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CONVERGENCE OF REDUCED EMPIRICAL AND QUANTILE PROCESSES WITH APPLICATION TO FUNCTIONS OF ORDER STATISTICS IN THE NON-I.I.D. CASE[1]

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Any triangular array of row independent rv's having continuous df's can be transformed naturally so that the empirical and quantile processes of the resulting rv's are relatively compact. Moreover, convergence (to a necessarily normal process) takes place if and only if a simple covariance function converges pointwise. Using these results we derive the asymptotic normality of linear combinations of functions of order statistics of non-i.i.d. rv's in the case of bounded scores.

1. Notation and results. Let $\alpha_{n1}, \cdots, \alpha_{nn}$, $n \geqq 1$ be a triangular array of row independent rv's having continuous df's $F_{n1}, \cdots, F_{nn}$, $n \geqq 1$. Let $\mathbb{F}_n$ denote the empirical df of $\alpha_{n1}, \cdots, \alpha_{nn}$. Using the left continuous version of the inverse of a df, we let

$$(1.1) \qquad \beta_{ni} = F_n(\alpha_{ni}) \quad \text{and} \quad G_{ni} = F_{ni} \circ F_n^{-1} \quad \text{where } F_n = n^{-1} \textstyle\sum_1^n F_{ni} .$$

Then β_{ni} has absolutely continuous df G_{ni} on [0, 1] (in fact $|G_{ni}(t) - G_{ni}(s)| \leqq n|t - s|$) and also

$$(1.2) \qquad n^{-1} \textstyle\sum_1^n G_{ni}(t) = t \qquad \text{for } 0 \leqq t \leqq 1 .$$

Let $\mathbb{G}_n$ denote the empirical df of $\beta_{n1}, \cdots, \beta_{nn}$; we will call $\mathbb{G}_n$ the *reduced empirical* df of $\alpha_{n1}, \cdots, \alpha_{nn}$. Define the *reduced empirical process* X_n by

$$(1.3) \qquad X_n(t) = n^{\frac{1}{2}}[\mathbb{G}_n(t) - t] \qquad \text{for } 0 \leqq t \leqq 1 .$$

Clearly X_n has mean value function 0 by (1.2). Also, if we define

$$(1.4) \qquad K_n(s, t) = s \wedge t - n^{-1} \textstyle\sum_1^n G_{ni}(s)G_{ni}(t) \qquad \text{for } 0 \leqq s, t \leqq 1 ,$$

then K_n is the covariance function of the X_n process since

$$\begin{aligned} \operatorname{Cov}[X_n(s), X_n(t)] &= n^{-1} \textstyle\sum_1^n [G_{ni}(s) \wedge G_{ni}(t) - G_{ni}(s)G_{ni}(t)] \\ &= K_n(s, t) . \end{aligned}$$

The *reduced quantile process* Y_n is defined by

$$(1.5) \qquad Y_n(t) = n^{\frac{1}{2}}[\mathbb{G}_n^{-1}(t) - t] \qquad \text{for } 0 \leqq t \leqq 1 .$$

Expressions for the exact mean value and covariance function of Y_n would be difficult; but the asymptotic behavior of Y_n is simply related to that of X_n.

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We wish to consider the weak convergence ($\Rightarrow$) of the X_n and Y_n processes. Let C denote the collection of all continuous functions on $[0, 1]$; and let $\rho(x, y) = \sup_t |x(t) - y(t)|$ for functions x and y on $[0, 1]$ or $(-\infty, \infty)$. *Weak convergence* and *relative compactness* of processes on (C, ρ) is as defined in Billingsley (1968).

We will write $X_n \rightarrow_{\text{f.d.}} X$ as $n \rightarrow \infty$ if the *finite dimensional distributions* of the X_n process *converge* to those of the X process. We will write $\rightarrow_e$ to denote convergence at every point of the probability space; call this *everywhere convergence.*

For technical reasons it is convenient to define $\bar{X}_n$ to be that process on (C, ρ) that equals X_n at each t in $[0, 1]$ corresponding to one of $\beta_{n1}, \cdots, \beta_{nn}$ and that is linear on the intervals between these observations. Note that X_n and Y_n are not processes on (C, ρ).

The theorems below show that convergence of the X_n process is equivalent to pointwise convergence of K_n; and the limiting process must necessarily be normal. Moreover, X_n converges if and only if Y_n does, and the limits are simply related.

THEOREM 1. (i) *For any triangular array of row independent* rv'*s* $\alpha_{n1}, \cdots, \alpha_{nn}$, $n \geqq 1$ *having continuous* df'*s* $F_{n1}, \cdots, F_{nn}$, $n \geqq 1$ *we have both*

$$\rho(X_n, \bar{X}_n) \leqq n^{-\frac{1}{2}} \rightarrow 0 \qquad \text{as } n \rightarrow \infty$$

and

$$\{\bar{X}_n : n \geqq 1\} \quad \text{is relatively compact on} \quad (C, \rho) .$$

(ii) *If*

$$K_n(s, t) \rightarrow K(s, t) \qquad \text{as } n \rightarrow \infty \text{ for all } 0 \leqq s, t \leqq 1 \tag{1.6}$$

for some function K, *then there exists a normal process* X *having sample paths in* C *and covariance function* K *for which*

$$\bar{X}_n \Rightarrow X \quad \text{on} \quad (C, \rho) \qquad \text{as } n \rightarrow \infty .$$

(iii) *Conversely, if* $X_n \rightarrow_{\text{f.d.}} X$ *as* $n \rightarrow \infty$, *then* X *is necessarily a normal process*; *and if* K *denote the covariance function of* X, *then* (1.6) *must hold.*

THEOREM 2. *Let* $F_{n1}, \cdots, F_{nn}$, $n \geqq 1$ *be a triangular array of continuous* df'*s for which* (1.6) *holds. Then there exists a triangular array of row independent* rv'*s* $\alpha_{n1}, \cdots, \alpha_{nn}$, $n \geqq 1$ *having* df'*s* $F_{n1}, \cdots, F_{nn}$, $n \geqq 1$ *and there exists a normal process* X *with sample paths in* C *and covariance function* K *for which*

$$\rho(X_n, X) \rightarrow_e 0 \qquad \text{as } n \rightarrow \infty$$

and

$$\rho(Y_n, -X) \rightarrow_e 0 \qquad \text{as } n \rightarrow \infty .$$

Moreover, all finite dimensional moments $(E[X_n(t_1)^{m_1} \cdots X_n(t_k)^{m_k}]$ *with* $0 \leqq t_1, \cdots, t_k \leqq 1$ *and* $k, m_1, \cdots, m_k$ *integers) of* X_n *converge to the corresponding moments of the limiting* X *process as* $n \rightarrow \infty$.

A *Brownian bridge* is a normal process on (C, ρ) having mean value function 0 and covariance function $s \wedge t - st$ for $0 \leqq s, t \leqq 1$.

COROLLARY 1. *Let $F_{n1}, \cdots, F_{nn}$, $n \geqq 1$ be a triangular array of continuous* df's *for which*

$$(\max_{1\leqq i,j\leqq n})\rho(F_{ni}, F_{nj}) \to 0 \qquad as\ n \to \infty\,. \tag{1.7}$$

There there exists a triangular array of row independent rv's $\alpha_{n1}, \cdots, \alpha_{nn}$ $n \geqq 1$ *having* df's $F_{n1}, \cdots, F_{nn}$, $n \geqq 1$ *and there exists a Brownian bridge X for which*

$$\rho(X_n, X) \to_e 0 \qquad and \qquad \rho(Y_n, -X) \to_e 0 \qquad as\ n \to \infty\,.$$

REMARK 1. Koul (1970) uses the methods of Billingsley (1968) to generalize the usual result on weak convergence of the empirical process of Uniform (0, 1) rv's. We use a different technique (the introduction of S_n below bypasses Theorems 12.1 and 12.2 of Billingsley) on a special case of Koul's problem and obtain stronger and cleaner results for our X_n (we could have considered the more general problem). The parallel results on Y_n are new and should prove highly useful. Application to the limiting distribution of linear combinations of functions of order statistics in the non i.i.d. case (see Shorack (1972) for the i.i.d. case) is discussed in Section 3.

2. Proofs. See pages 33, 34 and 56 of Hájek and Šidák (1967) for useful properties of F^{-1}.

PROOF OF THEOREM 1. (i) Let $0 \leqq s \leqq t \leqq 1$. Let

$$\begin{aligned}\pi_{ni} &= 1 - p_{ni} &&\text{if } s < \beta_{ni} \leqq t\\ &= -p_{ni} &&\text{if not,}\end{aligned}$$

where $p_{ni} = G_{ni}(t) - G_{ni}(s)$. Then using (1.2) we have

$$\begin{aligned}E[X_n(t) - X_n(s)]^4 &= n^{-2}E[\textstyle\sum_i^n \pi_{ni}]^4\\ &= n^{-2}[\textstyle\sum_1^n E\pi_{ni}^4 + 3\sum\sum_{i\neq j} E\pi_{ni}^2 E\pi_{nj}^2]\\ &= n^{-2}[3(\textstyle\sum_1^n E\pi_{ni}^2)^2 + \sum_1^n (E\pi_{ni}^4 - 3E^2\pi_{ni}^2)]\\ &= 3[n^{-1}\textstyle\sum_1^n p_{ni}(1 - p_{ni})]^2\\ &\qquad + n^{-2}\textstyle\sum_1^n [p_{ni}(1 - p_{ni})(1 - 6p_{ni} + 6p_{ni}^2)]\\ &\leqq 3[n^{-1}\textstyle\sum_1^n p_{ni}]^2 + n^{-1}[n^{-1}\sum_1^n p_{ni}]\\ &= 3(t - s)^2 + (t - s)/n\,.\end{aligned} \tag{2.1}$$

Define S_n on (C, ρ) by setting $S_n(i/n) = X_n(i/n)$ for $0 \leqq i \leqq n$ and by letting S_n be linear on the intervals $[(i-1)/n, i/n]$. We will demonstrate below that S_n satisfies

$$E|S_n(t) - S_n(s)|^4 \leqq 144|t - s|^2 \qquad \text{for all } 0 \leqq s, t \leqq 1 \tag{2.2}$$

for all $n \geqq 1$. For the time being we take (2.2) to be true. Thus $\{S_n : n \geqq 1\}$ is relatively compact on (C, ρ); see page 95 of Billingsley or (better yet for simplicity, and appreciation of S_n) page 28 of Varadhan (1968). (The improvement of (2.2) over (2.1) is the most interesting part of this proof. Note (5.3) of Sen (1970).)

Now

$$\bar{X}_n = (\bar{X}_n - X_n) + (X_n - S_n) + S_n \,. \tag{2.3}$$

Note that $\rho(\bar{X}_n, X_n) \leqq n^{-\frac{1}{2}}$ so that

$$\rho(\bar{X}_n, X_n) \to_p 0 \qquad \text{as } n \to \infty \,. \tag{2.4}$$

Also for $n > (4/\varepsilon)^2$ we have $n^{\frac{1}{2}}|t - i/n| < \varepsilon/4$ for all $|t - i/n| \leqq 1/n$; hence (recall (1.3) with $\mathbb{G}_n$ increasing)

$$\begin{aligned} P(\rho(X_n, S_n) > \varepsilon) &= P(\max_{1 \leqq i \leqq n} (\sup_{i-1/n \leqq t \leqq i/n})|X_n(t) - S_n(t)| > \varepsilon) \\ &\leqq P(\max_{0 \leqq i \leqq n} |\sup_{|t - i/n| \leqq 1/n} |X_n(t) - X_n(i/n)| > \varepsilon) \\ &\leqq P(\max_{1 \leqq i \leqq n} n^{\frac{1}{2}}|\mathbb{G}_n(i/n) - \mathbb{G}_n((i-1)/n)| > 3\varepsilon/4) \\ &\leqq P(\max_{1 \leqq i \leqq n} |X_n(i/n) - X_n((i-1)/n)| > \varepsilon/2) \\ &\leqq \textstyle\sum_1^n P(|X_n(i/n) - X_n((i-1)/n)| > \varepsilon/2) \,, \end{aligned}$$

where the first inequality depends on the linearity of S_n in that

$$|X_n(t) - S_n(t)| \leqq |X_n(t) - S_n(i/n)| \vee |X_n(t) - S_n((i-1)/n)|$$

for all $(i-1)/n \leqq t \leqq i/n$. Thus by (2.1) we have

$$\begin{aligned} P(\rho(X_n, S_n) > \varepsilon) &\leqq 16 \textstyle\sum_i^n E[X_n(i/n) - X_n((i-1)/n)]^4/\varepsilon^4 \\ &\leqq (16/\varepsilon^4)n(4/n^2) \to 0 \,. \end{aligned}$$

Thus

$$\rho(X_n, S_n) \to_p 0 \qquad \text{as } n \to \infty \,. \tag{2.5}$$

Let ψ denote a bounded real functional on C that is uniformly continuous in the ρ-metric. Then on any subsequence n' where $S_{n'}$ converges weakly (let S denote the limit) we have

$$\begin{aligned} |E\psi(\bar{X}_{n'}) - E\psi(S)| &\leqq E|\psi(\bar{X}_{n'}) - \psi(X_{n'})| + E|\psi(X_{n'}) - \psi(S_{n'})| \\ &\quad + |E\psi(S_{n'}) - E\psi(S)| \\ &\to 0 + 0 + 0 = 0 \qquad \text{as } n' \to \infty \,; \end{aligned}$$

here (2.4), (2.5) and the uniform continuity of bounded ψ yield the first two zeroes. Thus $\bar{X}_{n'} \Rightarrow S$ on (C, ρ) as $n' \to \infty$ by page 12 of Billingsley. Thus $\{\bar{X}_n : n \geqq 1\}$ is relatively compact on (C, ρ).

To complete the proof of (i), it remains only to establish (2.2). Let $n \geqq 1$ and $0 \leqq s,\ t \leqq 1$, be arbitrary, but fixed. Choose the integers i and j so that

$$(i-1)/n \leqq s \leqq i/n \qquad \text{and} \qquad (j-1)/n \leqq t \leqq j/n \,.$$

Let $\Delta_{km} = |S_n(m/n) - S_n(k/n)|$ for integers k and m; and note from (2.1) that

$$E\Delta_{km}^4 \leqq [3|(\mathrm{m} - k)/n|^2 + |(m-k)/n|/n] \leqq 4[(m-k)/n]^2 \,.$$

Let $e_{km} = 4[(m-k)/n]^2$. We also let $\Delta_{uv} = |S_n(v) - S_n(u)|$.

Case 1. $i < j - 1$. Then

$$\Delta_{st} \leqq \Delta_{i,j-1} \vee \Delta_{ij} \vee \Delta_{i-1,j-1} \vee \Delta_{i-1,j} \,,$$

so that

$$E\Delta_{st}^4 \leqq e_{i,j-1} + e_{ij} + e_{i-1,j-1} + e_{i-1,j} \leqq 4e_{i-1,j}$$
$$= 16[(j-(i-1))/n]^2 \leqq 144(t-s)^2 .$$

Case 2. $i = j$. Since the change of a linear function on an interval of length $t - s$ equals the slope times $t - s$ we have

$$\Delta_{st} \leqq n\Delta_{i-1,i}(t-s) ,$$

so that

$$E\Delta_{st}^4 \leqq n^4(t-s)^4 e_{i-1,i} \leqq 4n^2(t-s)^4 \leqq 4(t-s)^2 .$$

Case 3. $i = j - 1$. Then

$$\Delta_{st} \leqq \Delta_{s,i/n} + \Delta_{i/n,t} \leqq 2(\Delta_{s,i/n} \vee \Delta_{i/n,t}) ,$$

so that by Case 2 we have

$$E\Delta_{st}^4 \leqq 2^4(E\Delta_{s,i/n}^4 + E\Delta_{i/n,t}^4) \leqq 2^4[4(i/n-s)^2 + 4(t-i/n)^2]$$
$$\leqq 2^7(t-s)^2 .$$

Thus (2.2) is established.

(ii) Suppose $K_n \to K$. The limit of covariance functions is necessarily a covariance function (page 468 of Loève (1963)); and since K is real there is a normal process X (we do not know yet that it has continuous sample paths) having covariance function K (page 467 of Loève). Thus for fixed k, $0 \leqq t_1, \cdots, t_k \leqq 1$ and real $a_1, \cdots, a_k$ the variance of the rv $\theta_n \equiv \sum_1^k a_i X_n(t_i)$ converges to the variance (denote it by c) of the rv $\theta \equiv \sum_1^k a_i X(t_i)$ as $n \to \infty$. If $c = 0$, then trivially $\theta_n \to_p 0$ as $n \to \infty$. If $c > 0$, then $\theta_n \to_d N(0, c)$ as $n \to \infty$ by the Liaponov central limit theorem (this is again trivial since the summands are uniformly bounded and the sum of their variances goes to ∞). We have shown that $X_n \to_{\text{f.d.}} X$ as $n \to \infty$ (see page 49 of Billingsley). Hence by (2.4) we have $\bar{X}_n \to_{\text{f.d.}} X$ as $n \to \infty$. But $\{\bar{X}_n : n \geqq 1\}$ is relatively compact by (i); and if $X_{n'} \Rightarrow Z$ on (C, ρ) as $n' \to \infty$ for some subsequence n', then X must have the same finite dimensional distributions as Z. Thus $\bar{X}_n \Rightarrow X$ on (C, ρ) as $n \to \infty$.

(iii) It is easy that for all $n \geqq 1$ and all $0 \leqq t \leqq 1$ we have $EX_n^{2m}(t) \leqq M_m$ for each $m \geqq 1$, where M_m is a constant. The Cauchy–Schwarz inequality can be used to bound any finite dimensional moment by a product of powers of one dimensional moments. Application of the Corollary on page 184 of Loève then establishes convergence of the finite dimensional moment. In particular, K_n converges pointwise to K as $n \to \infty$. Thus X is a normal process by (ii). □

Proof of Theorem 2. Using Theorem 1 (ii) and Item 3.1.1 of Skorokhod (1956) it is easy to construct the required α_{ni}'s and X for which $\rho(X_n, X) \to_e 0$ as $n \to \infty$. Now

$$Y_n = -X_n(\mathbb{G}_n^{-1}) + n^{\frac{1}{2}}(\mathbb{G}_n \circ \mathbb{G}_n^{-1} - I) , \tag{2.6}$$

where I denotes the identity function; so that

$$\begin{aligned}\rho(Y_n, -X) &\leqq \rho(X_n(\mathbb{G}_n^{-1}), X) + n^{\frac{1}{2}}\rho(\mathbb{G}_n \circ \mathbb{G}_n^{-1}, I)\\ &\leqq \rho(X_n(\mathbb{G}_n^{-1}), X(\mathbb{G}_n^{-1})) + \rho(X(\mathbb{G}_n^{-1}), X) + n^{-\frac{1}{2}}\\ &\leqq \rho(X_n, X) + \rho(X(\mathbb{G}_n^{-1}), X) + n^{-\frac{1}{2}};\end{aligned}$$

and thus it suffices to show that $\rho(X(\mathbb{G}_n^{-1}), X) \to_e 0$ as $n \to \infty$. But since all sample paths of X are uniformly continuous on $[0, 1]$, it suffices to show that $\rho(\mathbb{G}_n^{-1}, I) \to_e 0$ as $n \to \infty$. But by symmetry

$$\rho(\mathbb{G}_n^{-1}, I) = \rho(\mathbb{G}_n, I) = n^{-\frac{1}{2}}\rho(X_n, 0) \leqq n^{-\frac{1}{2}}[\rho(X_n, X) + \rho(X, 0)] \to_e 0$$

as $n \to \infty$, since every sample path of X is bounded. Thus $\rho(Y_n, -X) \to_e 0$ as $n \to \infty$.

For the moment convergence, see the proof of Theorem 1 (iii). □

Proof of Corollary 1. Condition (1.7) implies $(\max_{1 \leqq i \leqq n})\rho(F_{ni}, F_n) \to 0$ as $n \to \infty$. Hence $(\max_{1 \leqq i \leqq n})\rho(G_{ni}, I) \to 0$ as $n \to \infty$. Hence (1.6) holds with $K(s, t) = s \wedge t - st$. Now apply Theorem 2. □

3. Asymptotic normality of linear combinations of functions of order statistics in the non i.i.d. case. We use the notation of Section 1. Consider

$$T_n = n^{-1} \sum_{i=1}^n c_{ni} h_n(\eta_{ni}) + \sum_{k=1}^{\kappa} d_{nk} h_n(\eta_{n,[np_k]+1}) \tag{3.1}$$

where $\eta_{n1} \leqq \cdots \leqq \eta_{nn}$ are the ordered values of $\alpha_{n1}, \cdots, \alpha_{nn}$, where the h_n's are known functions, where $0 < p_1 < \cdots < p_\kappa < 1$ and [ ] denotes the greatest integer function and where $c_{n1}, \cdots, c_{nn}$, $n \geqq 1$ and $d_{n1}, \cdots, d_{n\kappa}$, $n \geqq 1$ are known constants. Let $g_n = h_n(F_n^{-1})$. Then

$$T_n = \int_0^1 g_n(\mathbb{G}_n^{-1}(t))J_n(t)\,dt + \sum_{k=1}^{\kappa} d_{nk} g_n(\mathbb{G}_n^{-1}(p_k)), \tag{3.2}$$

where J_n on $[0, 1]$ is defined by $J_n(t) = c_{ni}$ for $(i-1)/n < t \leqq i/n$ and $1 \leqq i \leqq n$ with $J_n(0) = c_{n1}$. Let

$$\mu_n = \int_0^1 g_n(t)J_n(t)\,dt + \sum_{k=1}^{\kappa} d_k g_n(p_k) \tag{3.3}$$

for given constants $d_1, \cdots, d_\kappa$.

Theorem 3. *Suppose that for functions g and J either Assumptions* 1, 2, 3, 4 *or Assumption* 1, 2′, 3′, 4 *of Shorack* (1972) *are satisfied; where $b_1 = b_2 = 0$ in Assumption* 1 (*thus J is a* bounded *function*). *Then*

$$n^{\frac{1}{2}}(T_n - \mu_n) \to_d -\int_0^1 JX\,dg - \sum_{k=1}^{\kappa} d_k g'(p_k)X(p_k) \qquad as\ n \to \infty.$$

The limiting rv *is* ($N(0, \sigma^2)$ *where the formula for σ^2 is* (3) *of Shorack* (1972) *with $s \wedge t - st$ replaced by the covariance function $K(s, t)$ of our present X process.*

Proof. Replace U_n, U, ξ_{ni} in the proofs of Theorem 1 and Corollary 2 of [6] by X_n, X, β_{n1}. For bounded J, Lemma A3 of [6] is not used to obtain a dominating function. □

The restriction to bounded J could be removed if an analog of Lemma A3 of

[6] could be proved for triangular arrays of continuous rv's; for some special triangular arrays such results are possible by deterministic manipulations of the G_{ni}'s. Many of the most interesting J's are bounded (those corresponding to trimmed, Winsorized and linearly weighted means for example).

Results similar to Theorem 3 obtained by the projection method appear in Stigler (1972).

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