{-p}) .$$

PROOF. From (3) and Jensen's inequality, it may be shown that for all distributions $0 \leqq E[|X - \text{med}\, X|] \leqq \nu_1 \leqq \nu_2^{\frac{1}{2}}$. Thus, it holds for the empirical distribution and hence

$$0 \leqq V_n \leqq 1 . \tag{6}$$

It is next shown that

$$|B_n - 1| \leqq n^{1/(n-1)} - 1 , \tag{7}$$

for all $x_1, \cdots, x_n$. Because B_n has a different definition for n odd than for n even, two cases must be considered. First let $n = 2k + 1$. Since $0 < \gamma(M) \leqq \gamma(y_i)$, for all i, $0 < \gamma^{n-1}(M)\gamma^{-n+1}(y_i) \leqq 1$ with equality holding when $i = k + 1$. Furthermore, $[(2i - n)(n + 2 - 2i)]^{-1}$ is negative if $i \neq k + 1$, and equals one if $i = k + 1$. Thus, $1 \geqq \sum_1^n \gamma^{n-1}(M)\gamma^{-n+1}(y_i)(n + 2 - 2i)^{-1}(2i - n)^{-1} \geqq n^{-1}$ and $n^{1/(n-1)} \geqq B_n \geqq 1$. Hence, inequality (7) holds.

If $n = 2k$ then as above it is shown that

$$0 \geqq \textstyle\sum_{i=1, i\neq n_1, i\neq n_2}^{n} \gamma^{n-1}(M)\gamma^{-n+1}(y_i)(2i - n)^{-1}(n + 2 - 2i)^{-1} \geqq n^{-1} - 2^{-1} .$$

Furthermore, $\gamma(M) = \sum_{j=1}^{n_1} (y_{n+1-j} - y_j) \geqq n_1(y_{n_2} - y_{n_1})$ and thus, $1 \geqq (n - 1)(y_{n_2} - y_{n_1})2^{-1}\gamma^{-1}(M) \geqq 0$. Therefore, $n^{1/(n-1)} \geqq B_n \geqq (\frac{2}{3})^{1/(n-1)}$. Since $(\frac{2}{3})^{1/(n-1)} - 1 \geqq [1 - n^{1/(n-1)}]$, inequality (7) holds.

Finally, $n^p|V_n - B_n V_n| \leqq |V_n|\,|1 - B_n|n^p < n^p(n^{1/(n-1)} - 1)$. If $0 < p < 1$, it may be shown by l'Hospital's rule that $\lim_{n\to\infty} n^p(n^{1/(n-1)} - 1) = 0$, and if $p < 0$ the limit is zero because both factors are bounded and at least one has limit zero. □

PROOF OF THEOREM 2. From Lemmas 1 and 2, it is sufficient to consider the asymptotic distribution of V_n. Furthermore, because $S\nu_2^{-\frac{1}{2}} \to_p 1$ and $n^{\frac{1}{2}}(V_n - \nu_1\nu_2^{-\frac{1}{2}}) = n^{\frac{1}{2}}(\gamma'(M)\nu_2^{-\frac{1}{2}} - \nu_1\nu_2^{-1}S)/(S\nu_2^{-\frac{1}{2}})$, it is sufficient to investigate the limiting distribution of $n^{\frac{1}{2}}(\gamma'(M)\nu_2^{-\frac{1}{2}} - \nu_1\nu_2^{-1}S)$.

As in proof of Lemma 1 it may be shown that

$$\gamma'(M) = \gamma'(\mu) + R_n , \tag{8}$$

where $n^{\frac{1}{2}}R_n \to_p 0$. From Taylor's theorem

$$S = \nu_2^{\frac{1}{2}} + 2^{-1}\nu_2^{-\frac{1}{2}}(n^{-1} \textstyle\sum_1^n (X_i - \mu)^2 - \nu_2) + R_n' , \tag{9}$$

where $n^{\frac{1}{2}}R_n' \to_p 0$. From (8) and (9), it follows that $n^{\frac{1}{2}}(V_n - \nu_1\nu_2^{-\frac{1}{2}})$ has the same limiting distribution as $n^{\frac{1}{2}}(\sum_1^n n^{-1}\{\nu_2^{-\frac{1}{2}}|X_i - \mu| - 2^{-1}\nu_1\nu_2^{-\frac{3}{2}}(X_i - \mu)^2\} - 2^{-1}\nu_1\nu_1^{-\frac{1}{2}})$. The result then follows from the central limit theorem. □

Finally, the limiting normal distribution could be used to approximate the critical region of the test. In the normal distribution $\nu_4/\nu_2^2 = 3$, $\nu_3/\nu_2^{\frac{3}{2}} = 2(2/\pi)^{\frac{1}{2}}$ and $\nu_1/\nu_2^{\frac{1}{2}} = (2/\pi)^{\frac{1}{2}}$. Thus, under H_0 $B_n V_n$ is approximately $n(.798, .045/n)$.

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