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UNIT CANONICAL CORRELATIONS BETWEEN FUTURE AND PAST

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Stationary vector ARMA processes $x(t)$, $t = 0, \pm 1, \pm 2, \ldots$, of n components are considered that are of full rank, and the situation where there are linear functions of the future $x(t), t > 0$, and the past $x(t), t \le 0$ (more properly the present and the past) that have unit correlation. It is shown that the number of linearly independent such pairs (i.e., the number of unit canonical correlations between future and past) is the number of zeros of the determinant of the transfer functions, from innovations to outputs, that lie on the unit circle, counting these with their multiplicities.

1. Vector ARMA processes and unit canonical correlations. Consider a vector ARMA, stationary process $x(t)$ of n components, i.e., a process generated by

$$\sum_0^p A(j)x(t-j) = \sum_0^q B(j)\varepsilon(t-j), \qquad A(0) = B(0), \tag{1.1}$$
$$E\{\varepsilon(s)\varepsilon(t)'\} = \delta_{st}\Sigma, \qquad \Sigma > 0.$$

The $\varepsilon(t)$ are to be the innovations so that $k(z) = a(z)^{-1}b(z)$, $a(z) = \sum A(j)z^j$, $b(z) = \sum B(j)z^j$, is rational and analytic for $|z| \le 1$ and $\det(k) \ne 0$, $|z| < 1$. The requirement that $\Sigma > 0$ could be dropped but we consider only the full rank case. For details concerning these well-known facts, see, for example, Hannan (1970).

Let $x(t+u|t)$ be the best linear predictor of $x(t+u)$ from $x(s)$, $s \le t$, taking "best" to mean best in the least-squares sense. It is known that the Hilbert space spanned by the $x_j(t+u|t)$, $j = 1, \ldots, n$, $u = 1, 2, \ldots$, is finite dimensional and is spanned by

$$x_j(t+u|t), \qquad u = 1, \ldots, d_j,\ j = 1, \ldots, n.$$

The d_j are the Kronecker indices and their sum $d = \sum d_j$ is called the McMillan degree. [For details see Kailath (1980).] Let

$$k(z) = \sum_0^\infty K(j)z^j, \qquad K(0) = I_n, \qquad \mathscr{H} = [K(i+j-1)]_{i,j=1,2,\ldots}. \tag{1.2}$$

Here $\mathscr{H}$ is the so-called Hankel matrix of the system, being an infinite matrix with $K(i+j-1)$ as the (i, j)th block. Then d is the rank of $\mathscr{H}$.

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Here we consider canonical correlation of the future and past, i.e., (stationary) linear functions $\xi_j(t)$ of $x(s)$, $s \le t$, and $\eta_j(t)$ of $x(s)$, $s > t$, such that

$$E\{\xi_j(t)\xi_k(t)\} = E\{\eta_j(t)\eta_k(t)\} = \delta_{jk}, \tag{1.3}$$
$$E\{\xi_j(t)\eta_j(t)\} = \rho_j, \qquad 1 \ge \rho_1 \ge \rho_2 \ge \cdots \ge 0.$$

The existence of such canonical correlations ρ_j and of the associated pairs of "discriminant function" processes $\xi_j(t), \eta_j(t)$ is established, in general, by a standard Hilbert space argument but we do not need to go into that here because the situation is simpler. Let H_∞ be the real Hilbert space spanned by $x_j(t)$, $j = 1, \dots, n$, $t = 0, -1, -2, \dots$, and H^∞ be spanned by $x_j(t)$, $j = 1, \dots, n$, $t = 1, 2, \dots$. In each case the inner product is given by the covariance function. If H is spanned by all $x_j(t)$ let P project in H on H_∞. If $\eta \in H^\infty$, $\xi \in H_\infty$, then $E\{\eta\xi\} = E\{P\eta\xi\} = E[\{P\Sigma a_{j,u}x_j(u)\}\{\Sigma b_{j,u}x_j(u|0)\}]$, where the sums are over $u = 1, \dots, d_j$, $j = 1, \dots, n$. Indeed, $P\eta$ must be of the form $\Sigma a_{j,u}x_j(u|0)$ and ξ may be decomposed into a sum of the same random variables and a component orthogonal to all of those random variables. Thus

$$E\{\eta\xi\} = E\left[\left\{\sum a_{j,u}x_j(u)\right\}\left\{\sum b_{j,u}x_j(u|0)\right\}\right].$$

If ξ is orthogonal to $x_j(u|0)$, $u = 1, \dots, d_j$, $j = 1, \dots, n$, in H_∞, then $E\{\eta\xi\} = 0$ and thus, in particular, if η is orthogonal to the space spanned by the $x_j(u)$, $u = 1, \dots, d_j$, $j = 1, \dots,$ and ξ is orthogonal to the $x_j(u|0)$, $u = 1, \dots, d_j$, $j = 1, \dots, n$, then $E\{\eta\xi\} = 0$. Thus the canonical correlation construction is reduced to that between two finite set of $d = \Sigma d_j$ random variables and is classical. It follows also that in (1.3) $\rho_j = 0$, $j > d$. Thus according to this classical theory, we may construct $\xi_j(0), \eta_j(0)$, $j = 1, \dots, d$, that are orthogonal and lie, respectively, in H_∞, H^∞ and satisfy $E\{\xi_j(0)\eta_j(0)\} = \rho_j, 1 \ge \rho_1 \ge \rho_2 \ge \cdots \ge \rho_d \ge 0$. Now $\xi_j(t), \eta_j(t)$ are obtained by time translation and (1.3) follows by stationarity.

Let $m(z)$ be the polynomial, with $m(0) = 1$, having as zeros precisely the zeros of $\det k(z)$ that lie on $|z| = 1$, with the same multiplicities. If $k(z) = a(z)^{-1}b(z)$ and $a(z)$, $b(z)$ are left coprime, then $\det a(z)$ and $\det b(z)$ are also coprime [again, see Kailath (1980) for details] and hence the zeros of $\det k(z)$ are those of $\det b(z)$. However, the matrix fraction description $k(z) = a(z)^{-1}b(z)$ is to some extent arbitrary so we continue to refer to $\det k(z)$. Returning to $m(z)$, it is easily established that, $\mu < d$ being the number of zeros counted with their multiplicities of $\det k(z)$ that are of unit modulus, and a being the number that are at $z = 1$, then $m(z)/m(z^{-1}) = (-1)^a z^\mu$.

Let $\xi_j(t), \eta_j(t)$ be two discriminant function processes. Then these correspond to transfer functions, $\xi_j(z), \eta_j(z)$ from $x(t)$ to $\xi_j(t), \eta_j(t)$, respectively. (The notation has virtues and vices but should cause no confusion.) Thus [Hannan (1970), Chapter II]

$$\xi_j(t) = \int_{-\pi}^{\pi} e^{-it\omega}\xi_j(e^{i\omega})'\,d\zeta(\omega), \qquad \eta_j(t) = \int_{-\pi}^{\pi} e^{-it\omega}\eta_j(e^{i\omega})'\,d\zeta(\omega).$$

Necessarily $\zeta_j(z)$ must be analytic for $|z| < 1$ and $\eta_j(z)$ for $|z| > 1$ because $\xi_j(t)$

must be expressible in terms only of $x(t-s), s \geq 0$, and $\eta_j(t)$ in terms of $x(t+s), s > 0$. Because $s = 0$ does not occur in the second case we must also have $\eta_j(z)$ converging to zero as $|z| \to \infty$. In fact, we shall show that any such pair, which we call $\xi(z), \eta(z)$ in general, for which $\rho_j = 1$, is of the form

$$\xi(z) = \eta(z) = m(z)^{-1}p(z),$$

where $p(z)$ is a vector of polynomials of degree $\mu - 1$ at most. There will be a different polynomial for each j value such that $\rho_j = 1$. Of course, $m(z)^{-1}p(z)$ is analytic for $|z| < 1$ but also $m(z) = m(z^{-1})(-1)^a z^{\mu}$ so $m(z)^{-1}p(z) = (-1)^a m(z^{-1})^{-1} z^{-\mu} p(z)$ and since $p(z)$ is of degree $\mu - 1$, then this is also analytic for $|z| > 1$ and has $\eta(z) \to 0$ as $|z| \to \infty$. Clearly, $\xi_j(t) \equiv \eta_j(t)$ a.s. if $\xi_j(z), \eta_j(z)$ are of this form and thus clearly $\rho_j = 1$. If $n = 1$ we may take $p(z)$ to be any polynomial of degree $\mu - 1$ and thus the dimension of the space of discriminant function pairs for unit canonical correlation is at least μ. In fact, the theorem that follows says it is μ. However, when $n > 1$ it is not evident that $\xi(z)'k(z)$ is square integrable on $|z| = 1$. For $n = 1$ there is no problem since $m(z)$ divides $k(z)$ but for $n > 1$, $m(z)$ does not necessarily divide all elements of $p(z)'k(z)$. Indeed, now the space of all $p(z)$ of degree $\mu - 1$ at most is $\eta\mu$, which is, in fact, too high and $p(z)$ must be constrained in a way indicated in the proof of the theorem. We illustrate by the case where $k(z) = I - Bz$ and there is a nonsingular matrix A so that ABA^{-1} is diagonal. Then $Ak(z)A^{-1} = \Lambda(z)$ is diagonal. Let us say there are two unit zeros of det $k(z)$, these being 1 and -1. We may arrange these to be in the first two places in the diagonal in $\Lambda(z)$ so that these elements are $(1-z), (1+z)$. Thus $m(z) = 1 - z^2, a = 1$ and $m(z)^{-1}k(z) = A^{-1}\{m(z)^{-1}\Lambda(z)\}A$. Now clearly $p(z)'A^{-1}$ must be such as to eliminate the poles on $|z| = 1$ so that $p(z)'A^{-1} = (a_1(1+z), (a_2(1-z), 0, 0, \ldots, 0)$. Thus the space of such $p(z)$ is two dimensional and there are two discriminant function process pairs with unit canonical correlation and transfer functions of the form $m(z)^{-1}p(z)$. The general case is described in the proof of the theorem and will not be explained in more detail here for the construction can hardly be of any utility, other than in the proof of the theorem.

THEOREM. *The number of canonical correlations that are unity is the number of zeros of* $\det\{k(z)\}$ *on* $|z| = 1$, *counting each such zero with its multiplicity.*

PROOF. Let $f(\omega)$ be the spectral density matrix. Then $2\pi f(\omega) = k(e^{i\omega})\Sigma k(e^{i\omega})^*$. This decomposition is unique, subject to the conditions on $k(z), \Sigma$ given previously. Since $f(-\omega) = \overline{f(\omega)}$ is again a rational spectral density matrix, then $2\pi f(-\omega) = l(e^{i\omega})\Omega l(e^{i\omega})^*$, $\Omega > 0$, where $l(0) = I_n$, $l(z)$ is analytic for $|z| \leq 1$ and $\det\{l(z)\} \neq 0, |z| > 1$. Since

$$\det\{f(\omega)\} = \det\{f(-\omega)\} = |\det k(e^{i\omega})|^2 \det(\Sigma/2\pi) = |\det l(e^{i\omega})|^2 \det(\Omega/2\pi)$$

and det k, det l are rational, analytic for $|z| \leq 1$, never zero for $z < 1$ and satisfy det $k(0)$ = det $l(0) = 1$, then $\det\{k(z)\} = \det\{l(z)\}$ and $\det(\Sigma) = \det((\Omega)$. Let ξ, η be linear functions, respectively, of $x(t), t \leq 0, x(t), t > 0$, having unit correlation. Then, as before,

$$\xi = \int_{-\pi}^{\pi} \xi(e^{i\omega})' \, d\zeta(\omega), \qquad \eta = \int_{-\pi}^{\pi} \eta(e^{i\omega})' \, d\zeta(\omega),$$

where

$$x(t) = \int_{-\pi}^{\pi} e^{-it\omega} \, d\zeta(\omega)$$

is the spectral representation of $x(t)$ and

$$\xi(z) = \sum_0^\infty \alpha(j) z^j, \qquad \eta(z) = \sum_1^\infty \beta(j) z^{-j}, \tag{1.4}$$

where the $\alpha(j)$, $\beta(j)$ are vectors of n components. Thus

$$1 = \int_{-\pi}^{\pi} \xi(e^{i\omega})' f(\omega) \eta(e^{-i\omega}) \, d\omega$$
$$\div \left\{ \int_{-\pi}^{\pi} \xi(e^{i\omega})' f(\omega) \xi(e^{-i\omega}) \, d\omega \int_{-\pi}^{\pi} \eta(e^{i\omega})' f(\omega) \eta(e^{-i\omega}) \, d\omega \right\}^{1/2}.$$

It follows from the obvious generalisation to vectors of Schwarz's inequality [Hardy, Littlewood and Pólya (1964), page 132] that

$$\xi(e^{i\omega})' k(e^{i\omega}) = \eta(e^{i\omega})' k(e^{i\omega}) \quad \text{a.e.} \ (d\omega), \tag{1.5}$$

or at least this can be made so by multiplying, say, each element of $\eta(z)'k(z)$ by a positive real constant. The Hardy space H_2 is the space of all functions analytic for $|z| < 1$ and square integrable on $|z| = 1$. [See Hoffman (1962), for example.] We shall also say that an n element vector of functions is in H_2 if each of its elements is so. Thus, from (1.5), $\eta(z)'k(z) \in H_2$, since $\xi(z)'k(z)$ certainly $\in H_2$. Also since $z^{-1}\eta(z^{-1})'$ contains only positive powers of z in its expansion, converging for $|z| < 1$ and $\eta(z)'l(z)$ is square integrable on $|z| = 1$, then $z^{-1}\eta(z^{-1})'l(z) \in H_2$, the factor z coming from the fact that $\beta(0) = 0$. Thus

$$\eta(z)'k(z) = h_1(z)', \qquad z\eta(z)'l(z^{-1}) = h_2(z^{-1})',$$

where $h_1(z), h_2(z) \in H_2$. Put $k(z)^{-1} = m(z)^{-1}r(z)$, $l(z)^{-1} = m(z)^{-1}s(z)$. Then $r(z)$, $s(z)$ are analytic for $|z| \le 1$. Remembering that $m(z)/m(z^{-1}) = (-1)^a z^\mu$, we have

$$\eta(z)' = m(z)^{-1} h_1(z)' r(z) = z^{-1} m(z^{-1})^{-1} h_2(z^{-1})' s(z^{-1}),$$

so that

$$h_1(z)' r(z) = (-1)^a z^{\mu - 1} h_2(z^{-1})' s(z^{-1}).$$

This is possible only if $h_1(z)'r(z)$, and $h_2'(z)s(z)$ are rows of polynomials of degree $\mu - 1$ at most. Thus

$$\xi(z)' = \eta(z)' = m(z)^{-1} p(z)' = (-1)^a z^{-\mu} m(z^{-1})^{-1} p(z)',$$

where $p(z)'$ is a row of polynomials of degree $\mu - 1$ at most. However, $\xi(z)'k(z) = \eta(z)'k(z)$ must also be square integrable on $|z| = 1$, i.e., each element of the vector must be square integrable. There exist matrices $u(z)$, $v(z)$, with constant, nonzero determinants, such that $k(z) = u(z)\lambda(z)v(z)$ and $\lambda(z)$ is diagonal with diagonal elements $n_i(z)/d_i(z)$, the numerator and denominator polynomials being prime to each other. Also $n_i(z)$ divides $n_{i+1}(z)$, and $d_{i+1}(z)$ divides $d_i(z)$. [Again, see Kailath (1980).] Thus square integrability of $\xi(z)'k(z)$ is equivalent to that of $\xi(z)'u(z)\lambda(z)$ since $v(z)$ is bounded on $|z| = 1$, with $\det\{v(z)\}$ a nonzero constant. In turn this is equivalent to the square integrability of $\xi(z)'u(z)\text{diag}\{m_i(z)\}$, where $\text{diag}\{m_i(z)\}$ is a diagonal matrix with $m_i(z)$ in the ith place and $m_i(z)$ with $m_i(0) = 1$ has zeros those of $n_i(z)$ on $|z| = 1$. This is because $m_i(z)^{-1}\{n_i(z)/d_i(z)\}$ is analytic for $|z| \le 1$ and never zero there. Put $\nu_i(z) = m(z)/m_i(z)$. [Clearly, $m_i(z)$ divides $m(z)$.] Then $p(z)'u(z)\nu(z)^{-1}$ must be square integrable, where $\nu(z)$ is diagonal with $\nu_i(z)$ in the ith place. Now $j(z) = u(z)\nu(z)^{-1}$ is a matrix of rational functions, analytic for $|z| < 1$. It is also the reciprocal of a matrix of polynomials. Put $j(z) = q(z) + r(z)$, where $q(z)$ is the matrix of quotients obtained by dividing each numerator of an element of $j(z)$ by its denominator. Now $r(z)$ is strictly proper (i.e., all numerator degrees are lower than denominator degrees) and thus has a canonical left-prime matrix fraction description $c(z)^{-1}d(z)$, corresponding to its Kronecker indices. [See Kailath (1980) or Deistler and Hannan (1981), where this is termed the echelon form.] Thus $j(z) = c(z)^{-1}\{c(z)q(z) + d(z)\}$ and again this is left prime. Thus $c(z)q(z) + d(z)$ is unimodular and the degree of det $c(z)$ is that of det $j(z)^{-1}$, namely, $(n-1)\mu$. Also the largest Kronecker index is no greater than μ since $m(z)j(z)$ is a polynomial and $m(z)$ is of degree μ. Let d_i be the ith Kronecker index of $r(z)$. Then $\Sigma'(\mu - d_i) = \mu$, where the sum may be taken over i for which $d_i < \mu$. For $p(z)'j(z)$ to be square integrable it is necessary and sufficient that it be polynomial, i.e., that $p(z)'r(z)$ be polynomial. For this it is necessary that $p(z)' = e(z)'c(z)$, where $e(z)$ is polynomial. Thus the number of linearly independent $p(z)$, of degree $\mu - 1$, for which $p(z)'j(z)$ is polynomial is the number of linearly independent vectors $e(z)$ for which $e(z)'c(z)$ is polynomial and of degree $\mu - 1$. The matrix $c(z)$ is characterised by the fact that the diagonal elements are monic and of degree d_i and the elements $c_{ij}(z)$ satisfy the following degree requirements: degree $c_{ij}(z) < d_j$, $i \ne j$; degree $c_{ij}(z) < d_i$, $j > i$; degree $c_{ij}(z) \le d_i$, $j \le i$. Now, clearly, we may allow $e_i(z)$, the ith element of $e(z)$ to vary freely over all polynomials of degree $\mu - d_i - 1$ since the ith row of $c(z)$ is of degree d_i. Here the polynomial is null if $(\mu - d_i - 1) < 0$, i.e., $d_i = \mu$. Thus if we show that these are the actual maximum degrees, then the theorem will be proved since there are $\Sigma'(\mu - d_i)$ freely varying coefficients in such $e(z)$. Let $\delta_i = \mu - 1 - d_i + a_i$, $a_i \le 0$, $d_i < \mu$; $\delta_i = a_i - 1$, $d_i = \mu$, $a_i \ge 0$, be the degrees of the elements of $e(z)$, where again the element is null if $\delta_i < 0$. Let j be the greatest column number for which a_i is greatest. Then the degrees n_i of $e_i(z)c_{ij}(z)$ are

$$i < j, \qquad n_i \le \delta_i + d_i - 1 \le \mu + a_i - 2 < \mu + a_j - 1,$$

$$i > j, \qquad n_i \le \delta_i + d_i \le \mu + a_i - 1 < \mu + a_j - 1,$$

$$i = j, \qquad n_j = \delta_j + d_j = \mu + a_j - 1.$$

Thus the jth element of $e(z)'c(z)$ is of degree $\mu + a_j - 1$ so $a_j = 0$. Thus the theorem is proved. □

2. Discussion. For a discussion of the general problem (i.e., not for ARMA processes alone) of canonical correlation of future and past the reader may consult Jewell and Bloomfield (1983) and references therein. There θ_1 defined by $\cos\theta_1 = \rho_1$, $0 \le \theta_1 \le \pi$, is called the angle between the present and past. A general condition, for $n = 1$, for this angle to be positive is given there. That condition shows easily that the angle is zero if $k(z)$ has a zero on $|z| = 1$. Of course, the condition that $x(t+1)$ be linearly perfectly predicted from the past is the condition, for $n = 1$, that

$$\int_{-\pi}^{\pi} \log f(\omega)\, d\omega = -\infty, \tag{2.1}$$

so that $f(\omega)$ must approach zero at some frequency. Of course, (2.1) cannot hold for a scalar ARMA process, and the angle between present and past can be zero only if $f(\omega)$ is zero at some ω.

The canonical correlations can be explained also in terms of $\mathscr{H}$. Thus let $x^{(t+1)} = (x(t+1)', x(t+2)', \dots)'$, $\varepsilon_t = (\varepsilon(t)', \varepsilon(t-1)', \dots)'$, with $\varepsilon^{(t+1)}$ defined as for $x^{(t+1)}$. Then it is trivial to show that

$$x^{(t+1)} = \mathscr{H}\varepsilon_t + \mathscr{K}\varepsilon^{(t+1)},$$

where $\mathscr{K}$ is a block Toeplitz matrix with $K(i-j)$ as the block in row i, column j, $i, j = 1, 2, \dots$. Let $\mathscr{L}$ have $L(j-i)$ as the block in row i column j, where

$$l(z) = \sum_0^{\infty} L(j) z^j.$$

In the case where $\det\{k(z)\} \neq 0$, $|z| \le 1$, then the canonical correlations are the singular values of

$$\mathscr{G} = \left(I_\infty \otimes \Omega^{-1/2}\right)\mathscr{L}^{-1}\mathscr{H}\left(I_\infty \otimes \Sigma^{1/2}\right),$$

where $I_\infty \otimes \Omega^{-1/2}$, for example, is a block diagonal matrix with all diagonal blocks $\Omega^{-1/2}$. The matrix $\mathscr{G}$ is again a Hankel matrix and has "symbol" $g(z) = \Omega^{-1/2} l(z^{-1})^{-1} k(z) \Sigma^{1/2}$ [which is just $k(z)/k(z^{-1})$ for $n = 1$]. By this it is meant that the typical block $G(i+j-1)$ of $\mathscr{G}$ is the coefficient of z^{i+j-1} in $g(z)$. However, $g(z)$ is not analytic for $|z| = 1$, though it is unitary there. When $\det\{k\}$ can be zero on $|z| = 1$, $\mathscr{L}$ does not have a bounded inverse. However, $\mathscr{G}$ is well defined via $g(z)$ as before. It must be possible to establish the theorem of this paper through a direct consideration of $g(z)$.

An explicit construction in a simple case may be of interest. This case also relates closely to the example discussed just before the statement of the theorem. Take $n = 1$ and $x(t) = \varepsilon(t) - \varepsilon(t-1)$. Then $\varepsilon(t)$ is the limit in mean square of

$$\sum_{j=0}^{N-1} \left(1 - \frac{j}{N}\right) x(t-j)$$

and $\varepsilon(t)$ has unit correlation, as is easily seen, with $\eta(t+1)$, which is the limit in mean square of

$$-\sum_{j=0}^{N-1} \left(1 - \frac{j}{N}\right) x(t+j+1).$$

Finally, Whittle (1984), Section 3.8, contends that processes of the type discussed previously will not be found in practice because of observation error. The results of this paper also suggest that from a statistical viewpoint ARMA models with zeros on $|z| = 1$ may be somewhat unreal, for one does not expect to be able to predict precisely any aspect of the future even using the infinite past. Of course, such a statement must be taken with a grain of salt since ARMA models are in any case unreal, in the sense that any model is an approximation to reality.

REFERENCES

DEISTLER, M. and HANNAN, E. J. (1981). Some properties of the parameterization of ARMA systems of unknown order. *J. Multivariate Anal.* **11** 474–484.

HANNAN, E. J. (1970). *Multiple Time Series*. Wiley, New York.

HARDY, G. H., LITTLEWOOD, J. E. and PÓLYA, G. (1964). *Inequalities*. Cambridge Univ. Press, Cambridge.

HOFFMAN, K. (1962). *Banach Spaces of Analytic Functions*. Prentice-Hall, Englewood Cliffs, N.J.

JEWELL, N. P. and BLOOMFIELD, P. (1983). Canonical correlation of past and future for time series: Definitions and theory. *Ann. Statist.* **11** 837–847.

KAILATH, T. (1980). *Linear Systems*. Prentice-Hall, Englewood Cliffs, N.J.

WHITTLE, P. (1984). *Prediction and Regulation by Linear Least-Squares Methods*, 2nd ed. Blackwell, Oxford.

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