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MOST POWERFUL INVARIANT TESTS FOR BINORMALITY

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We investigate the problem of testing multinormality against alternatives invariant with respect to the affine group of transformations G and against left-bounded alternatives defined by Szkutnik. The last problem remains invariant under a suitably chosen subgroup G^* of G. Using Wijsman's theorem we find general forms of the most powerful G- and G^*-invariant tests for multinormality which opens the way to an extension of the one-dimensional results of Uthoff to the bivariate case. We find explicit forms of tests against bivariate exponential and bivariate uniform alternatives. A Monte Carlo approximation of the power of these tests is given. This provides us with upper bounds for the power of all invariant tests for binormality against the alternatives considered. The maximin property of the tests obtained is also studied.

1. Preliminaries. Let X be a (p, n) random matrix. The columns X_i of X are assumed to be i.i.d. random vectors with a probability distribution absolutely continuous with respect to Lebesgue measure in $\mathbb{R}^p$. Let $\mathrm{GL}(p)$, $\mathrm{UT}(p)$ and $\mathscr{P}(n)$ denote, respectively, the set of nonsingular (p, p) matrices, the set of (p, p) upper triangular matrices with positive diagonal and the set of (n, n) permutation matrices. $G = \mathrm{GL}(p) \times \mathscr{P}(n) \times \mathbb{R}^p$ denotes the group of transformations $\mathbb{R}^{pn} \to \mathbb{R}^{pn}$ of the form $gX = CXP + b1_n^T$, where $C = [c_{ij}] \in \mathrm{GL}(p)$, $P \in \mathscr{P}(n)$, $b = (b_1, \dots, b_p)^T \in \mathbb{R}^p$ and $1_n^T = (1, \dots, 1) \in \mathbb{R}^n$. Similarly, $G^* = \mathrm{UT}(p) \times \mathscr{P}(n) \times \mathbb{R}^p$ denotes the group of transformations of the same form but with $C \in \mathrm{UT}(p)$.

Following Szkutnik (1987) we assume:

DEFINITION. The distribution of a random p-vector $(Y_1, \dots, Y_p)$ is called left-bounded if:

1. The marginal distribution of Y_p is bounded from the left, i.e., there exists α such that $\alpha \le Y_p$ with probability 1.
2. The conditional distributions of $Y_i | Y_{i+1} = a_{i+1}, \dots, Y_p = a_p$ are bounded from the left for $i = 1, \dots, p-1$ and for any fixed values $a_{i+1}, \dots, a_p$.

The group G^* is shown by Szkutnik (1987) to be the maximal subgroup of G preserving left-boundedness of distributions of columns of X.

By $(\mathscr{N}_p, \mathscr{F}, G)$ we denote the problem of testing H_0: $X_i \sim \mathscr{N}_p(m, \Sigma)$, $i = 1, \dots, n$, against H_1: $X_1 \sim \mathscr{F}[U(X_i - m)]$, $i = 1, \dots, n$, where $U \in \mathrm{GL}(p)$, $m \in \mathbb{R}^p$ and $\mathscr{F}$ is a distribution function of a random p-vector. $(\mathscr{N}_p, \mathscr{F}, G^*)$ denotes the problem of testing H_0 against H_1 which are of the preceding form but with

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$U \in \mathrm{UT}(p)$ and $\mathscr{F}$ being a distribution function of a left-bounded random vector. Clearly, $(\mathscr{N}_p, \mathscr{F}, G)$ remains G-invariant, $(\mathscr{N}_p, \mathscr{F}, G^*)$ remains G^*-invariant and any G- or G^*-invariant test does not depend on, commonly unknown, nuisance parameters m, Σ under H_0.

In the next section we use two easily verifiable lemmas.

LEMMA 1. *A density of a left-invariant Haar measure $\mathscr{V}^*$ on G^* with respect to a measure μ, which is the product of Lebesgue measure in the space $\mathbb{R}^{p(p+1)/2}$ containing* $\mathrm{UT}(p)$, *Lebesgue measure in the space $\mathbb{R}^p$ and counting measure in the $n!$-element space $\mathscr{P}(n)$, may be expressed as*

$$d\mathscr{V}^*/d\mu = (n!)^{-1} \prod_{i=1}^{p} c_{ii}^{i-p-2}.$$

LEMMA 2. *A density of a left-invariant Haar measure on G with respect to the product of Lebesgue measure on the space $\mathbb{R}^{p^2}$ containing* $\mathrm{GL}(p)$, *Lebesgue measure on $\mathbb{R}^p$ and counting measure on $\mathscr{P}(n)$ may be expressed as* $d\mathscr{V}/d\mu = (n!)^{-1}|\det C|^{-p-1}$.

2. General form of the most powerful tests. Wijsman's theorem [for the discussion of applicability of the theorem, see Wijsman (1985)] implies the following form of the most powerful G-invariant test φ for $(\mathscr{N}_p, \mathscr{F}, G)$:

$$\varphi(X) = \begin{cases} 1, & \text{for } F_0/F_1 \le c_{\mathrm{cr}}, \\ 0, & \text{for } F_0/F_1 > c_{\mathrm{cr}}, \end{cases} \tag{2.1}$$

where

$$F_i(X) = \int_G p_i(CXP + b1_n^T)|\det C|^n \, d\mathscr{V}(g), \qquad i = 0, 1, \tag{2.2}$$

and p_i is the probability density function of X provided H_i is true ($i = 0, 1$).

$F_0(X)$ has been studied in the context of detection of multivariate outliers and has been computed by Sinha (1984). However, there is a mistake in his result caused by an incorrect form of the left-invariant Haar measure on G. The correct result should have the form

$$F_0(X) \propto |S|^{-(n-1)/2}, \tag{2.3}$$

where $n^{-1}S$ is the sample covariance matrix based on X. For $p = 1$ we have $F_0(X) \propto \hat{\sigma}^{-(n-1)}$, which conforms with the result of Uthoff (1970).

Applying Lemma 2 to (2.2), making use of the equality $p_1(CXP + b1_n^T) = p_1(CX + b1_n^T)$ implied by the independence of columns of CX and performing the integration over $\mathscr{P}(n)$, we get

$$F_1(X) = \int_{\mathbb{R}^{p^2}} \int_{\mathbb{R}^p} p_1(CX + b1_n^T)|\det C|^{n-p-1} \, db \, dC. \tag{2.4}$$

In the remaining part of this paper we assume that $\mathscr{F}$, which defines H_1, is a distribution function of a random vector having i.i.d. marginals. Because of the invariance the class of distributions constituting H_1 contains also distributions with dependent marginals. Under the preceding assumptions we have $p_1(X) = f(X^{(1)}) \cdots f(X^{(p)})$, where $X^{(i)}$ denotes the ith row of X. Formula (2.4) becomes then

$$(2.5) \quad F_1(X) = \int_{\mathbb{R}^{p^2}} \int_{\mathbb{R}^p} \prod_{i=1}^{p} f\left(c_{i1}X^{(1)} + \cdots + c_{ip}X^{(p)} + b_i 1_n^T\right) |\det C|^{n-p-1} \, db \, dC$$

and the integration over $\mathbb{R}^{p^2}$ may be restricted to the set of C's for which $c_{i1} = 0$, $i = 1, \dots, p$.

Let us introduce in (2.5) new integration variables $\overline{U} = [u_{ij}]$ by $u_{i1} = c_{i1}$, $u_{i2} = c_{i2}/c_{i1}, \dots, u_{ip} = c_{ip}/c_{i1}$, $i = 1, \dots, p$. The Jacobian $|\partial C/\partial \overline{U}| = |u_{11} \cdots u_{p1}|^{p-1}$ may be found by straightforward calculation. Denote by U the matrix $\overline{U}$ in which $u_{i1} = 1$ for $i = 1, \dots, p$ and note that $\det C = u_{11} \cdots u_{p1} \det U$. After some elementary calculation we get

$$(2.6) \quad F_1(X) = \int_{\mathbb{R}^{p(p-1)}} L\left[(UX)^{(1)}\right] \cdots L\left[(UX)^{(p)}\right] |\det U|^{n-p-1} \, dU,$$

where $(UX)^{(i)}$ is the ith row of the matrix UX and for a vector $Y = (y_1, \dots, y_n)$,

$$L(Y) = \int_{\mathbb{R}^2} f(uy_1 + v, \dots, uy_n + v) |u|^{n-2} \, du \, dv.$$

The integral $L(Y)$ has been analysed in the theory of one-dimensional invariant tests for hypotheses concerning the shape of distribution and computed for several f (compare Hájek and Šidák [(1967), page 51], Uthoff (1970, 1973) and Franck (1981)). For symmetric f it can be transformed to an integral over the half-plane $u \geq 0$.

Now, let us consider the most powerful G^*-invariant test for $(\mathcal{N}_p, \mathcal{F}, G^*)$. As in the preceding, we have to find

$$F_i^*(X) = \int_{G^*} p_i\left(CXP + b1_n^T\right) |\det C|^n \, d\mathcal{V}^*(g), \qquad i = 0, 1.$$

Applying Lemma 1 we get, similarly to (2.4),

$$(2.7) \quad F_i^*(X) = \int_{\mathrm{UT}(p)} \int_{\mathbb{R}^p} p_i\left(CX + b1_n^T\right) \prod_{j=1}^{p} c_{jj}^{n+j-p-2} \, db \, dC.$$

For H_0 we have $p_0(Y) \propto \exp\{-\frac{1}{2} \operatorname{tr} YY^T\}$, $Y = CX + b1_n^T$. After some standard transformations [compare Sinha (1984)] we get

$$(2.8) \quad F_0^*(X) \propto |S|^{-(n-p)/2} \prod_{i=1}^{p} w_{ii}^{2(1-i)},$$

where $W = [w_{ij}] \in \mathrm{UT}(p)$ and $WW^T = S$.

We now proceed to calculate $F_1^*(X)$. Using (2.7) we write a formula analogous to (2.5), define $\overline{V} = [v_{ij}] \in \mathrm{UT}(p)$ and replace the integration variable C by $\overline{V}$ according to $v_{ii} = c_{ii}$, $v_{ii+1} = c_{ii+1}/c_{ii}, \dots, v_{ip} = c_{ip}/c_{ii}$, $i = 1, \dots, p$. The

Jacobian of this transformation is $|\partial C/\partial \bar{V}| = \prod_{i=1}^{p} v_{ii}^{p-i}$. Let V be a matrix $\bar{V}$ with $v_{ii} = 1$, $i = 1, \ldots, p$. After some elementary computations we get

$$F_1^*(X) = \int_{\mathbb{R}^{p(p-1)/2}} L^*\left[(VX)^{(1)}\right] \cdots L^*\left[(VX)^{(p)}\right] dV, \tag{2.9}$$

where $(VX)^{(i)}$ is the ith row of VX and for a vector $Y = (y_1, \ldots, y_n)$,

$$L^*(Y) = \int_{-\infty}^{\infty} \int_0^{\infty} f(uy_1 + v, \ldots, uy_n + v) u^{n-2}\, du\, dv.$$

The function $L^*(Y)$ has been analysed in one-dimensional theory of invariant tests and computed for the normal and exponential distributions by Uthoff (1970).

Formulas (2.1)–(2.3), (2.6), (2.8) and (2.9) give the general form of the tests considered.

3. Testing binormality against bivariate exponential. We assume here $\mathscr{F}$ to be a probability function of a two-dimensional random vector with a probability density function of the form $\psi(\xi_1, \xi_2) = \exp\{-(\xi_1 + \xi_2)\}$ for $\xi_1 \geq 0$, $\xi_2 \geq 0$ and find the most powerful G^*-invariant test for $(\mathscr{N}_2, \mathscr{F}, G^*)$.

Formulas (2.1), (2.2), (2.8) and (2.9) give for $p = 2$ the critical regions

$$\hat{\sigma}_2^2\left(\hat{\sigma}_1\hat{\sigma}_2\sqrt{1 - r_{12}^2}\right)^{n-2} L^*(X^{(2)}) \int_{-\infty}^{\infty} L^*(X^{(1)} + sX^{(2)})\, ds \geq c_{\text{cr}}, \tag{3.1}$$

where $\hat{\sigma}_1^2$, $\hat{\sigma}_2^2$ and r_{12} are, respectively, sample variances for the first and for the second row of X and the sample correlation coefficient. For the sake of simplicity of further considerations, we replace the matrix X by the matrix Y which results from X after a permutation of its columns putting the second row in an increasing sequence. Of course, it does not change the value of the left-hand side of (3.1).

Following Uthoff (1970) we have $L^*(Y^{(2)}) \propto (\bar{Y}^{(2)} - Y^{(2)}_{\min})^{-(n-1)}$, where $\bar{Y}^{(2)}$ denotes the sample mean and $Y^{(2)}_{\min}$ the minimal element of $Y^{(2)}$. Consequently, $L^*(Y^{(1)} + sY^{(2)})$ is proportional to $[\bar{Y}^{(1)} + s\bar{Y}^{(2)} - (Y^{(1)} + sY^{(2)})_{\min}]^{-(n-1)}$. Let $Y = [y_{ij}]$. We have $(Y^{(1)} + sY^{(2)})_{\min} = y_{1l} + sy_{2l}$ if and only if $y_{1l} + sy_{2l} < y_{1i} + sy_{2i}$ for $i \neq l$, which may be expressed in the form $s > (y_{1l} - y_{1i})/(y_{2i} - y_{2l})$ for $i > l$ and $s < (y_{1l} - y_{1i})/(y_{2i} - y_{2l})$ for $i < l$. Denote

$$\begin{aligned} \underline{\alpha}_l &= \max_{i>l} (y_{1l} - y_{1i})/(y_{2i} - y_{2l}), \qquad l = 1, \ldots, n-1, \\ \bar{\alpha}_l &= \min_{i<l} (y_{1l} - y_{1i})/(y_{2i} - y_{2l}), \qquad l = 2, \ldots, n, \end{aligned} \tag{3.2}$$

and assume $\underline{\alpha}_n = -\infty$, $\bar{\alpha}_1 = \infty$ and $(\pm\infty)^{-1} = 0$.

Let further $\Gamma = \{l: \underline{\alpha}_l < \bar{\alpha}_l, l = 1, \ldots, n\}$, $z_{1l} = \bar{Y}^{(1)} - y_{1l}$ and $z_{2l} = \bar{Y}^{(2)} - y_{2l}$, $l = 1, \ldots, n$. The integral in (3.1) is then computed as the sum of integrals over the intervals $(\underline{\alpha}_l, \bar{\alpha}_l)$, $l \in \Gamma$. This leads after some transformations to the following formula which is equivalent to (3.1):

$$T_E^* = \frac{\bar{Y}^{(2)} - Y^{(2)}_{\min}}{\hat{\sigma}_1\hat{\sigma}_2\sqrt{1 - r_{12}^2}} \left[\frac{\hat{\sigma}_1\sqrt{1 - r_{12}^2}}{\hat{\sigma}_2 I_E}\right]^{1/(n-1)} \leq c_{\text{cr}}, \tag{3.3}$$

where $I_E = \sum_{l \in \Gamma} z_{2l}^{-1}[(z_{1l} + \underline{\alpha}_l z_{2l})^{-(n-2)} - (z_{1l} + \bar{\alpha}_l z_{2l})^{-(n-2)}]$ and c_{cr} is a suitably chosen constant. This test is analysed in Section 6.

4. G^*-invariant testing binormality against bivariate uniformity. In this section we find the most powerful G^*-invariant test for $(\mathscr{N}_2, \mathscr{F}, G^*)$, $\mathscr{F}$ being a distribution function of a random vector uniformly distributed over $[0,1]^2$. In this case we have [see Uthoff (1970)] $L^*(Y^{(2)}) \propto (Y^{(2)}_{\max} - Y^{(2)}_{\min})^{-(n-1)}$ and (3.1) holds. Hence, $L^*(Y^{(1)} + sY^{(2)}) \propto [(Y^{(1)} + sY^{(2)})_{\max} - (Y^{(1)} + sY^{(2)})_{\min}]^{-(n-1)}$. In Section 3 we have defined $\underline{\alpha}_l, \bar{\alpha}_l$ and found that $(Y^{(1)} + sY^{(2)})_{\min} = y_{1l} + sy_{2l}$ if and only if $\underline{\alpha}_l < s < \bar{\alpha}_l$. Similarly, one can prove that $(Y^{(1)} + sY^{(2)})_{\max} = y_{1l} + sy_{2l}$ if and only if $\underline{\eta}_l < s < \bar{\eta}_l$, where

$$(4.1)\qquad \begin{aligned} \underline{\eta}_l &= \max_{i<l} (y_{1l} - y_{1i})/(y_{2i} - y_{2l}), \qquad l = 2,\dots,n, \\ \bar{\eta}_l &= \min_{i>l} (y_{1l} - y_{1i})/(y_{2i} - y_{2l}), \qquad l = 1,\dots,n-1, \\ \bar{\eta}_n &= \infty, \qquad \underline{\eta}_1 = -\infty. \end{aligned}$$

As in the preceding, we assume the conventional equality $(\pm\infty)^{-1} = 0$. Let us further denote

$$(4.2)\qquad \begin{aligned} &\underline{\theta}_{kl} = \max\{\underline{\eta}_k, \underline{\alpha}_l\}, \qquad \bar{\theta}_{kl} = \min\{\bar{\eta}_k, \bar{\alpha}_l\}, \\ &\Delta = \{(k,l)\colon \underline{\theta}_{kl} < \bar{\theta}_{kl};\ k,l = 1,\dots,n\}. \end{aligned}$$

Then the integral in (3.1) becomes the sum of integrals over the intervals $(\underline{\theta}_{kl}, \bar{\theta}_{kl})$, $(k,l) \in \Delta$, and after some transformations we may put (3.1) in the equivalent form

$$(4.3)\qquad T_U^* = \frac{Y^{(2)}_{\max} - Y^{(2)}_{\min}}{\hat{\sigma}_1\hat{\sigma}_2\sqrt{1 - r_{12}^2}} \left[\frac{\hat{\sigma}_1\sqrt{1 - r_{12}^2}}{\hat{\sigma}_2 I_U}\right]^{1/(n-1)} \le c_{\mathrm{cr}},$$

where

$$I_U = \sum_{(k,l)\in\Delta} \frac{1}{y_{2k} - y_{2l}} \Big\{\big[(y_{1k} - y_{1l}) + \underline{\theta}_{kl}(y_{2k} - y_{2l})\big]^{-(n-2)} - \big[(y_{1k} - y_{1l}) + \bar{\theta}_{kl}(y_{2k} - y_{2l})\big]^{-(n-2)}\Big\}.$$

For further investigation of this test see Section 6.

5. G-invariant testing binormality against bivariate uniformity. As a third example we find the most powerful G-invariant test for $(\mathscr{N}_2, \mathscr{F}, G)$, where $\mathscr{F}$ is a distribution function of a random vector uniformly distributed over $[0,1]^2$. Formulas (2.1)–(2.3) and (2.6) give in this case

$$(5.1)\qquad \left(\hat{\sigma}_1\hat{\sigma}_2\sqrt{1 - r_{12}^2}\right)^{n-1} \int_{\mathbf{R}^2} L(Y^{(1)} + vY^{(2)}) L(Y^{(1)} + tY^{(2)}) |t - v|^{n-3}\, dt\, dv \ge c_{\mathrm{cr}}$$

and, following Uthoff (1970), we have $L(Y^{(i)}) \propto L^*(Y^{(i)}) \propto (Y^{(i)}_{\max} - Y^{(i)}_{\min})^{-(n-1)}$.

Applying the results of the previous section and the definitions (4.1) and (4.2) we may write

$$L(Y^{(1)} + vY^{(2)})L(Y^{(1)} + tY^{(2)}) \propto \left[(y_{1k} - y_{1l}) + v(y_{2k} - y_{2l})\right]^{-(n-1)} \times \left[(y_{1m} - y_{1r}) + t(y_{2m} - y_{2r})\right]^{-(n-1)}$$

for $v \in (\underline{\theta}_{kl}, \bar{\theta}_{kl})$ and $t \in (\underline{\theta}_{mr}, \bar{\theta}_{mr})$, $(k, l) \in \Delta$, $(m, r) \in \Delta$. It follows from definitions (4.1) and (4.2) that

$$\bigcup_{(k,l)\in\Delta} \left[\underline{\theta}_{kl}, \bar{\theta}_{kl}\right] = \mathbb{R}$$

and the intervals $[\underline{\theta}_{kl}, \bar{\theta}_{kl}]$, $(k, l) \in \Delta$, have only endpoints in common. Let us order these intervals with respect to increasing values of $\underline{\theta}_{kl}$, denote the ith interval by I_i and define $k(i)$, $l(i)$ by $I_i = [\underline{\theta}_{k(i)l(i)}, \bar{\theta}_{k(i)l(i)}]$. Note that

$$\mathbb{R}^2 = \bigcup_{i=1}^{\alpha} \bigcup_{j=1}^{\alpha} M_{ij},$$

where α is the number of elements of Δ and $M_{ij} = I_i \times I_j$. Note further that the function under the integral in (5.1) is symmetric with respect to the straight line $v = t$, define $\overline{M}_{ii} = \{(v, t): (v, t) \in M_{ii} \text{ and } t \geq v\}$ and denote by T_{ij} the integral over M_{ij}, $i \neq j$, and by T_{ii} the integral over $\overline{M}_{ii}$. We may then rewrite (5.1) equivalently as

$$\hat{\sigma}_1\hat{\sigma}_2\sqrt{1 - r_{12}^2}\left(\sum_{i=1}^{\alpha-1}\sum_{j=i+1}^{\alpha} T_{ij} + \sum_{i=1}^{\alpha} T_{ii}\right)^{1/(n-1)} \geq c_{\text{cr}}. \tag{5.2}$$

To evaluate T_{ij} denote $a_i = y_{1k(i)} - y_{1l(i)}$, $b_i = y_{2k(i)} - y_{2l(i)}$, $\underline{\varepsilon}_i = a_i + \underline{\theta}_{k(i)l(i)}b_i$, $\bar{\varepsilon}_i = a_i + \bar{\theta}_{k(i)l(i)}b_i$, $x = a_i + vb_i$ and $y = a_i + tb_i$. Introducing in T_{ii} $1/x$ and $1/y$ as new integration variables we get finally

$$T_{ii} = \frac{1}{(n-2)(n-1)b_i^{n-1}}\left(\frac{1}{\underline{\varepsilon}_i} - \frac{1}{\bar{\varepsilon}_i}\right)^{n-1}. \tag{5.3}$$

Similarly, for $i \neq j$,

$$T_{ij} = (b_ib_j)^{2-n}\int_{\underline{\varepsilon}_i}^{\bar{\varepsilon}_i}\int_{\underline{\varepsilon}_j}^{\bar{\varepsilon}_j}(b_iy - b_jx - \delta_{ij})^{n-3}(xy)^{1-n}\,dy\,dx,$$

where $\delta_{ij} = b_ia_j - b_ja_i$. For $\delta_{ij} = 0$ this integral may be computed analogously to T_{ii} as

$$T_{ij} = \frac{1}{(n-1)(n-2)}\left\{\left[\frac{1}{\bar{\varepsilon}_ib_j} - \frac{1}{\underline{\varepsilon}_jb_i}\right]^{n-1} - \left[\frac{1}{\bar{\varepsilon}_ib_j} - \frac{1}{\bar{\varepsilon}_jb_i}\right]^{n-1} + \left[\frac{1}{\underline{\varepsilon}_ib_j} - \frac{1}{\bar{\varepsilon}_jb_i}\right]^{n-1} - \left[\frac{1}{\underline{\varepsilon}_ib_j} - \frac{1}{\underline{\varepsilon}_jb_i}\right]^{n-1}\right\}. \tag{5.4}$$

For $\delta_{ij} \neq 0$ we introduce new variables, say $\xi = \delta_{ij}(b_j x)^{-1}$, $\eta = \delta_{ij}(b_i y)^{-1}$, and use the trinomial theorem on $(\xi - \eta - \xi\eta)^{n-3}$. Then the integral becomes a sum of products of elementary integrals. Changing summation variables we get finally

$$(5.5)\quad \begin{aligned} T_{ij} &= (n-2)^{-1}\delta_{ij}^{1-n} \sum_{k=1}^{n-2} \sum_{l=0}^{k-1} \binom{n-2}{k}\binom{k-1}{l} \frac{(-1)^{k+1}}{n-l-2} \\ &\quad \times \left[\left(\frac{\delta_{ij}}{b_i \underline{\varepsilon}_j}\right)^k - \left(\frac{\delta_{ij}}{b_i \bar{\varepsilon}_j}\right)^k\right]\left[\left(\frac{\delta_{ij}}{b_j \underline{\varepsilon}_i}\right)^{n-l-2} - \left(\frac{\delta_{ij}}{b_j \bar{\varepsilon}_i}\right)^{n-l-2}\right]. \end{aligned}$$

Note that formulas (5.3)–(5.5) remain true for i or j equal to 1 or n under the convention $(\pm\infty)^{-1} = 0$ and for $b_i < 0$. If $b_i < 0$, then $\underline{\varepsilon}_i > \bar{\varepsilon}_i$, but the variables x and y are always positive. The common factor $(n-2)^{-1}$ may be omitted. Formulas (5.2)–(5.5) give the critical regions of the test which will be denoted by T_U.

6. Properties of the tests and some approximations.

As an almost immediate consequence of the Hunt–Stein theorem, the existence theorem for maximin tests and the results of Sections 3 and 4, we have the following

THEOREM. *The tests T_E^* and T_U^* are maximin tests for testing problems considered in Sections* 3 *and* 4, *respectively.*

To show that UT(p) satisfies the assumptions of the Hunt–Stein theorem a slight modification of considerations from Lehmann (1950) is sufficient. For a similar result for $p = 1$ see Hájek and Šidák [(1967), page 80]. It is a well known fact that GL(p) does not satisfy the Hunt–Stein assumptions and it is not clear whether T_U is also a maximin test.

The tests T_E^*, T_U^* and T_U have rather complicated form and it seems to be very difficult to find their exact critical points. Instead, a preliminary Monte Carlo analysis was carried out. Empirical critical points and empirical powers were found in each case on the basis of 500 generated samples. Table 1 shows the power of the test T_E^* for $n = 10$ against the bivariate exponential alternative and, for comparison, the power of the two-dimensional Shapiro–Wilk test W_2 derived by Malkovich and Afifi (1973). Table 2 shows, for $n = 25$, the power of

TABLE 1
$\exp_2$, $n = 10$

α	0.05	0.10	0.15	0.20	0.25
T_E^*	0.84	0.90	0.95	0.96	0.97
W_2	0.41	0.57	0.64	0.70	0.72

TABLE 2
U_2, $n = 25$

α	0.05	0.10	0.15	0.20	0.25
T_U^*	0.84	0.94	0.98	0.99	0.99
CM	0.50	0.61	0.68	0.75	0.77

TABLE 3
U_2, $n = 10$

α	0.05	0.10	0.15	0.20	0.25
T_U	0.18	0.33	0.43	0.53	0.59

the test T_U^* and the power of the best of tests studied by Malkovich and Afifi (1973) against the bivariate uniform alternative described in Section 4.

Numerical analysis of the test T_U caused some trouble. Formula (5.5) proved to be numerically very unstable and even for $n = 10$ double precision had to be used. The powers of the test T_U for $n = 10$ against bivariate uniformity are given in Table 3. For $n = 25$ even double precision was insufficient to compute T_{ij} according to (5.5) properly and we had to look for an approximate version of T_U.

Let us transform the matrix X and define a new observation matrix $B_x = M_x L_x(X - \tilde{X})$, where $L_x \in \mathrm{UT}(p)$ is the matrix defined by $L_x^T L_x = nS^{-1}$, $n\tilde{X} = X 1_n 1_n^T$ and M_x is an orthogonal matrix defined in Szkutnik (1987). The approximate test statistic will be based on B_x which is affine invariant [compare Szkutnik (1987)]. Hence, it will be G-invariant. Because of the invariance the values of the test statistic computed for X and for B_x are equal. Note that for B_x we have $\hat{\sigma}_1 = \hat{\sigma}_2 = 1$, $r_{12} = 0$ and the formula (5.2) becomes simpler. We omit the details here and mention only that the approximate equality $a^k - b^k \simeq k[(a+b)/2]^{k-1}(a-b)$ is used for different a, b and k and, for large n, we assume that $(a_1^n + \cdots + a_k^n)^{1/n} \simeq \max_{j=1,\dots,k} a_j$ for $a_j \geq 0$, $j = 1, \dots, k$.

The statistic of the approximate version of the test T_U is found to be

$$\tilde{T}_U = \max\left\{ V_i, \frac{W_{ji} - W_{ij} - \delta_{ij}/2}{2W_{ij}W_{ji}}; i, j = 1, \dots, \alpha \text{ and } j > i \right\},$$

where

$$W_{ij} = b_j/(1/\underline{\varepsilon}_i + 1/\bar{\varepsilon}_i), \qquad V_i = (1/\underline{\varepsilon}_i - 1/\bar{\varepsilon}_i)/b_i$$

are computed for the matrix B_x. The powers of this approximate test are given in Table 4.

Comparing the first row of Table 4 with Table 3 we see that our approximation performs very well even for $n = 10$.

TABLE 4
U_2, test $\tilde{T}_U$

n \ α	0.05	0.10	0.15	0.20	0.25
10	0.19	0.32	0.41	0.52	0.59
15	0.32	0.51	0.62	0.70	0.76
20	0.60	0.75	0.85	0.90	0.92
25	0.83	0.92	0.96	0.97	0.98

Some approximations of the tests T_E^* and T_U^*, similar to $\tilde{T}_U$, may also be constructed and show in simulation practically the same power as original tests. Their critical regions have the form

$$\tilde{T}_E^* = y_{21}\max\{y_{1l} + \alpha_l y_{2l};\ l \in \Gamma\} < c_{\mathrm{cr}},$$

where $\alpha_1 = \underline{\alpha}_1$, $\alpha_n = \bar{\alpha}_n$ and $\alpha_l = (\underline{\alpha}_l + \bar{\alpha}_l)/2$ in the remaining cases and

$$\tilde{T}_U^* = (y_{2n} - y_{21})\min\{y_{1k} - y_{1l} + \theta_{kl}(y_{2k} - y_{2l});\ (k,l) \in \Delta\} < c_{\mathrm{cr}},$$

where $\theta_{1n} = \bar{\theta}_{1n}$, $\theta_{n1} = \underline{\theta}_{n1}$ and $\theta_{kl} = (\underline{\theta}_{kl} + \bar{\theta}_{kl})/2$ in the remaining cases. Both these statistics are based on the matrix $B_x' = L_x(X - \tilde{X})$ and, hence, they are G^*-invariant [compare Szkutnik (1987)]. The matrix Y and all notation are the same as in Sections 3–5.

7. Final remarks. The powers of the tests T_E^*, T_U^* and T_U presented in this paper are approximate upper bounds for the power of any G^*- or G-invariant test for binormality against the alternatives studied. Tables 1 and 2 show a great difference, in small samples, between the power of some known tests and of optimal ones.

Since G^* is a subgroup of G, the upper bounds for the power of G^*-invariant tests are, at the same time, upper bounds for G-invariant tests although, in this case, they may be inaccessible. On the other hand, the group G^* itself is of interest. If we are looking for a test which would be sensitive against some specific types of nonnormality, the G-invariance may not always be desired [compare Szkutnik (1987)]. From this point of view the test T_E^* may be considered as a test directed to strongly asymmetric alternatives and some results of our simulations, not presented here, support this thesis. In some cases protection against such kinds of nonnormality is highly desirable. It is known, for example, that the size of the Hotelling T^2 test is much influenced by the asymmetry of the parent distribution while symmetric departures from normality are not so crucial [Mardia (1970)].

There were two reasons for choosing the uniform and exponential alternatives. First, they may be considered examples of, respectively, symmetric and nonsymmetric alternatives and second, expressing in an exact form the most powerful invariant tests against other alternatives is a very difficult task. Formulas (2.6) and (2.9) show that we have to find the tests for $p = 1$ first [or, equivalently, compute the function $L^*(Y)$ or $L(Y)$] and then be able to perform the integration in (2.6) or (2.9). For $p = 1$ only two other alternatives have been successfully analysed, namely the Cauchy and Laplace [Franck (1981) and Uthoff (1973)], leading to $L^*(Y)$ functions which prove to be practically intractable in (2.6) or (2.9).

Similarly, formula (2.9) shows, after simple transformation, that calculation of $F_1^*(X)$ for, e.g., $p = 3$ is, in fact, equivalent to the integration with respect to, say, u of the "two-dimensional" $F_1^*(Y)$ with $X^{(1)} + uX^{(2)}$ as the first and $X^{(3)}$ as the second row of Y. The "two-dimensional" F_1^* functions are, however, too complicated to admit such an operation.

Though the test statistics obtained in this paper seem to be complicated, the algorithms for computing their values are, save T_U, simple and fast.

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