The Annals of Statistics
1973, Vol. 1, No. 1, 67-73

LIMITING DISTRIBUTIONS OF KOLMOGOROV-SMIRNOV TYPE STATISTICS UNDER THE ALTERNATIVE[1]

BY M. RAGHAVACHARI

Indian Institute of Management, Ahmedabad

Let $X_1, X_2, \cdots$ be a sequence of independent and identically distributed random variables with the common distribution being uniform on [0, 1]. Let $Y_1, Y_2, \cdots$ be a sequence of i.i.d. variables with continuous cdf $F(t)$ and with [0, 1] support. Let $F_n(t, \omega)$ denote the empirical distribution function based on $Y_1(\omega), \cdots, Y_n(\omega)$ and let $G_m(t, \omega)$ the empirical cdf pertaining to $X_1(\omega), \cdots, X_m(\omega)$. Let $\sup_{0 \leq t \leq 1} |F(t) - t| = \lambda$ and $D_n = \sup_{0 \leq t \leq 1} |F_n(t, \omega) - t|$. The limiting distribution of $n^{\frac{1}{2}}(D_n - \lambda)$ is obtained in this paper. The limiting distributions under the alternative of the corresponding one-sided statistic in the one-sample case and the corresponding Smirnov statistics in the two-sample case are also derived. The asymptotic distributions under the alternative of Kuiper's statistic are also obtained.

1. Introduction and summary. Let $X_1, X_2, \cdots$ be a sequence of independently and identically distributed random variables on a probability space $(\Omega, \mathscr{A}, P)$. Assume that X_n has a uniform distribution over [0, 1]. Let $Y_1, Y_2, \cdots$ be another sequence of independent and identically distributed rv's with a continuous cdf $F(t)$. Assume further that $0 \leqq Y_n(\omega) \leqq 1$, $\omega \in \Omega$. This is actually no restriction since one can realize this by a continuous transformation. Further let the X's and Y's be mutually independent. The empirical (or sample) distribution function $F_n(t, \omega)$ corresponding to the points $Y_1(\omega), \cdots, Y_n(\omega)$ is defined for $0 \leqq t \leqq 1$, as n^{-1} times the number of $i \leqq n$ for which $Y_i(\omega) \leqq t$. Denote the empirical distribution function corresponding to the points $X_1(\omega), \cdots, X_m(\omega)$ by $G_m(t)$. The following results are well known for the case $F(t) \equiv t$, $0 \leqq t \leqq 1$.

$$P\{\omega : \sup_{0 \leqq t \leqq 1} |n^{\frac{1}{2}}(F_n(t, \omega) - t)| \leqq \alpha\} \to_{n\to\infty} \psi(\alpha) = 1 - 2 \sum_{k=1}^{\infty} (-1)^{k+1} \exp(-2k^2\alpha^2)\,, \qquad \alpha \geqq 0\,. \tag{1}$$

$$P\{\omega : \sup_{0 \leqq t \leqq 1} n^{\frac{1}{2}}(F_n(t, \omega) - t) \leqq \alpha\} \to_{n\to\infty} \xi(\alpha) = 1 - \exp(-2\alpha^2)\,, \qquad \alpha \geqq 0 \tag{2}$$

$$P\left\{\omega : \sup_{0 \leqq t \leqq 1} \left|\left(\frac{mn}{m+n}\right)^{\frac{1}{2}} (F_n'(t, \omega) - G_m(t, \omega))\right| \leqq \alpha\right\} \tag{3}$$

tends to $\psi(\alpha)$ for $\alpha \geqq 0$ as m and n tend to infinity such that $m/(m+n) \to \tau$, $0 < \tau < 1$.

$$P\left\{\omega : \sup_{0 \leqq t \leqq 1} \left(\frac{mn}{m+n}\right)^{\frac{1}{2}} (F_n(t, \omega) - G_m(t, \omega)) \leqq \alpha\right\} \to \xi(\alpha)\,, \qquad \alpha \geqq 0 \tag{4}$$

as m and n tend to infinity in such a way that $m/(m+n) \to \tau$, $0 < \tau < 1$.

Received April 8, 1971; revised May 9, 1972.

[1] This research was supported in part by the National Science Foundation under Grant GP-22595 while the author was at the Carnegie-Mellon University.

The result (1) was obtained by Kolmogorov [6] and was also later derived by Doob [4] and Donsker [3] using the theory of weak convergence of probability measures and the so-called "invariance principle." Smirnov [8], Gnedenko and Korolyuk [5] derived the distributions of (2), (3) and (4). The limiting distributions of these statistics when F is not uniform has not been obtained before. This paper is devoted to the investigation of this problem. It is shown that the limiting distributions exist and it is shown that they are distributions of some appropriate functionals of the Wiener Process. Section 2 deals with the one-sample case and Section 3 the two-sample case. Another similar statistic usually called the Kuiper statistic is also discussed in Sections 2 and 3. The proofs for the theorems are given in Section 4.

2. One-sample case. Let

$$D_n = \sup_{0\leqq t\leqq 1} |F_n(t, \omega) - t| \,. \tag{5}$$

Unless otherwise stated we shall assume throughout that $F(t)$ is not identically equal to t, $0 \leqq t \leqq 1$. Let

$$\sup_{0\leqq t\leqq 1} |F(t) - t| = \lambda \tag{6}$$

and let K be the set

$$K\colon \{t\colon 0 \leqq t \leqq 1 \ \text{ and } \ |F(t) - t| = \lambda\} \,. \tag{7}$$

Define

$$D_n^* = \sup_{0\leqq t\leqq 1} |F_n(t, \omega) - t| - \lambda \,. \tag{8}$$

D_n is the Kolmogorov two-sided test statistic. Define

$$D_n^+ = \sup_{0\leqq t\leqq 1} (F_n(t, \omega) - t) \,. \tag{9}$$

Let $\sup_{0\leqq t\leqq 1} (F(t) - t) = \lambda^+$ and K^+ be the set $\{t\colon 0 \leqq t \leqq 1 \text{ and } F(t) - t = \lambda^+\}$. Let $\inf_{0\leqq t\leqq 1} (F(t) - t) = \lambda^-$ and let K^- be the set $\{t\colon 0 \leqq t \leqq 1 \text{ and } F(t) - t = \lambda^-\}$.

The following two theorems give the limiting distribution of $n^{\frac{1}{2}} D_n^*$ and the limiting distribution of $n^{\frac{1}{2}}(D_n^+ - \lambda^+)$: Throughout the paper W^{01} denotes the Gaussian random element of $\mathscr{C}(0, 1]$ with $E(W_t^{01}) = 0$ and $\operatorname{Cov}(W_s^{01} W_t^{01}) = F(s)(1 - F(t))$, $s \leqq t$.

THEOREM 1. $\lim_{n\to\infty} P\{n^{\frac{1}{2}}(D_n^+ - \lambda^+) \leqq \alpha\} = P\{\sup_{t\in K^+} W_t^{01} \leqq \alpha\}$

THEOREM 2. $\lim_{n\to\infty} P(n^{\frac{1}{2}} D_n^* \leqq \alpha) = P(\sup_{t\in K_1} W_t^{01} \leqq \alpha;\ \inf_{t\in K_2} W_t^{01} \geqq -\alpha)$ *where the sets* K_1 *and* K_2 *are defined by* (23) *and* (24).

Kuiper [7] proposed a modification of Kolmogorov statistic as follows. He defined

$$V_n = \sup_{0\leqq t\leqq 1} (F_n(t, \omega) - t) - \inf_{0\leqq t\leqq 1} (F_n(t, \omega) - t) \tag{10}$$

and obtained the limiting distribution of $n^{\frac{1}{2}} V_n$ when $F(t) \equiv t$, $0 \leqq t \leqq 1$. The following theorem gives the limiting distribution of V_n when $F(t) \not\equiv t$.

THEOREM 3. $\lim_{n\to\infty} P\{n^{\frac{1}{2}}(V_n - \lambda^+ + \lambda^-) \leqq \alpha\}$
$= P\{(\sup_{t\in K^+} W_t^{01} - \inf_{t\in K^-} W_t^{01}) \leqq \alpha\}$.

3. Two-sample case. Let

$$D_{m,n} = \sup_{0\leqq t\leqq 1} |F_n(t, \omega) - G_m(t, \omega)| - \lambda \, , \tag{11}$$

$$D^+_{m,n} = \sup_{0\leqq t\leqq 1} (F_n(t, \omega) - G_m(t, \omega)) - \lambda^+ \, , \tag{12}$$

$$V_{m,n} = \sup_{0\leqq t\leqq 1} (F_n(t, \omega) - G_m(t, \omega)) - \inf_{0\leqq t\leqq 1} (F_n(t, \omega) - G_m(t, \omega)) - \lambda^+ + \lambda^- \, . \tag{13}$$

(11) and (12) are the usual Kolmogorov–Smirnov two-sample statistics and (13) is the Kuiper's two-sample statistic. Let K, K^+, K^-, λ, λ^+ and λ^- be defined as in Section 2. Let us assume throughout this section that m and n tend to infinity in such a way that $m/(m+n) \to \tau$ with $0 < \tau < 1$.

We state the following theorem which gives the limiting distribution of (11), (12) and (13).

THEOREM 4.

(14) (i) $$\lim_{m\to\infty, n\to\infty} P\left\{\left(\frac{mn}{m+n}\right)^{\frac{1}{2}} D_{m,n} \leqq \alpha\right\} = P\{\sup_{t\in K_1} (\tau^{\frac{1}{2}} W_t^{01} - (1-\tau)^{\frac{1}{2}} W_t^{02}) \leqq \alpha \, ; \ \sup_{t\in K_2} ((1-\tau)^{\frac{1}{2}} W_t^{02} - \tau^{\frac{1}{2}} W_t^{01}) \leqq \alpha\}$$

(15) (ii) $$\lim_{m\to\infty, n\to\infty} P\left\{\left(\frac{mn}{m+n}\right)^{\frac{1}{2}} D^+_{m,n} \leqq \alpha\right\} = P\{\sup_{t\in K^+} (\tau^{\frac{1}{2}} W_t^{01} - (1-\tau)^{\frac{1}{2}} W_t^{02}) \leqq \alpha\}$$

(16) (iii) $$\lim_{m\to\infty, n\to\infty} P\left\{\left(\frac{mn}{m+n}\right)^{\frac{1}{2}} V_{m,n} \leqq \alpha\right\} = P\{[\sup_{t\in K^+} (\tau^{\frac{1}{2}} W_t^{01} - (1-\tau)^{\frac{1}{2}} W_t^{02}) - \inf_{t\in K^-} (\tau^{\frac{1}{2}} W_t^{01} - (1-\tau)^{\frac{1}{2}} W_t^{02})] \leqq \alpha\}$$

where W^{02} is the Brownian bridge and W^{01} and W^{02} are independent.

Kuiper [7] obtained the limiting distribution of $V_{m,n}$ (i.e., the case $m = n$) in the "null" case ($F(t) \equiv t$) and the following Theorem obtains the limiting distribution for general m, n. An adaptation of the proof of Theorem 5 also shows directly the equality of limiting distributions in (1) and (3) and the equality of limiting distributions in (2) and (4).

THEOREM 5. *For the case $F(t) \equiv t$, $0 \leqq t \leqq 1$,*

$$\lim_{m\to\infty, n\to\infty} P\left\{\left(\frac{mn}{m+n}\right)^{\frac{1}{2}} V_{m,n} \leqq \alpha\right\} = P\{(\sup_{0\leqq t\leqq 1} W_t^{02} - \inf_{0\leqq t\leqq 1} W_t^{02}) \leqq \alpha\} \, , \qquad \alpha \geqq 0$$

with W_t^{02} the same as in Theorem 4.

4. Proofs. The following lemma is needed to prove some of the theorems.

LEMMA 1. *Let ε and δ be any two positive numbers and $t^* \in (0, 1)$, then for all sufficiently large n we have*

$$P\{\omega : \sup_{t:|F(t)-F(t^*)|\leqq\delta} n^{\frac{1}{2}}|F_n(t, \omega) - F(t) - F_n(t^*, \omega) + F(t^*)| \geqq \varepsilon\} \leqq c\delta^2 \tag{17}$$

where c is a constant independent of δ, n and t^*.

PROOF. This follows easily from Billingsley [1] pages 106–108.

PROOF OF THEOREM 1. Let $Z_n^+ = \sup_{t\in K^+}(F_n(t, \omega) - F(t))$ and define $D_n^{++} = D_n^+ - \lambda^+ - Z_n^+$. First we will show that $n^{\frac{1}{2}}D_n^{++} \to_P 0$ as $n \to \infty$. From the definition of K^+, it follows easily that K^+ is a compact subset of $[0, 1]$. For any positive integer k, there exist points $t_1, t_2, \cdots, t_\gamma$ in K^+ ($\gamma \leqq k$) such that $K^+ \subset \bigcup_{i=1}^{\gamma} S(t_i, k)$ where

$$S(t_i, k) = \{t : 0 \leqq t \leqq 1 \text{ and } |F(t) - F(t_i)| < k^{-1}\}.$$

Denote by $M_{k,\gamma}$ the set $\bigcup_{i=1}^{\gamma} S(t_i, k)$, its closure by $\bar{M}_{k,\gamma}$ and its complement by $M_{k,\gamma}^c$. Choose any $\varepsilon > 0$. We have

$$\begin{aligned} P\{n^{\frac{1}{2}}D_n^{++} \geqq \varepsilon\} &= P\{n^{\frac{1}{2}}(\sup_{0\leqq t\leqq 1}(F_n(t, \omega) - t) - \lambda^+ - Z_n^+) \geqq \varepsilon\} \\ &\leqq P\{n^{\frac{1}{2}}(\sup_{t\in\bar{M}_{k,\gamma}}(F_n(t, \omega) - t) - \lambda^+ - Z_n^+) \geqq \varepsilon\} \\ &\quad + P\{n^{\frac{1}{2}}(\sup_{t\in M_{k,\gamma}^c}(F_n(t, \omega) - t) - \lambda^+ - Z_n^+) \geqq \varepsilon\}. \end{aligned} \tag{18}$$

Now

$$\begin{aligned} &P\{n^{\frac{1}{2}}(\sup_{t\in\bar{M}_{k,\gamma}}(F_n(t, \omega) - t) - \lambda^+ - Z_n^+) \geqq \varepsilon\} \\ &\quad\leqq \textstyle\sum_{i=1}^{\gamma} P\{\sup_{t\in\overline{S(t_i,k)}} n^{\frac{1}{2}}((F_n(t, \omega) - t) - \lambda^+ - Z_n^+) \geqq \varepsilon\} \\ &\quad\leqq \textstyle\sum_{i=1}^{\gamma} P\{\sup_{t\in\overline{S(t_i,k)}} n^{\frac{1}{2}}((F_n(t; \omega) - t) - \lambda^+ \\ &\qquad - F_n(t_i, \omega) + F(t_i)) \geqq \varepsilon\}. \end{aligned} \tag{19a}$$

Writing $F_n(t, \omega) - t = F_n(t, \omega) - F(t) + F(t) - t$ and noting that $F(t) - t \leqq F(t_i) - t_i = \lambda^+$, we have

$$\begin{aligned} &\leqq \textstyle\sum_{i=1}^{\gamma} P\{\sup_{t\in\overline{S(t_i,k)}} n^{\frac{1}{2}}(F_n(t, \omega) - F(t) - F_n(t_i, \omega) + F(t_i)) \geqq \varepsilon\} \\ &\leqq \textstyle\sum_{i=1}^{\gamma} ck^{-2}, \qquad \text{for all sufficiently large } n, \text{ by Lemma 1} \\ &\leqq c/k, \qquad \text{for all sufficiently large } n. \end{aligned} \tag{19b}$$

Note that on $M_{k,\gamma}^c$, $F(t) - t$ is bounded above by a number ρ with $0 < \rho < \lambda^+$. This follows from the continuity of $F(t)$, compactness of K^+ and the fact that $K^+ \subset M_{k,\gamma}$

Choose an $\eta > 0$ such that $\eta < \lambda^+ - \rho$. By the Glivenko–Cantelli Theorem we have for all sufficiently large n,

$$\begin{aligned} &\sup_{t\in M_{k,\gamma}^c}(F_n(t, \omega) - t) - \lambda^+ \\ &\qquad < \sup_{t\in M_{k,\gamma}^c}(F(t) - t) - \lambda^+ + \eta < \rho - \lambda^+ + \eta < 0 \end{aligned}$$

with probability one. Thus it follows that

$$\lim_{n\to\infty} P\{\sup_{t\in M_{k,\gamma}^c} n^{\frac{1}{2}}((F_n(t, \omega) - t) - Z_n^+ - \lambda^+) \geqq \varepsilon\} = 0.$$

Hence

$$\limsup_{n\to\infty} P\{n^{\frac{1}{2}}D_n^{++} \geqq \varepsilon\} \leqq c/k\,. \tag{20}$$

Since k is an arbitrary positive integer, $P\{n^{\frac{1}{2}}D_n^{++} \geqq \varepsilon\} \to 0$ as $n \to \infty$. Further

$$\begin{aligned} D_n^{++} &= \sup_{0\leqq t\leqq 1}(F_n(t,\omega) - t) - \lambda^+ - \sup_{t\in K^+}(F_n(t,\omega) - F(t)) \\ &\geqq \sup_{t\in K^+}(F_n(t,\omega) - t) - \lambda^+ - \sup_{t\in K^+}(F_n(t,\omega) - F(t)) \\ &= \sup_{t\in K^+}(F_n(t,\omega) - t - F(t) + t) - \sup_{t\in K^+}(F_n(t,\omega) - F(t)) \\ &= 0\,. \end{aligned}$$

Thus (20) implies that $n^{\frac{1}{2}}D_n^{++} \to_P 0$. $n^{\frac{1}{2}}Z_n^+$, however, has a limiting distribution as $n \to \infty$ and the limiting distribution is the same as the distribution of $\sup_{t\in K^+} W_t^{01}$. This is a consequence of Donsker's Theorem [2] and the well-known convergence of the empirical distribution process to the appropriate Wiener process. See, for example, Billingsley [1]. This completes the proof of Theorem 1.

PROOF OF THEOREM 2. For brevity, we give only the main ideas of the proof. Let

$$Z_n = \sup_{t\in K}|F_n(t,\omega) - t| - \lambda \tag{21}$$

so that

$$D_n^* - Z_n = \sup_{0\leqq t\leqq 1}|F_n(t,\omega) - t| - \sup_{t\in K}|F_n(t,\omega) - t|\,. \tag{22}$$

We will show first that $n^{\frac{1}{2}}(D_n^* - Z_n) \to 0$ as $n \to \infty$. Define the sets K_1 and K_2, (one of them possibly empty)

$$K_1 = \{t\colon t\in K \text{ and } F(t) - t = \lambda\} \tag{23}$$

$$K_2 = \{t\colon t\in K \text{ and } F(t) - t = -\lambda\}\,. \tag{24}$$

By the Glivenko–Cantelli Theorem $\sup_{0\leqq t\leqq 1}|F_n(t,\omega) - F(t)| \to 0$ with probability one as $n \to \infty$. Thus by Egoroff's Theorem, for any given small $\delta > 0$ and $\eta > 0$, there exists a set A with $P(A) \geqq 1 - \delta$ and a positive integer N such that for all $n \geqq N$, $|F_n(t,\omega) - F(t)| < \eta$ for all $t \in [0, 1]$ and all $\omega \in A$. Choose any $\varepsilon > 0$. Now for $n \geqq N$

$$\begin{aligned} P\{n^{\frac{1}{2}}(D_n^* - Z_n) \geqq \varepsilon\} &= P\{[n^{\frac{1}{2}}(D_n^* - Z_n) \geqq \varepsilon] \cap A\} \\ &\quad + P\{[n^{\frac{1}{2}}(D_n^* - Z_n) \geqq \varepsilon] \cap A^c\}\,. \end{aligned} \tag{25}$$

As in the proof of Theorem 1, define the sets $M_{k,\gamma,1}$ and $M_{k,\gamma,2}$ corresponding to K_1 and K_2 respectively. Note that, by choosing k large, we have for all sufficiently large n and all $\omega \in A$, $F_n(t,\omega) - t > 0$ for all $t \in \bar{M}_{k,\gamma,1}$ and $F_n(t,\omega) - t < 0$ for all $t \in \bar{M}_{k,\gamma,1}$. Thus

$$\begin{aligned} &P\{[n^{\frac{1}{2}}(D_n^* - Z_n) \geqq \varepsilon] \cap A\} \\ &\quad \leqq P\{n^{\frac{1}{2}}[\sup_{t\in\bar{M}_{k,\gamma,1}}(F_n(t,\omega) - t) - \sup_{t\in K_1}(F_n(t,\omega) - t)] \geqq \varepsilon\} \\ &\qquad + P\{n^{\frac{1}{2}}[\sup_{t\in\bar{M}_{k,\gamma,2}}(t - F_n(t,\omega) - \sup_{t\in K_2}(t - F_n(t,\omega))] \geqq \varepsilon\} \\ &\qquad + P\{n^{\frac{1}{2}}[\sup_{t\in(M_{k,\gamma,1},\cup M_{k,\gamma,2})^c}|F_n(t,\omega) - t| \\ &\qquad - \sup_{t\in K}|F_n(t,\omega) - t|] \geqq \varepsilon\}\,. \end{aligned} \tag{26}$$

Using arguments similar to the ones employed in the proof of Theorem 1, we can show (details omitted) that (26) $\to 0$ as $n \to \infty$. Since $P(A^c) \leqq \delta$ and δ is arbitrary, we have that $P\{n^{\frac{1}{2}}(D_n^* - Z_n) \geqq \varepsilon\} \to 0$ as $n \to \infty$. Since $D_n^* \geqq Z_n$, it follows that $n^{\frac{1}{2}}(D_n^* - Z_n) \to_P 0$ as $n \to \infty$. We have for $n \geqq N$,

$$P\{Z_n \leqq \alpha\} = P\{(Z_n \leqq \alpha) \cap A\} + P\{(Z_n \leqq \alpha) \cap A^c\} . \tag{27}$$

Consider

$$\begin{aligned} P\{(Z_n \leqq \alpha) \cap A\} &= P\{[\max(n^{\frac{1}{2}} \sup_{t \in K_1} (F_n(t) - F(t)); \\ &\qquad n^{\frac{1}{2}} \sup_{t \in K_2} (F(t) - F_n(t)) \leqq \alpha] \cap A\} \\ &= P\{(Z_n^* \leqq \alpha) \cap A\} , \quad \text{say.} \end{aligned} \tag{28}$$

It follows from (27) and (28) that

$$\begin{aligned} \lim_{n\to\infty} \sup P\{Z_n \leqq \alpha\} &\leqq P\{\sup_{t \in K_1} W_t^{01} \leqq \alpha, \sup_{t \in K_2} (-W_t^{01}) \leqq \alpha\} + P(A^c) \\ &\leqq P\{\sup_{t \in K_1} W_t^{01} \leqq \alpha, \sup_{t \in K_2} (-W_t^{01}) \leqq \alpha\} + \delta . \end{aligned}$$

Further

$$\begin{aligned} P\{Z_n \leqq \alpha\} &\geqq P\{(Z_n \leqq \alpha) \cap A\} \\ &= P\{(Z_n^* \leqq \alpha) \cap A\} \\ &= P(Z_n^* \leqq \alpha) + P(A) - P\{(Z_n^* \leqq \alpha) \cup A\} . \end{aligned}$$

Hence

$$\begin{aligned} &\lim_{n\to\infty} \inf P\{Z_n \leqq \alpha\} \\ &\qquad \geqq P\{\sup_{t \in K_1} W_t^{01} \leqq \alpha, \sup_{t \in K_2} (-W_t^{01}) \leqq \alpha\} + P(A) - 1 \\ &\qquad \geqq P\{\sup_{t \in K_1} W_t^{01} \leqq \alpha, \sup_{t \in K_2} (-W_t^{01}) \leqq \alpha\} - \delta . \end{aligned}$$

Since δ is arbitrary, we have

$$\lim_{n\to\infty} P\{Z_n \leqq \alpha\} = P\{\sup_{t \in K_1} W_t^{01} \leqq \alpha, \sup_{t \in K_2} (-W_t^{01}) \leqq \alpha\} .$$

This completes the proof of Theorem 2.

PROOF OF THEOREM 3. The proof of this theorem is omitted as the proof parallels closely that of Theorem 1.

INDICATION OF THE PROOF OF THEOREM 4. Define

$$\begin{aligned} Z_{m,n} &= \sup_{t \in K} |F_n(t, \omega) - G_m(t, \omega)| - \lambda \\ Z_{m,n}^+ &= \sup_{t \in K^+} (F_n(t, \omega) - F(t) - G_m(t, \omega) + t) \\ V_{m,n}^* &= \sup_{t \in K^+} (F_n(t, \omega) - F(t) - G_m(t, \omega) + t) \\ &\qquad - \inf_{t \in K^-} (F_n(t, \omega) - F(t) - G_m(t, \omega) + t) . \end{aligned}$$

Using ideas similar to the ones used in proving Theorems 1 and 2 we can show first that

$$\begin{aligned} (mn/(m+n))^{\frac{1}{2}}(D_{m,n} - Z_{m,n}) &\to_P 0 \\ (mn/(m+n))^{\frac{1}{2}}(D_{m,n}^+ - Z_{m,n}^+) &\to_P 0 \\ (mn/(m+n))^{\frac{1}{2}}(V_{m,n} - V_{m,n}^*) &\to_P 0 . \end{aligned}$$

The equality of the limiting distributions of $(mn/(m+n))^{\frac{1}{2}}Z_{m,n}$, $(mn/(m+n))^{\frac{1}{2}}Z^{+}_{m,n}$ and $(mn/(m+n))^{\frac{1}{2}}V^{*}_{m,n}$ and the distributions given on the right of (14), (15) and (16) respectively follows immediately.

PROOF OF THEOREM 5. We have for the case $F(t) \equiv t$, $0 \leqq t \leqq 1$,

$$V_{m,n} = \sup_{0\leqq t\leqq 1}(F_n(t,\omega) - t + t - G_m(t,\omega)) - \inf_{0\leqq t\leqq 1}(F_n(t,\omega) - t + t - G_m(t,\omega)),$$

so that

$$\left(\frac{mn}{m+n}\right)^{\frac{1}{2}} V_{m,n} = \sup_{0\leqq t\leqq 1}\left(\left(\frac{m}{m+n}\right)^{\frac{1}{2}} n^{\frac{1}{2}}(F_n(t,\omega) - t) - \left(\frac{n}{m+n}\right)^{\frac{1}{2}} m^{\frac{1}{2}}(G_m(t,\omega) - t)\right) - \inf_{0\leqq t\leqq 1}\left(\left(\frac{m}{m+n}\right)^{\frac{1}{2}} n^{\frac{1}{2}}(F_n(t,\omega) - t) - \left(\frac{n}{m+n}\right)^{\frac{1}{2}} m^{\frac{1}{2}}(G_m(t,\omega) - t)\right).$$

Thus we have

$$P\left\{\left(\frac{mn}{m+n}\right)^{\frac{1}{2}} V_{m,n} \leqq \alpha\right\} \to P\{\sup_{0\leqq t\leqq 1}(\tau^{\frac{1}{2}}W_t^{02} - (1-\tau)^{\frac{1}{2}}W_t^{03}) - \inf_{0\leqq t\leqq 1}(\tau^{\frac{1}{2}}W_t^{02} - (1-\tau)^{\frac{1}{2}}W_t^{03}) \leqq a\}$$

where W^{02} and W^{03} are independent Brownian bridges. It can be verified that $\tau^{\frac{1}{2}}W^{02} - (1-\tau)^{\frac{1}{2}}W^{03}$ is also the Brownian bridge. This proves the theorem.

Acknowledgment. The author is grateful to Prof. J. Wolfowitz for his comments on an earlier draft of this paper.

REFERENCES

[1] BILLINGSLEY, P. (1968). *Convergence of Probability Measures*. Wiley, New York.

[2] DONSKER, N. (1951). An invariance principle for certain probability limit theorems. *Mem. Amer. Math. Soc.* **6**.

[3] DONSKER, M. (1952). Justification and extension of Doob's heuristic approach to the Kolmogorov-Smirnov theorems. *Ann. Math. Statist.* **23** 277–281.

[4] DOOB, J. L. (1949). Heuristic approach to the Kolmogorov-Smirnov theorems. *Ann. Math. Statist.* **20** 393–403.

[5] GNEDENKO, B. V. and KOROLYUK (1951). On the maximum deviation between two empirical distributions. *Dokl. Akad. Nauk. SSSR* **80** 525–528.

[6] KOLMOGOROV, A. N. (1933). Sulla determinzione empirica di una legge di distribuzione. *Giorn. Ist. Ital. Attuari* **4** 83–91.

[7] KUIPER (1960). Tests concerning random points on a circle. *Proc. Konink. Ned. Akad. Van Wettenschaffen A* **63** 38–47; *Indag. Math.* **22** 38–47.

[8] SMIRNOV, N. (1939). On the estimation of the discrepancy between empirical curves of distribution for two independent samples. *Bull. Math. Univ. Moscow* **2** No. 2, 3–14.

INDIAN INSTITUTE OF MANAGEMENT
VASTRAPUR
AHMEDABAD-15, INDIA