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ROBUST FIXED SIZE CONFIDENCE PROCEDURES FOR A RESTRICTED PARAMETER SPACE

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Robust fixed size confidence procedures are derived for the location parameter θ of a sample of N i.i.d. observations of a scalar random variable Z with CDF $F(z - \theta)$. Here, θ is restricted to a closed interval Ω and the uncertainty in F is modeled by an uncertainty class $\mathscr{F}$. These robust confidence procedures are, in turn, based on the solution of a related robust minimax decision problem that is characterized by a zero–one loss function, the parameter space Ω and the uncertainty class $\mathscr{F}$. Sufficient conditions for the existence of robust minimax and robust median-minimax estimators are delineated. Sufficient conditions on $\mathscr{F}$ are obtained such that (i) both types of rules are minimax within the class of nonrandomized odd monotone procedures and (ii) subject to additional conditions, both types of rules are globally minimax admissible Bayes procedures. The paper concludes with an examination of the asymptotic behavior of the robust median-minimax estimators and their extensions to the robust α-minimax rules, which are based on the α-trimmed mean.

1. Introduction. The standard statement of a minimax location parameter estimation problem includes as given: a parameter space Ω, a space of actions $\mathscr{A}$, a loss function L defined on $\mathscr{A} \times \Omega$ and a CDF F. If the underlying CDF is imprecisely known, then this standard minimax decision model must be reformulated to account for this additional uncertainty. Statistical decision rules which are applicable in this more general problem setting are referred to as robust procedures.

This paper considers robust fixed size confidence procedures for a restricted parameter space. These robust confidence procedures are based, in turn, on the solution of a related robust minimax decision problem.

Let $\mathbf{Z}$ denote a vector of N i.i.d. observations of a scalar random variable with CDF $F(z - \theta)$, where $F \in \mathscr{F}$, a given uncertainty class. Let $\Omega = \mathscr{A} = [-d, d]$ and define a zero–one loss function L on $\mathscr{A} \times \Omega$,

$$L(a, \theta) = \begin{cases} 0, & |a - \theta| \le e, \\ 1, & |a - \theta| > e, \end{cases} \tag{1.1}$$

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where $e > 0$ is given. Further, let $R(\delta, \theta, F) = E[L(\delta, \theta)|\theta, F]$ denote the risk function of the decision rule δ given $\theta \in \Omega$ and $F \in \mathscr{F}$.

DEFINITION 1.1. An estimator δ^* is said to be a robust minimax estimator for θ, if for all δ,

$$\sup_{\substack{\theta\in\Omega \\ F\in\mathscr{F}}} R(\delta^*, \theta, F) \le \sup_{\substack{\theta\in\Omega \\ F\in\mathscr{F}}} R(\delta, \theta, F).$$

Based on these definitions and assumptions, we seek a robust minimax estimator δ^* for θ. For brevity, we restrict our consideration to the case when d/e is an integer greater than or equal to 2.

OBSERVATION 1.1. The connection between the robust minimax rule $\delta^*(\mathbf{Z})$ and a robust fixed size confidence procedure is obtained by noting that

$$C^*(\mathbf{Z}) = [\delta^*(\mathbf{Z}) - e, \delta^*(\mathbf{Z}) + e]$$

can be interpreted as a robust confidence procedure of size $2e$ which has the highest confidence coefficient $\inf_{\theta, F} P_{\theta, F}[\theta \in C^*(\mathbf{Z})]$.

The remainder of this paper is organized as follows.

Section 2 reviews the solution to the related minimax estimation problem where $F = \mathscr{N}(0, \sigma^2)$ and σ is given. These results provide the basis for the solution to the robust minimax estimation problem where $\mathscr{F} = \{F = \mathscr{N}(0, \sigma^2): \sigma \le \sigma_u\}$.

Section 3 extends these robust minimax results to uncertainty classes which contain non-Gaussian, asymmetric and discontinuous CDF's. Robust median-minimax rules are introduced and evaluated. Section 3 concludes with an examination of the asymptotic behavior of robust median-minimax rules and their extensions to robust α-minimax rules, which are based on the α-trimmed mean.

Appendix 1 contains proofs or sketches of proofs for the main results in Sections 2 and 3.

2. Robust minimax rules and Gaussian uncertainty classes.

2.1. *Preliminary minimax results.* Zeytinoglu and Mintz (1984) addresses the related minimax estimation problem where $F = \mathscr{N}(0, \sigma^2)$ and σ is given. The main result requires Definition 2.1 and is summarized by Theorem 2.1.

DEFINITION 2.1. Let $\mathscr{C}$ denote the class of nonrandomized odd monotone nondecreasing decision rules δ: $E^1 \to \mathscr{A}$. Let $\Delta \subset \mathscr{C}$ denote the set of rules $\delta(t)$

defined for $t \geq 0$ by

$$(2.1) \qquad \delta(t) = \begin{cases} d - e, & c + a_n + 2ne \leq t, \\ \vdots & \\ t - a_2, & c + a_2 + 2e \leq t < c + a_2 + 4e, \\ 2e + c, & c + a_1 + 2e \leq t < c + a_2 + 2e, \\ t - a_1, & c + a_1 \leq t < c + a_1 + 2e, \\ c, & c \leq t < c + a_1, \\ t, & 0 \leq t < c, \end{cases}$$

where $0 \leq a_1 \leq a_2 \leq \cdots \leq a_n < \infty$, $d = (2n + 1)e + c$ and c equals zero (e) if d is an odd (even) multiple of e. [Due to the existing symmetry, all function definitions are stated for nonnegative arguments.]

THEOREM 2.1. *Let L denote the loss function* (1.1). *If $Z \in \mathcal{N}(\theta, \sigma^2)$, where σ is given, then for any $N \geq 1$ there exists a (globally) minimax admissible rule $\delta^* \in \Delta$ which is Bayes with respect to a least favorable prior distribution λ^*. Further, δ^* depends on $\mathbf{Z}$ through the sample mean $\bar{Z}_N$.*

For the proof, see Zeytinoglu and Mintz (1984).

$R(\delta, \theta, F)$, $R(\delta^*, \theta, F)$ and λ^* have the following characteristics:

1. If $\delta \in \Delta$, then $R(\delta, \theta, F)$ is a piecewise constant function of θ over the sets of a finite partition of Ω.
2. The minimax rules δ^* are "almost" equalizer rules, in the sense that the nondegenerate piecewise constant segments of the risk function are equalized at the minimax risk M.
3. λ^* is defined by a density function which is symmetric and piecewise constant.

2.2. *Robust minimax rules.*

DEFINITION 2.2. Let

$$(2.2) \qquad \mathcal{F} = \{F = \mathcal{N}(0, \sigma^2) : \sigma \leq \sigma_u\}$$

denote an uncertainty class of Gaussian distributions.

DEFINITION 2.3. The CDF $F_u = \mathcal{N}(0, \sigma_u^2)$ defines the upper-envelope of $\mathcal{F}$ (2.2) in the sense that $F(x) \leq F_u(x)$ for all $F \in \mathcal{F}$ and $x \leq 0$.

Theorem 2.2, which is the main result of this section, extends the results of Theorem 2.1 to the robust minimax estimation problem.

THEOREM 2.2. *Let $\mathcal{F}$ denote the uncertainty class* (2.2) *with upper-envelope $F_u = \mathcal{N}(0, \sigma_u^2)$. Let δ^* denote the minimax rule obtained through Theorem* 2.1 *based on a sample size N and* CDF *F_u. There exists a bound $B(d/e, N, F_u)$ such* *that if $e \geq B$, then δ^* is a robust minimax admissible Bayes rule in the sense of Definition* 1.1. *Further, δ^* depends on $\mathbf{Z}$ through $\bar{Z}_N$.*

For the proof, see Appendix 1.

EXAMPLE 2.1. Let $d = 3e$, $e = 0.1$, $N = 49$, $\sigma_u = 2$, $F_u = \mathscr{N}(0, \sigma_u^2)$ and $G = \mathscr{N}(0, \sigma_u^2/N)$. Applying Theorem 2.2, the minimax rule $\delta^*(\bar{Z}_N)$ and the risk function $R(\delta^*, \theta, F_u)$ corresponding to F_u are

$$\delta^*(\bar{Z}_N) = \begin{cases} 2e, & a_1 + 2e \leq \bar{Z}_N, \\ \bar{Z}_N - a_1, & a_1 \leq \bar{Z}_N < a_1 + 2e, \\ 0, & 0 \leq \bar{Z}_N < a_1, \end{cases} \tag{2.3}$$

$$R(\delta^*, \theta, F_u) = \begin{cases} G(a_1 - e), & e < \theta \leq 3e, \\ G(-a_1 - e), & \theta = e, \\ 2G(-a_1 - e), & 0 \leq \theta < e, \end{cases} \tag{2.4}$$

where a_1 satisfies

$$G(a_1 - e) = 2G(-a_1 - e). \tag{2.5}$$

In this example, $a_1 = 0.092$ and the corresponding minimax risk is 0.49. The bound B, which is derived in Appendix 1, is

$$B(d/e, N, F_u) = -\left(1/\sqrt{4N}\right)F_u^{-1}(1/4) = 0.0966.$$

Here, $e \geq B$, and thus δ^* (2.3) is a robust minimax rule.

3. Robust minimax rules and non-Gaussian uncertainty classes.

3.1. *Preliminaries.* This section extends the robust minimax results of Section 2 to uncertainty classes which contain non-Gaussian, asymmetric and discontinuous CDF's.

DEFINITION 3.1. Let $\mathscr{F}$ denote an uncertainty class with upper-envelope F_u,

$$\mathscr{F} = \{F\colon F(x^-) \leq F_u(x),\ x \leq 0; \text{ and } F(x) \geq F_u(x),\ x > 0\}, \tag{3.1}$$

where F_u has a density function which is unimodal and symmetric about zero. [$F(x^-)$ denotes the left-hand limit.]

REMARK 3.1. We allow F_u to be substochastic, i.e., F_u can have less than unit probability mass. Thus, the usual ε-contamination models can be represented by $\mathscr{F}$ (3.1).

The main results of this section are based on Theorem 3.1, which addresses the existence and construction of $\mathscr{C}$-minimax and minimax rules for single-sample decision problems. [A rule is (robust) $\mathscr{D}$-minimax if it is (robust) minimax within the class $\mathscr{D}$. A rule is $\mathscr{D}$-Bayes if it is Bayes within the class $\mathscr{D}$. A rule is $\mathscr{D}$-admissible if it is admissible within the class $\mathscr{D}$.]

THEOREM 3.1. *Let $N = 1$. If the* CDF F *has a density function which is unimodal and symmetric about zero, then there exists a $\mathscr{C}$-minimax rule $\delta^* \in \Delta$. Further, if F possesses a (strictly) monotone likelihood ratio, then δ^* is a minimax (admissible) Bayes rule.*

For the proof, see Zeytinoglu and Mintz (1984).

3.2. *The single-sample case.* Theorem 3.2 extends the results of Theorem 3.1 to the single-sample robust $\mathscr{C}$-minimax estimation problem.

THEOREM 3.2. *Let $N = 1$, $\mathscr{F}$ denote the uncertainty class* (3.1) *with upper-envelope F_u and δ^* denote the $\mathscr{C}$-minimax rule obtained through Theorem* 3.1 *based on* CDF F_u. *There exists a bound $B(d/e, N = 1, F_u)$ such that if $e \geq B$, then δ^* is a robust $\mathscr{C}$-minimax rule. Further, if F_u possesses a (strictly) monotone likelihood ratio, then δ^* is a robust minimax (admissible) Bayes rule.*

For the proof, see the sketch in Appendix 1.

EXAMPLE 3.1 (An ε-contamination model). Let $d = 3e$ and $\mathscr{F}$ denote the uncertainty class

$$\mathscr{F} = \{F\colon F = (1 - \varepsilon)\Phi + \varepsilon H\}, \tag{3.2}$$

where $\Phi = \mathscr{N}(0, 1)$, the CDF H is symmetric about zero and $0 < \varepsilon < 1/2$. The corresponding (substochastic) upper-envelope is

$$F_u = (1 - \varepsilon)\Phi + \varepsilon/2. \tag{3.3}$$

In this example, $B(d/e, N = 1, F_u)$ is

$$\begin{aligned} B(d/e, N = 1, F_u) &= -(1/2)F_u^{-1}(1/4) \\ &= -(1/2)\Phi^{-1}((1 - 2\varepsilon)/(4 - 4\varepsilon)). \end{aligned} \tag{3.4}$$

[See Appendix 1.]

Applying Theorem 3.2, the $\mathscr{C}$-minimax rule $\delta^*(Z)$ and the risk function $R(\delta^*, \theta, F_u)$ corresponding to F_u are

$$\delta^*(Z) = \begin{cases} 2e, & a_1 + 2e \leq Z, \\ Z - a_1, & a_1 \leq Z < a_1 + 2e, \\ 0, & 0 \leq Z < a_1, \end{cases} \tag{3.5}$$

$$R(\delta^*, \theta, F_u) = \begin{cases} F_u(a_1 - e), & e < \theta \leq 3e, \\ F_u(-a_1 - e), & \theta = e, \\ 2F_u(-a_1 - e), & 0 \leq \theta < e, \end{cases} \tag{3.6}$$

where a_1 satisfies

$$F_u(a_1 - e) = 2F_u(-a_1 - e) \tag{3.7}$$

or, equivalently,

$$\Phi(a_1 - e) = 2\Phi(-a_1 - e) + \varepsilon/(2 - 2\varepsilon). \tag{3.8}$$

Thus, if $e \geq B$, then δ^* (3.5) is a robust $\mathscr{C}$-minimax rule for this ε-contamination model.

This solution is easily extended to other values of d/e and nominal distributions. The required calculations include the computation of the vector $\mathbf{a}$ which parametrizes the underlying $\mathscr{C}$-minimax rule δ^* and the computation of the bound $B(d/e, N = 1, F_u)$—which are each readily obtained by means of a Newton–Raphson algorithm.

3.3. *The multisample case.* This section extends the robust $\mathscr{C}$-minimax results of Theorem 3.2 to the multisample problem by restricting the class of estimators to rules of the form $\delta(T(\mathbf{Z}))$, where $\delta \in \mathscr{C}$, T: $E^N \to E^1$ and $T(\mathbf{Z})$ possesses a density function which is unimodal and symmetric about θ. Examples of candidate T statistics include the sample mean, the sample median and other symmetric linear combinations of order statistics. In the remainder of this section, we consider the sample median. Since the sample median or other symmetric linear combinations of order statistics have no global optimality properties, we compare, by example, the performance of these restricted decision rules to an alternative—the highest posterior density (HPD) credible set [Berger (1985)].

DEFINITION 3.2. Let Z_M denote the median of the N observations $\mathbf{Z}$. [If N is even, $Z_M = (Z_{[N/2]} + Z_{[(N/2)+1]})/2$.] The decision rule $\delta^*(Z_M)$, defined by the composition $\delta^* \circ Z_M$, is said to be a median-minimax estimator for θ, if δ^* is a minimax rule in the usual sense. The respective definitions of robust median-minimax rules, $\mathscr{C}$-median-minimax rules and robust $\mathscr{C}$-median-minimax rules are obtained as before.

THEOREM 3.3. *Let $N > 1$. If the* CDF *F has a density function which is unimodal and symmetric about zero, then there exists a $\mathscr{C}$-median-minimax rule $\delta^* \in \Delta$. Further, if the* CDF *of $(Z_M - \theta)$ possesses a (strictly) monotone likelihood ratio, then δ^* is a median-minimax (median-admissible) median-Bayes rule.*

For the proof, see the sketch in Appendix 1.

EXAMPLE 3.2. Let $d = 3e$, $e = 0.2133$, $N = 3$ and F denote the CDF of the double exponential distribution

$$F(x) = \begin{cases} 1 - (1/2)\exp(-x), & x \geq 0, \\ (1/2)\exp(x), & x < 0. \end{cases} \tag{3.9}$$

Let F' denote the CDF of the centered sample median $Z_M - \theta = T(\mathbf{Z}) - \theta$. In this example, $F'(t) = F^2(t)(3 - 2F(t))$. Applying Theorem 3.3, the $\mathscr{C}$-median-minimax rule $\delta^*(T)$ and risk function $R(\delta^*, \theta, F)$ are

$$\delta^*(T) = \begin{cases} 2e, & a_1 + 2e \le T, \\ T - a_1, & a_1 \le T < a_1 + 2e, \\ 0, & 0 \le T < a_1, \end{cases} \tag{3.10}$$

$$R(\delta^*, \theta, F) = \begin{cases} F'(a_1 - e), & e < \theta \le 3e, \\ F'(-a_1 - e), & \theta = e, \\ 2F'(-a_1 - e), & 0 \le \theta < e, \end{cases} \tag{3.11}$$

where a_1 satisfies

$$F'(a_1 - e) = 2F'(-a_1 - e). \tag{3.12}$$

In this example, $a_1 = e = 0.2133$, and the corresponding $\mathscr{C}$-median-minimax risk is 0.50.

It is illustrative to compare the $\mathscr{C}$-median-minimax risk of δ^* with the maximum risk of the decision rule δ_0 defined by the midpoint of the HPD credible set of size $2e$ with respect to the uniform prior distribution on Ω. The decision rule δ_0 is readily expressed in terms of the order statistics $\{Z_{[i]}: i = 1, 2, 3\}$ and sample mean $\bar{Z}_3$ of $\mathbf{Z}$, in conjunction with $\gamma_1 = (Z_{[1]} + Z_{[2]} + e)/2$ and $\gamma_2 = (Z_{[2]} + Z_{[3]} - e)/2$. Let $\mathscr{H}$ denote the midpoint of the HPD credible set of size $2e$ with respect to the uniform prior distribution on E^1. There are two cases to consider.

CASE 1 ($Z_{[2]} \le \bar{Z}_3$).

$$\mathscr{H}(\mathbf{Z}) = \begin{cases} \bar{Z}_3, & \bar{Z}_3 \le \gamma_1, \\ \gamma_1, & Z_{[2]} \le \gamma_1 \le \bar{Z}_3, \\ Z_{[2]}, & \gamma_1 \le Z_{[2]}. \end{cases} \tag{3.13}$$

CASE 2 ($Z_{[2]} \ge \bar{Z}_3$).

$$\mathscr{H}(\mathbf{Z}) = \begin{cases} \bar{Z}_3, & \gamma_2 \le \bar{Z}_3, \\ \gamma_2, & \bar{Z}_3 \le \gamma_2 \le Z_{[2]}, \\ Z_{[2]}, & Z_{[2]} \le \gamma_2. \end{cases} \tag{3.14}$$

Further, let $\Psi \in \Delta$ denote the truncation function

$$\Psi(y) = \begin{cases} 2e, & 2e \le y, \\ y, & -2e \le y \le 2e, \\ -2e, & y \le -2e. \end{cases} \tag{3.15}$$

TABLE 1

$d = 3e$		$\delta^*(T)$		$\Psi(\mathscr{H})$	$\delta^*(\mathscr{H})$	
d	e	a_1	$R_{\max}$	$R_{\max}$	a_1	$R_{\max}$
0.6399	0.2133	0.2133	0.50	0.71	0.2133	0.50
1.5000	0.5000	0.1988	0.31	0.43	0.1885	0.30
3.0000	1.0000	0.1864	0.13	0.16	0.1668	0.12
4.5000	1.5000	0.1806	0.05	0.05	0.1565	0.04
6.0000	2.0000	0.1775	0.02	0.02	0.1451	0.01

Then

$$\delta_0(\mathbf{Z}) = \Psi(\mathscr{H}(\mathbf{Z})). \tag{3.16}$$

In this example, the corresponding maximum risk of δ_0, obtained by a Monte Carlo calculation, is 0.71. However, since the CDF of $\mathscr{H}(\mathbf{Z})$ possesses a density function which is unimodal and symmetric about θ, the risk performance of $\Psi(\mathscr{H})$ can be improved considerably by replacing Ψ with some alternative $\delta \in \Delta$. Specifically, the decision rule $\delta^*(\mathscr{H})$, where δ^* is (3.10) and $a_1 = 0.2133$, is an "almost" equalizer rule with a maximum risk which is very closely approximated by 0.50. This latter risk evaluation is obtained by a Monte Carlo calculation.

Table 1 displays the maximum risk values of these decision rules for several values of d.

THEOREM 3.4. *Let $N > 1$, $\mathscr{F}$ denote the uncertainty class* (3.1) *with upper-envelope F_u and δ^* denote the $\mathscr{C}$-median-minimax rule obtained through Theorem* 3.3 *based on* CDF F_u. *There exists a bound $B(d/e, N, F_u)$ such that if $e \geq B$, then δ^* is a robust $\mathscr{C}$-median-minimax rule. Further, if the upper-envelope* CDF *of $(Z_M - \theta)$ possesses a (strictly) monotone likelihood ratio, then δ^* is a robust median-minimax (median-admissible) median-Bayes rule.*

For the proof, see the sketch in Appendix 1.

EXAMPLE 3.3 (Example 3.2 revisited and extended). Let $d = 3e$, $e \geq 0.2133$, $N = 3$, F_u denote the CDF of the double exponential distribution (3.9) and $\mathscr{F}$ denote (3.1). In this example, $B(d/e, N, F_u)$ is

$$B(d/e, N, F_u) = -(1/2)G^{-1}(1/4) = 0.2133, \tag{3.17}$$

where $G = F_u^2(3 - 2F_u)$. [See Appendix 1.] Since $e \geq B$, it follows from Theorem 3.4 that δ^* (3.10) is a robust $\mathscr{C}$-median-minimax rule with maximum risk $R_{\max}$. See columns 1–4 of Table 1 for the corresponding values of d, e, a_1 and $R_{\max}$.

3.4. *Large sample approximations.* This section examines the asymptotic behavior of robust median-minimax rules and their extensions to robust α-minimax rules, which are based on the α-trimmed mean $T(\alpha, \mathbf{Z})$.

OBSERVATION 3.1 [Bickel (1965)]. If $Z_1, \dots, Z_N$ are i.i.d. with CDF $F(z - \theta)$, where F is continuous, symmetric about zero, strictly increasing and possesses a density f which is continuous and strictly positive on its convex support $\{x: 0 < F(x) < 1\}$, then if $0 \le \alpha < 1/2$,

$$\lim_{N \to \infty} \mathscr{L}\big(N^{1/2}(T(\alpha, \mathbf{Z}) - \theta)\big) = \mathscr{N}\big(0, \sigma^2(\alpha)\big),$$

where

$$\sigma^2(\alpha) = (1 - 2\alpha)^{-2}\left[2\int_0^{x(1-\alpha)} t^2\, dF(t) + 2\alpha x^2(1 - \alpha)\right]$$

and

$$\lim_{\alpha \to 1/2} \sigma^2(\alpha) = 1/(2f(0))^2.$$

DEFINITION 3.3. Let $\mathscr{F}_s$ denote the uncertainty class (3.1), where each $F \in \mathscr{F}_s$ also satisfies the conditions of Observation 3.1.

OBSERVATION 3.2. Let $\mathscr{F}_s$ denote an uncertainty class with upper-envelope F_u (with corresponding density f_u) subject to Definition 3.3. Then, for suitably large N, an approximation to the robust median-minimax rule δ^*, obtained through Theorem 3.4, can be achieved by substituting $\mathscr{N}(0, 1/(4Nf_u^2(0)))$ for the upper-envelope CDF of $(Z_M - \theta)$. [Since $(Z_M - \theta)$ is asymptotically normal, the $\mathscr{C}$ can be omitted in the designation of δ^*.]

OBSERVATION 3.3. Observation 3.2 can be extended by replacing Z_M with $T(\alpha, \mathbf{Z})$. The corresponding upper-envelope asymptotic approximation is $\mathscr{N}(0, \sigma^2(\alpha)/N)$. The resulting rules, which are obtained in this way through Theorem 3.4, are referred to as robust α-minimax rules. Further, since α offers a degree of freedom in selecting the decision rule, we can optimize the choice of α by solving the following minimax problem. Let $V(\alpha, F)$ denote the asymptotic variance of $T(\alpha, \mathbf{Z})$ based on $F \in \mathscr{F}_s$. Determine $\alpha^* \in [0, 1/2]$ such that for all $\alpha \in [0, 1/2]$,

$$\sup_{F \in \mathscr{F}_s} V(\alpha^*, F) \le \sup_{F \in \mathscr{F}_s} V(\alpha, F).$$

An existence theorem for the minimax solution to this problem appears in Gastwirth and Rubin (1969).

EXAMPLE 3.4. Huber (1981) considers the ε-contaminated $\mathscr{N}(0,1)$ distribution and obtains the value of α_ε^* as well as the corresponding least informative CDF F_ε^* for a given level of contamination ε. For example, if $\varepsilon = 0.3$, then $\alpha_\varepsilon^* = 0.323$ and $\sigma^2(\alpha_\varepsilon^*) = 2.822$; whereas, $\lim_{\alpha \to 1/2} \sigma^2(\alpha) = 3.206$. Further, if $\mathscr{K} = \{K = \mathscr{N}(0, V(\alpha_\varepsilon^*, F_\varepsilon)): F_\varepsilon \in \mathscr{F}_s\}$, then the least informative distribution $F_\varepsilon^* \in \mathscr{F}_s$ corresponds to the upper-envelope $\mathscr{N}(0, V(\alpha_\varepsilon^*, F_\varepsilon^*))$ of $\mathscr{K}$.

3.5. *Two special cases.* There are two limiting cases which are worthy of special mention.

CASE 1 ($d = 2e$). If $d = 2e$, then Theorems 2.2, 3.2 and 3.4 apply for all $e > 0$, i.e., in each instance B is zero. The robust rules obtained in Theorems 2.2, 3.2 and 3.4 are, respectively, $\bar{Z}_N$, Z and Z_M truncated to $[-e, e]$ with respective minimax risks $F_u(-\sqrt{N}e)$, $F_u(-e)$ and $G_u(-e)$, where G_u denotes the CDF of the median based on F_u.

CASE 2 ($d \to \infty$). If $\Omega = E^1$, then Theorems 2.2, 3.2 and 3.4 apply for all $e > 0$, i.e., in each instance B is again zero. However, in this case the resulting robust rules are extended Bayes. The robust rules obtained in Theorems 2.2, 3.2 and 3.4 are, respectively, $\bar{Z}_N$, Z and Z_M with respective minimax risks $2F_u(-\sqrt{N}e)$, $2F_u(-e)$ and $2G_u(-e)$, where G_u again denotes the CDF of the median based on F_u.

These limiting cases provide useful upper and lower bounds for the minimax risk for intermediate values of d. Further discussion of these limiting cases appears in Appendix 1.

APPENDIX 1

This appendix contains a proof for Theorem 2.2, the derivation of the bound $B(d/e, N, F_u)$ which appears in Example 2.1, sketches of proofs for Theorems 3.2, 3.3 and 3.4 and a remark pertaining to Section 3.5. We begin with the following observations.

OBSERVATION A.1 [Zeytinoglu and Mintz (1984)]. If $N = 1$, $\delta \in \Delta$ and F is any continuous CDF which is symmetric about zero, then

$$(\text{A.1}) \quad R(\delta, \theta, F) = \begin{cases} F(a_n - e), & d - 2e < \theta \le d, \\ F(a_{n-1} - e), & \theta = d - 2e, \\ F(-a_n - e) + F(a_{n-1} - e), & d - 4e < \theta < d - 2e, \\ \vdots & \\ F(-a_2 - e) + F(a_1 - e), & c + e < \theta < c + 3e, \\ F(-a_2 - e) + (c/e)F(-e) & \\ \quad +(1 - c/e)F(-a_1 - e), & \theta = c + e, \\ F(-a_1 - e) + (c/e)F(-e) & \\ \quad +(1 - c/e)F(-a_1 - e), & 0 < \theta < c + e, \\ 2F(-a_1 - e), & \theta = 0. \end{cases}$$

OBSERVATION A.2. The risk expression (A.1) can be readily modified to include CDF's F which are both asymmetric and discontinuous. The generalized (asymmetric) risk function $R(\delta, \theta, F)$ is again a piecewise constant function of θ over the sets of the finite partition of Ω expressed in (A.1).

OBSERVATION A.3. Let $N = 1$, $\delta \in \Delta$ and $\mathscr{F}$ denote (3.1) with upper-envelope F_u. If $a_n \leq e$, then

$$R(\delta, \theta, F) \leq R(\delta, \theta, F_u) \tag{A.2}$$

for all $F \in \mathscr{F}$ and $\theta \in \Omega$.

If F is symmetric about zero, then (A.2) is a consequence of the nonpositivity of the argument of each F on the right-hand side of (A.1). When F is asymmetric, (A.2) is established from the functional form of the generalized (asymmetric) risk function $R(\delta, \theta, F)$ by similar means.

OBSERVATION A.4. Let $N > 1$ and $F \in \mathscr{F}$ (3.1) with upper-envelope F_u. If G and G_u denote, respectively, the CDF's of the medians based on F and F_u, then $G(x^-) \leq G_u(x)$, $x \leq 0$, and $G(x) \geq G_u(x)$, $x > 0$. Further, G_u possesses a density function which is unimodal and symmetric about zero.

PROOF OF THEOREM 2.2. There are two cases.

CASE 1 ($N = 1$). If $\delta \in \Delta$ and $F \in \mathscr{F}$ (2.2), then $R(\delta, \theta, F)$ is obtained from (A.1) by replacing $F(\cdot)$ with $\Phi(\cdot/\sigma)$. The minimax rule δ^*, based on the upper-envelope $\mathscr{N}(0, \sigma_u^2)$ and obtained through Theorem 2.1, is constructed by determining the parameter vector $\mathbf{a}^* \in E^n$ which equalizes the piecewise constant segments of $R(\delta^*, \theta, F_u)$ over the $n + 1$ nondegenerate subintervals of $[0, d]$. If $a_n^* \leq e$, then

$$R(\delta^*, \theta, F) \leq R(\delta^*, \theta, F_u) \leq \sup_{\theta \in \Omega} R(\delta^*, \theta, F_u) \tag{A.3}$$

for all $F \in \mathscr{F}$ (2.2) and $\theta \in \Omega$. The left-hand inequality in (A.3) follows from Observation A.3. Thus, if $a_n^* \leq e$, then δ^* is a robust minimax rule.

We next obtain a bound $B(d/e, N = 1, F_u)$ such that if $e \geq B$, then $a_n^* \leq e$. Let M denote the minimax risk of δ^* based on F_u. Here, M depends on d/e. Observe that for fixed d/e, the minimax risk M is a nonincreasing function of e. Let $B(d/e, N = 1, F_u)$ denote the smallest value of e such that $M(e) = 0.5$. Thus, the determination of B is equivalent to determining the values of e and $\mathbf{a}$ which equalize the right-hand side of (A.1) to 0.5 over the nondegenerate subintervals of $[0, d]$. It is shown in Zeytinoglu and Mintz (1984) that these equations can always be solved whenever F_u possesses a density function which is unimodal and symmetric about zero. Thus, if $e \geq B$, then δ^* is a robust minimax rule.

Further, we observe that the rule δ^* is Bayes with respect to the product measure $\mu = \lambda^* \times \nu$ on $\Omega \times \mathscr{F}$, where λ^* denotes the least favorable prior distribution on Ω associated with the solution to the minimax estimation problem based on F_u, and ν denotes the distribution on $\mathscr{F}$ which assigns probability 1 to F_u. Since δ^* is unique up to equivalence, it is admissible.

CASE 2 ($N > 1$). Since the proof of the Theorem 2.2 for the multisample case is quite similar to the single sample case, it suffices to observe that we follow the previous approach, but replace Z with $\bar{Z}_N$ and σ with $\sigma/\sqrt{N}$. The bound $B(d/e, N, F_u)$ is again obtained by equalizing $R(\delta^*, \theta, F_u)$ to 0.5 over the nondegenerate subintervals of $[0, d]$.

THE BOUND IN EXAMPLE 2.1. Let $d = 3e$, $e = 0.1$, $N = 49$, $\sigma_u = 2$, $F_u = \mathscr{N}(0, \sigma_u^2)$ and $G = \mathscr{N}(0, \sigma_u^2/N)$. The risk equalization condition becomes

$$G(a_1 - e) = 2G(-a_1 - e) = 1/2. \tag{A.4}$$

The solution to (A.4) is

$$a_1 = e = -\left(1/\sqrt{4N}\right)F_u^{-1}(1/4) = B(d/e, N, F_u) = 0.0966. \tag{A.5}$$

PROOF OF THEOREM 3.2: A SKETCH. The $\mathscr{C}$-minimax rule δ^*, based on the upper-envelope F_u and obtained through Theorem 3.1, is an "almost" equalizer rule parametrized by $\mathbf{a}^*$. The bound $B(d/e, N = 1, F_u)$ is established by following the technique in the proof of Case 1 of Theorem 2.2. If $e \geq B$, then $a_n^* \leq e$. Thus, by Observation A.3, δ^* satisfies (A.3) for all $F \in \mathscr{F}$ (3.1) and $\theta \in \Omega$. Based on Theorem 3.1, it can be shown that if F_u possesses a (strictly) monotone likelihood ratio, then δ^* is an (admissible) Bayes rule with respect to a product measure μ, which is identical in structure to the measure obtained in the proof of Case 1 of Theorem 2.2.

PROOF OF THEOREM 3.3: A SKETCH. A proof of Theorem 3.3 can be based on Theorem 3.1 and the second part of Observation A.4, i.e., if F possesses a density function which is unimodal and symmetric about zero, then the CDF of the median based on F possesses a density function which is unimodal and symmetric about zero. The following alterations are required in the application of Theorem 3.1: Replace Z with Z_M and replace F with the CDF of $(Z_M - \theta)$.

PROOF OF THEOREM 3.4: A SKETCH. A proof of Theorem 3.4 can be based on Theorems 3.2, 3.3 and the first part of Observation A.4. The following alterations are required in the application of Theorem 3.2: Replace Z with Z_M and replace $\mathscr{F}$ with $\mathscr{G}$, the set of CDF's of the medians based on $\mathscr{F}$.

A REMARK PERTAINING TO SECTION 3.5. The appropriate bound B in the limiting cases presented in Section 3.5 is zero in each case, since the arguments of the risk functions for the individual rules obtained in each instance are nonpositive for all $e > 0$.

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