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AVERAGE RUN LENGTHS OF AN OPTIMAL METHOD OF DETECTING A CHANGE IN DISTRIBUTION[1]

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Suppose one is able to observe sequentially a series of independent observations $X_1, X_2, \ldots,$ such that $X_1, X_2, \ldots, X_{\nu-1}$ are i.i.d. with known density f_0 and $X_\nu, X_{\nu+1}, \ldots$ are i.i.d. with density f_θ where ν is unknown. Define

$$R(n,\theta) = \sum_{k=1}^{n} \prod_{i=k}^{n} \frac{f_\theta(X_i)}{f_0(X_i)}.$$

It is known that rules, which call for stopping and raising an alarm the first time n that $R(n,\theta)$ or a mixture thereof exceeds a prespecified level A, are optimal methods of detecting that the density of the observations is not f_0 any more.

Practical applications of such stopping rules require knowledge of their operating characteristics, whose exact evaluation is difficult. Here are presented asymptotic ($A \to \infty$) expressions for the expected stopping times of such stopping rules (a) when $\nu = \infty$ and (b) when $\nu = 1$. We assume that the densities f_θ form an exponential family and that the distribution of $\log(f_\nu(X_1)/f_0(X_1))$ is (strongly) nonlattice.

Monte Carlo studies indicate that the asymptotic expressions are very good approximations, even when the expected sample sizes are small.

1. Introduction. Suppose one accumulates independent observations from a certain process. Initially, the process is at state 0. At some unknown point in time something occurs (e.g., a "breakdown") which puts the process in state 1, and consequently, the stochastic behavior of the observations changes. It is of interest to declare that a change took place (to "raise an alarm") as soon as possible after its occurrence, subject to a restriction on the rate of false detections. It is assumed that the aforementioned observations are the only information one has about the process, and the problem is to construct a good detection scheme.

Practical examples of this problem arise in areas such as health, quality control, ecological monitoring, etc. For instance, consider surveillance for congenital malformations in newborn infants. Under normal circumstances, the percentage of babies born with a certain type of malformation has a known value p_0. Should something occur (such as an environmental change, the introduction of a new drug to the market, etc.) the percentage may increase (e.g., the thalidomide episode of the 1960s). One would want to raise an alarm as quickly

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as possible after a change takes place, subject to an acceptable rate of false alarms. Generally, the problem arises wherever surveillance is being done.

A solution to the problem depends on what is known in advance about the distributions of the observations. Let f_0 denote the density of observations with respect to a σ-finite measure μ when the process is in state 0, let f_θ denote the density of observations with respect to μ when the process is in state 1 and let ν denote the unknown point in time when the first observation from state 1 is made. Thus, one has a sequence of independent observations $X_1, X_2, \dots,$ such that $X_1, X_2, \dots, X_{\nu-1}$ are i.i.d. with density f_0 and $X_\nu, X_{\nu+1}, \dots$ are i.i.d. with density f_θ, where $1 \le \nu \le \infty$ is unknown. It will be assumed here that f_0, f_θ belong to an exponential family of distributions and that f_0 is known.

Solutions for the problem that are in current use are known as CUSUM procedures. For a survey, see, for instance, Johnson and Leone (1962, 1963, 1964). Lorden (1971) proved a first-order asymptotic optimality property of a certain class of procedures for reacting to a change in distribution. When f_θ is known, this class includes most of the standard appropriate CUSUM techniques as special cases. When f_θ is unknown, Lorden (1971) suggests a first-order asymptotically optimal procedure. [Asymptotic operating characteristics of this and related procedures are given in Pollak and Siegmund (1975). Further refinements can be obtained using results of Lai and Siegmund (1977).]

Shiryayev (1963, 1978) solved the problem in a Bayesian framework in the case that f_θ is known. He also proposed a solution in a certain classical setting.

A solution that is second-order asymptotically optimal in a classical framework is presented in Pollak (1985). The statistic underlying this solution for the case that f_θ is known was considered by Shiryayev (1963) and Roberts (1966). Asymptotic operating characteristics of this and related procedures are the subject under study here.

Without loss of generality, let the assumed exponential family be defined by

$$f_y(x) = e^{yx-\psi(y)}, \qquad y \in \Omega,$$

where Ω is an interval on the real line, $0 = \psi(0) = \psi'(0)$. Let F be a probability measure on Ω with $F(\{0\}) = 0$. Let $0 < A < \infty$. Define

$$R(n, y) = \sum_{k=1}^{n} \prod_{i=k}^{n} \frac{f_y(X_i)}{f_0(X_i)} = \sum_{k=1}^{n} \exp\left(y \sum_{i=k}^{n} X_i - (n-k+1)\psi(y) \right),$$

$$R(n, F) = \int R(n, y)\, dF(y),$$

$$N(A, y) = \min\{n | R(n, y) \ge A\},$$

$$N(A, F) = \min\{n | R(n, F) \ge A\}.$$

Raising an alarm at time $N(A, \theta)$ is a procedure with optimality properties when the value θ (of the parameter of the distribution after a change occurred) is known and raising an alarm at time $N(A, F)$ has optimality properties when θ is unknown (Pollak, 1985).

In order to evaluate and compare between procedures, one needs to formalize a restriction on false detections as well as to formalize an expression for the speed of detection of a change after its occurrence. The restriction on false detections is usually formalized as a requirement that the expected number of observations until a false alarm (assuming that $\nu = \infty$) exceed a prespecified value B. This suggests a need for evaluating $E\{N(A, y)|\nu = \infty\}$, $E\{N(A, F)|\nu = \infty\}$. The quality of a procedure with regard to the speed of detection of a change after its occurrence is often measured by the supremum (or essential supremum) of the expected number of observations that it takes to detect a change after its occurrence, given that no false alarms have previously been raised [see Lorden (1971) and Pollak and Siegmund (1975)]. This suggests a need for evaluating $E(N(A, \theta) - \nu|\nu = 1, \theta)$, $E(N(A, F) - \nu|\nu = 1, \theta)$. These operating characteristics are difficult to compute. [For simulations see Roberts (1966).]

In this article, asymptotic expressions ($A \to \infty$) for these operating characteristics are presented. Monte Carlo studies indicate that these expressions are very good approximations, even when the expected sample sizes are small.

2. The average run length when $\nu = \infty$. Denote by P_ν^y, E_ν^y the probability, expectation, respectively, when $1 \le \nu < \infty$, $X_1, \dots, X_{\nu-1}$ are i.i.d. with density f_0 and are independent of $X_\nu, X_{\nu+1}, \dots$, which are i.i.d. with density f_y. Let P_∞, E_∞ denote probability, expectation, respectively, when $\nu = \infty$. Let F be a probability measure on Ω with $F(\{0\}) = 0$. Denote

$$Z_i^y = \log \frac{f_y(X_i)}{f_0(X_i)} = yX_i - \psi(y),$$

$$I(\theta) = E_1^\theta Z_1^\theta,$$

$$M(B, y) = \min\left\{n \,\middle|\, \sum_{i=1}^n Z_i^y \ge B\right\}, \qquad M(B, y) = \infty, \text{if no such } n \text{ exists},$$

$$\gamma(y) = 1 \Big/ \lim_{B \to \infty} E_1^y \exp\left\{-\left[\sum_{i=1}^{M(B, y)} Z_i^y - B\right]\right\} = 1 \Big/ \lim_{B \to \infty} BP_\infty\{M(B, y) < \infty\},$$

$$\gamma(F) = 1 \Big/ \int (1/\gamma(y))\, dF(y).$$

The computations of $\gamma(y)$ and $\gamma(F)$ are applications of renewal theory and have been calculated in other contexts. [See Feller (1971), Siegmund (1975) and (1985), Lai and Siegmund (1977) and Theorem 6.2 of Woodroofe (1982).]

THEOREM 1. (i) *$E_\infty N(A, y) \ge A$ for all $y \in \Omega$. If $I(y) < \infty$, then $E_\infty N(A, y) = O(A)$, where $O(A)/A$ is bounded as $A \to \infty$.*

(ii) *If* $y \in \Omega$, $I(y) < \infty$ *and the* P_1^y*-distribution of* Z_1^y *is nonlattice, then*

$$E_\infty N(A, y) = A\gamma(y)(1 + o(1)),$$

where $o(1) \to 0$ *as* $A \to \infty$.

THEOREM 2. *Suppose that the* P_1^y*-distribution of* X_1 *is strongly nonlattice* [*see Stone* (1965)] *for all* $y \in \Omega$. *Then*

(i) $E_\infty N(A, F) \geq A$. *If* $F(\{y | I(y) < \infty\}) = 1$, *then* $E_\infty N(A, F) = O(A)$, *where* $O(A)/A$ *is bounded as* $A \to \infty$.

(ii) *If* $F(\{y | I(y) < \infty\}) = 1$, *then*

$$E_\infty N(A, F) = A\gamma(F)(1 + o(1)),$$

where $o(1) \to 0$ *as* $A \to \infty$.

3. Proof of Theorem 1. For simplicity's sake, we will denote $Z_i = Z_i^y$, $R(n) = R(n, y)$ and $N_A = N(A, y)$ throughout this section.

The idea of the proof of Theorem 1 can be described as follows. Note that under P_∞, $R(n) - n$ is a martingale with zero expectation with respect to $\mathscr{F}(X_1, \dots, X_n)$. Consequently, by the optional sampling theorem, $E_\infty N_A = E_\infty R(N_A) = AE_\infty R(N_A)/A$. Therefore, the proof becomes an analysis of the asymptotic behavior of $E_\infty R(N_A)/A$ as $A \to \infty$.

Note that $R(n + 1) = [1 + R(n)]\exp\{Z_{n+1}\}$ and so

$$\log R(n + r) = \sum_{i=n+1}^{n+r} Z_i + \log R(n) + \sum_{i=0}^{r-1} \log[1 + 1/R(n + i)]. \tag{1}$$

When $R(n)$ is large, $\log[1 + 1/R(n)]$ is small, so the increments of $\log(R(n)$ begin to act like i.i.d. random variables (with negative drift under P_∞);

$$\log R(n + r) \approx \sum_{i=n+1}^{n+r} Z_i + \log R(n) \overset{\text{def}}{=} U(n, r)$$

for small values of r. Note that $\sum_{i=n+1}^{n+r} Z_i$ can be regarded as the log-likelihood ratio of the observations following X_n for H_0: $\theta = 0$ versus H_1: $\theta = y$. Let $C > 0$ be a (large) constant, and consider (1) for $n = N_{A/C}$. Since Z_i has negative drift (under P_∞) the process $U(N_{A/C}, r)$, $r = 1, 2, \dots,$ will either exceed $\log A$ "soon" or not at all. If it does, then $\log R(N_A) - \log A \approx \sup_{r \geq 1} U(N_{A/C}, r) - \log A$ can be handled in the usual way, as the excess over the boundary of the log-likelihood ratio statistic defining a power one test of H_0: $\theta = 0$ versus H_1: $\theta = y$ [as suggested by the approach of Lorden and Eisenberger (1973) and Siegmund (1975); see also Pollak (1985)]. If it doesn't, wait until the next time $R(n)$ crosses A/C, and reapply the same rationale. The behavior of $R(n)$ here will be almost independent of its behavior in the vicinity of the previous crossing of A/C. Continue this until the first time $R(n)$ crosses A/C and "soon" thereafter crosses A. This will account for Theorem 1. The formal details are now spelled out.

PROOF OF THEOREM 1(i). Denote $\Pi_A = \min\{n|\max_{1\le k\le n}\exp\{\sum_{i=k}^{n} Z_i\} \ge A\}$. It is well known that $E_\infty \Pi_A < \infty$. Since $N_A \le \Pi_A$, $1 \le k \le n$ $E_\infty N_A < \infty$. Hence $E_\infty[R(N_A) - N_A]$ exists. Since $|R(n)| < A$ on $\{N_A > n\}$ it is easy to see that $\liminf_{n\to\infty} \int_{\{N_A > n\}} |R(n) - n|\, dP_\infty = 0$. Therefore, the martingale optional sampling theorem [cf. Chow, Robbins and Siegmund (1971), Theorem 2.3] applies to yield $E_\infty(R(N_A) - N_A) = 0$. Hence, $E_\infty N_A = E_\infty R(N_A) \ge A$.

For the second part of Theorem 1(i), let $S_0 = 0$ and define S_i recursively by

$$S_i = \min\left\{ n|n > S_{i-1},\ \sum_{j=S_{i-1}+1}^{n} Z_j \notin (0, \log A) \right\}.$$

Let

$$d = \min\left\{ i \,\middle|\, \sum_{j=S_{i-1}+1}^{S_i} Z_j \in [\log A, \log(2A)] \right\}.$$

Clearly, $\Pi_A \le \sum_{i=1}^{d} S_i$. By Wald's lemma,

$$E_\infty \Pi_A \le E_\infty S_1 \Big/ P_\infty\left\{ \sum_{j=1}^{S_1} Z_j \in [\log A, \log(2A)] \right\}.$$

Now

$$\begin{aligned} P_\infty&\left\{ \sum_{j=1}^{S_1} Z_j \in [\log A, \log(2A)] \right\} \\ &= \sum_{n=1}^{\infty} \int_{\{S_1 = n,\ d=1\}} f_0(x_1, \dots, x_n)\, d\mu(x_1, \dots, x_n) \\ &\ge [1/(2A)] \sum_{n=1}^{\infty} \int_{\{S_1 = n,\ d=1\}} f_y(x_1, \dots, x_n)\, d\mu(x_1, \dots, x_n) \\ &= [1/(2A)]\, P_1^y\left\{ \sum_{j=1}^{S_1} Z_j \in [\log A, \log(2A)] \right\}. \end{aligned}$$

As $A \to \infty$, $\limsup E_\infty S_1 < \infty$, and, by the renewal theorem,

$$\liminf P_1^y\left\{ \sum_{j=1}^{S_1} Z_j \in [\log A, \log(2A)] \right\} > 0,$$

from which Theorem 1(i) now follows. □

PROOF OF THEOREM 1(ii). Let $A > C > 1$. Define $L_0 = 0$. For $j = 1, 2, \dots$ define

$$L_j = \min\left\{n | n > L_{j-1}, \sum_{k=L_{j-1}+1}^{n} \exp\left\{\sum_{i=k}^{n} Z_i\right\} \ge A/C\right\},$$

$$\begin{aligned} Q(j,n) &= \sum_{k=L_{j-1}+1}^{n} \exp\left\{\sum_{i=k}^{n} Z_i\right\}, \quad \text{for } n > L_{j-1}, \\ &= 0, \qquad \text{for other } n, \end{aligned}$$

$$V(j,n) = \begin{cases} \exp\left\{\sum_{i=L_j+1}^{n} Z_i\right\} Q(j, L_j), & \text{if } n > L_j, \\ Q(j, L_j), & \text{if } n = L_j, \end{cases}$$

$$\begin{aligned} H_j &= \min\{n | n \ge L_j, V(j,n) \ge A\}, \\ &= \infty, \quad \text{if no such } n \text{ exists}, \\ M_j &= H_j \wedge L_{j+1}, \\ J &= \min\{j | V(j, M_j) \ge A\}, \\ M(B, y) &= \min\left\{n | \sum_{i=1}^{n} Z_i^y \ge B\right\}, \quad = \infty, \quad \text{if no such } n \text{ exists}, \\ \mathscr{F}(n) &= \mathscr{F}(X_1, \dots, X_n), \\ \Xi(\cdot) &= \text{the indicator function of the set } (\cdot). \end{aligned}$$

LEMMA 1. *Let $0 < \eta < 1$. There exist $A_\eta^{(1)}$ and $C_\eta^{(1)}$ such that if $A > C \ge C_\eta^{(1)}$ and $A \ge A_\eta^{(1)}$, then*

$$\begin{aligned} (1-\eta)\left[E_\infty Q(j, L_j)/A\right]/\gamma(y) &\le P_\infty\{H_j < \infty\} \\ &\le (1+\eta)\left[E_\infty Q(j, L_j)/A\right]/\gamma(y). \end{aligned}$$

PROOF.

$$\begin{aligned} P_\infty\{H_j < \infty\} &= E_\infty P_\infty\{H_j < \infty | \mathscr{F}(L_j)\} \\ &= E_\infty P_\infty\left\{M\left(\log\left[A/Q(j, L_j)\right], y\right) < \infty | \mathscr{F}(L_j)\right\}. \end{aligned}$$

By virtue of Theorem 1(i), C can be chosen to be large enough so as to make $P_\infty\{Q(j, L_j) > A/\sqrt{C}\}$ arbitrarily small, independently of A. If C is large enough and $Q(j, L_j) \le A/\sqrt{C}$, then by the definition of $\gamma(y)$

$$\begin{aligned} &P_\infty\left\{M\left(\log\left[A/Q(j, L_j)\right], y\right) < \infty | \mathscr{F}(L_j)\right\} \\ &\quad = \left\{\left[Q(j, L_j)/A\right]/\gamma(y)\right\}\left(1 + o_p(1)\right), \end{aligned}$$

where $o_p(1) \to_P 0$ as $C \to \infty$. Lemma 1 now follows. □

LEMMA 2. *Let $0 < \eta < 1$. There exist $A_\eta^{(2)}$ and $C_\eta^{(2)}$ such that if $A > C \geq C_\eta^{(2)}$ and $A \geq A_\eta^{(2)}$, then*

$$(1-\eta)A\gamma(y) \leq E_\infty\{V(j, H_j)|H_j < \infty\} \leq (1+\eta)A\gamma(y).$$

PROOF. $E_\infty\{V(j, H_j);\ H_j < \infty\} = E_\infty Q(j, L_j)$, so that

$$E_\infty\{V(j, H_j)|H_j < \infty\} = E_\infty Q(j, L_j)/P_\infty(H_j < \infty).$$

Lemma 1 now accounts for Lemma 2. □

LEMMA 3. *Let $0 < \eta < 1$. There exist $A_{\eta,c}^{(3)}$ and $C_\eta^{(3)}$ such that if $C \geq C_\eta^{(3)}$ is fixed and $A \geq A_{\eta,c}^{(3)}$, then*

$$(1-\eta)A\gamma(y) \leq E_\infty\{V(j, M_j)|V(j, M_j) \geq A\} \leq (1+\eta)A\gamma(y).$$

PROOF. Clearly,

$$\begin{aligned} &E_\infty\{V(j, M_j)|V(j, M_j) \geq A\} \\ &\quad = \frac{E_\infty\{V(j, M_j);\ V(j, M_j) \geq A\}}{P_\infty\{V(j, M_j) \geq A\}} \\ &\quad = \frac{E_\infty\{V(j, M_j);\ H_j < \infty\} - E_\infty\{V(j, M_j);\ L_{j+1} < H_j < \infty\}}{P_\infty\{H_j < \infty\} - P_\infty\{L_{j+1} < H_j < \infty\}}. \end{aligned} \tag{2}$$

When C is fixed, $P_\infty\{x \leq H_j - L_j < \infty\} \to_{x\to\infty} 0$ uniformly in $A \geq C$. Therefore, by Theorem 1(i), as $A \to \infty$

$$\begin{aligned} P_\infty\{L_{j+1} \leq H_j < \infty\} &\leq P_\infty\{\log A \leq L_{j+1} - L_j \leq H_j - L_j < \infty\} \\ &\quad + P_\infty\{L_{j+1} - L_j \leq \log A \leq H_j - L_j < \infty\} \\ &\quad + P_\infty\{L_{j+1} - L_j \leq H_j - L_j \leq \log A\} \\ &\leq 2P_\infty\{\log A \leq H_j - L_j < \infty\} + P_\infty\{L_{j+1} - L_j \leq \log A\} \\ &= o(1)P_\infty\{H_j < \infty\} + P_\infty(L_1 \leq \log A) \qquad (3) \\ &\leq o(1)P_\infty\{H_j < \infty\} + \sum_{n=1}^{\log A} P_\infty\{R(n) \geq A/C\} \\ &\leq o(1)P_\infty\{H_j < \infty\} + (\log A)^2 C/A \\ &= o(1)P_\infty\{H_j < \infty\}. \end{aligned}$$

Also, when C is fixed, as $A \to \infty$

$$
\begin{aligned}
E_\infty\{V(j, M_j);\ L_{j+1} < H_j < \infty\} &\le AP_\infty(L_{j+1} < H_j < \infty)\\
&= AP_\infty(H_j < \infty)o(1)\\
&= AP_\infty(H_j \le L_{j+1})o(1)\\
&\le E_\infty\{V(j, M_j);\ H_j < \infty\}o(1).
\end{aligned}
\tag{4}
$$

Since

$$
\begin{aligned}
&E_\infty\{V(j, M_j);\ H_j < \infty\}\\
&\quad = E_\infty\{V(j, H_j);\ H_j < \infty\} - E_\infty\{V(j, H_j) - V(j, M_j);\ M_j < H_j < \infty\}\\
&\quad = E_\infty\{V(j, H_j);\ H_j < \infty\},
\end{aligned}
$$

(2), (3), (4) and Lemma 2 account for Lemma 3. □

LEMMA 4. *Let $0 < \eta < 1$. There exist $A^{(4)}_{\eta, C}$ and $C^{(4)}_\eta$ such that if $C \ge C^{(4)}_\eta$ is fixed and $A \ge A^{(4)}_{\eta, C}$, then*

$$1 - \eta \le E_\infty V(J, M_J)/[A\gamma(y)] \le 1 + \eta.$$

PROOF. Denote $V(0, M_0) = V(-1, M_{-1}) = 0$. Note that V_{j+2} is independent of $\mathscr{F}(M_j)$.

$$
\begin{aligned}
&E_\infty V(J, M_J)\\
&\quad = \sum_{j=1}^\infty \int_{J=j} V(j, M_j)\, dP_\infty\\
&\quad = \sum_{j=1}^\infty \int_{\{V(i, M_i) < A,\ i=0,\dots,j-1;\ V(j, M_j) \ge A\}} V(j, M_j)\, dP_\infty\\
&\quad = \sum_{j=1}^\infty \int_{\{V(i, M_i) < A,\ i=-1,0,\dots,j-2;\ V(j, M_j) \ge A\}} V(j, M_j)\, dP_\infty\\
&\qquad - \sum_{j=2}^\infty \int_{\{V(i, M_i) < A,\ i=0,\dots,j-2;\ V(j-1, M_{j-1}) \ge A,\ V(j, M_j) \ge A\}} V(j, M_j)\, dP_\infty.
\end{aligned}
\tag{5}
$$

The remainder of the proof will be an analysis of the two expressions on the right side of (5). We will first show the latter expression to be asymptotically negligible. It is enough to show that it is $O(A/C)$, since C can be arbitrarily large and obviously $E_\infty V(J, M_J) \geq A$. So

$$
\begin{aligned}
&\sum_{j=2}^{\infty} \int_{\{V(i, M_i)<A,\, i=0,\dots,j-2;\ V(j-1, M_{j-1})\geq A,\ V(j, M_j)\geq A\}} V(j, M_j)\, dP_\infty \\
&\quad = \sum_{j=2}^{\infty} \int_{\{V(i, M_i)<A,\, i=0,\dots,j-2;\ V(j-1, M_{j-1})\geq A\}} Q(j, L_j)\, dP_\infty \\
&\quad = \sum_{j=2}^{\infty} \int_{\{V(i, M_i)<A,\, i=0,\dots,j-2;\ V(j-1, M_{j-1})\geq A\}} \\
&\qquad\qquad \times \left\{ \left[Q(j, L_j) - Q(j, M_{j-1}) \right] + Q(j, M_{j-1}) \right\} dP_\infty \\
&\quad \leq \sum_{j=2}^{\infty} \int_{\{V(i, M_i)<A,\, i=0,\dots,j-2;\ V(j-1, M_{j-1})\geq A\}} \\
&\qquad\qquad \times \left[E_\infty Q(1, L_1) + Q(j, M_{j-1}) \right] dP_\infty \\
&\quad \leq E_\infty Q(1, L_1) \qquad (6)\\
&\qquad + \sum_{j=2}^{\infty} \int_{\{V(i, M_i)<A,\, i=0,\dots,j-2;\ V(j-1, M_{j-1})\geq A\}} Q(j, M_{j-1})\, dP_\infty \\
&\quad \leq E_\infty Q(1, L_1) \\
&\qquad + \sum_{j=2}^{\infty} \int_{\{V(i, M_i)<A,\, i=0,\dots,j-3;\ V(j-1, M_{j-1})\geq A\}} Q(j, M_{j-1})\, dP_\infty \\
&\quad \leq E_\infty Q(1, L_1) + \int_{\{V(1, M_1)\geq A\}} Q(2, M_1)\, dP_\infty \\
&\qquad\qquad \times \sum_{j=2}^{\infty} P_\infty\{ V(i, M_i) < A,\ i = 0, \dots, j-3 \} \\
&\quad \leq E_\infty Q(1, L_1) + \int_{\{V(1, M_1)\geq A\}} Q(2, M_1)\, dP_\infty [1 + E_\infty J].
\end{aligned}
$$

We will first show that $\int_{\{V(1, M_1)\geq A\}} Q(2, M_1)\, dP_\infty = O(\log A)$. Let $\tau_c = M(\log C, y)$. By the proof of Lemma 2 and by Lemma 3 of Pollak and Siegmund (1975), for any $\eta > 0$, there exists $\lambda > 0$ such that for $k = 1, 2, \dots$ and $A > 1$

$$
\begin{aligned}
&P_1^y\{ |\bar{X}_k - \psi'(y)| > \eta/|y| \} < \exp\{-\lambda k\}, \\
(7)\qquad &P_\infty\{ |\bar{X}_k| > \eta/|y| \} < \exp\{-\lambda k\}, \\
&P_1^y\{ \tau_A < (\log A)/[2I(y)] \} < \exp\{-\lambda \log A\}.
\end{aligned}
$$

Now for $C > 1,\ B > 1$

$$
\begin{aligned}
&E_\infty\{R(\tau_c);\ R(\tau_c - 1) < B,\ \tau_c \le m\}\\
&\quad = \sum_{n=1}^{m} \sum_{k=1}^{n} E_\infty\left[\exp\left\{\sum_{i=k}^{n} Z_i^y\right\}\Xi\{\tau_c = n,\ R(n-1) < B\}\right]\\
&\quad = \sum_{n=1}^{m} \sum_{k=1}^{n} E_\infty\left[\exp\left\{\sum_{i=k}^{m} Z_i^y\right\}\Xi\{\tau_c = n,\ R(n-1) < B\}\right]\\
&\quad = \sum_{k=1}^{m} E_\infty\left[\exp\left\{\sum_{i=k}^{m} Z_i^y\right\}\sum_{n=k}^{m} \Xi\{\tau_c = n,\ R(n-1) < B\}\right] \qquad (8)\\
&\quad = \sum_{k=1}^{m} E_\infty\left[\exp\left\{\sum_{i=k}^{m} Z_i^y\right\}\Xi\{k \le \tau_c \le m,\ R(\tau_c - 1) < B\}\right]\\
&\quad = \sum_{k=1}^{m} E_\infty P_k^y\{k \le \tau_c \le m,\ R(\tau_c - 1) < B | \mathscr{F}(k-1)\}.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&E_\infty\{R(\tau_c);\ R(\tau_c - 1) < B,\ \tau_c < \infty\}\\
&\quad = \sum_{k=1}^{\infty} E_\infty P_k^y\{k \le \tau_c,\ R(\tau_c - 1) < B | \mathscr{F}(k-1)\}.
\end{aligned}
$$

Let $0 < \eta < \min\{\psi(y), I(y)\}/2$. Denote $\bar{Z}_j^y = \sum_{i=1}^{j} Z_i^y/j$. For

$$k \ge (4 \log B)/[\psi(y) - \eta] + 1,$$

by virtue of (7),

$$
\begin{aligned}
&E_\infty P_k^y\{k \le \tau_c,\ R(\tau_c - 1) < B | \mathscr{F}(k-1)\}\\
&\quad = E_\infty\{P_k^y\{k \le \tau_c,\ R(\tau_c - 1) < B | \mathscr{F}(k-1)\};\ |\bar{X}_{k-1}| > \eta/|y|\} \qquad (9)\\
&\qquad + E_\infty\{P_k^y\{k \le \tau_c,\ R(\tau_c - 1) < B | \mathscr{F}(k-1)\};\ |\bar{X}_{k-1}| \le \eta/|y|\}\\
&\quad \le P_\infty\{|\bar{X}_{k-1}| > \eta/|y|\} \qquad (10)\\
&\qquad + P_k^y\left\{\sum_{i=k}^{\tau_c - 1} Z_i^y < \log B | k \le \tau_c,\ \bar{Z}_{k-1}^y = -\psi(y) + \eta\right\}\\
&\quad \le \exp\{-\lambda(k-1)\}\\
&\qquad + P_k^y\left\{\sum_{i=k}^{\tau_c - 1} Z_i^y < \log B, \tau_c - (k-1) > [\log C + (k-1)(\psi(y) - \eta)]/\right.\\
&\qquad \left.[2I(y)] | k \le \tau_c,\ \bar{Z}_{k-1}^y = -\psi(y) + \eta\right\}
\end{aligned}
$$

$$
\begin{aligned}
&+ P_k^y\{\tau_c - (k-1) \le [\log C + (k-1)(\psi(y) - \eta)]/[2I(y)] \,|\, k \le \tau_c,\\
&\qquad \bar{Z}_{k-1}^y = -\psi(y) + \eta\}\\
&\le \exp\{-\lambda(k-1)\}\\
&+ P_k^y\left\{ \sum_{i=k}^{\tau_c - 1} Z_i^y/[\tau_c - 1 - k] < I(y)/2,\ \tau_c - (k-1) \right.\\
&\qquad > [\log C + (k-1)(\psi(y) - \eta)]/\\
&\qquad \left. [2I(y)] \,|\, k \le \tau_c,\ \bar{Z}_{k-1}^y = -\psi(y) + \eta \right\}\\
&+ \exp\{-\lambda[\log C + (k-1)(\psi(y) - \eta)]\}
\end{aligned}
$$

$$
\begin{aligned}
&\le \exp\{-\lambda(k-1)\} + \sum_{j=[\log C + (k-1)(\psi(y)-\eta)]/[2I(y)]}^{\infty} \exp\{-\lambda j\} \qquad (11)\\
&+ \exp\{-\lambda[\log C + (k-1)(\psi(y) - \eta)]\}.
\end{aligned}
$$

It follows that

$$E_\infty\{R(\tau_c);\ R(\tau_c - 1) < B,\ \tau_c < \infty\} \le (4\log B)/[\psi(y) - \eta] + D(C),$$

where $D(C) < \infty$, $D(C)$ is decreasing in C. Therefore,

$$
\begin{aligned}
\int_{\{V(1, M_1) \ge A} Q(2, M_1)\, dP_\infty &\le \int_{\{H_1 < \infty,\ Q(2, H_1 - 1) < A/C)} Q(2, H_1)\, dP_\infty \qquad (12)\\
&\le [4\log(A/C)]/[\psi(y) - \eta] + D(1).
\end{aligned}
$$

Next, we will deal with the $[1 + E_\infty J]$ term on the right side of (6). Note that $V(j+1, M_{j+1})$ is not independent of $V(j, M_j)$, but is independent of $V(j-1, M_{j-1})$. To exploit this independence denote

$$
\begin{aligned}
J_{\text{odd}} &= \min\{n | n \text{ odd},\ V(n, M_n) \ge A\}, \qquad (13)\\
J_{\text{even}} &= \min\{n | n \text{ even},\ V(n, M_n) \ge A\}.
\end{aligned}
$$

Note that $\frac{1}{2}J_{\text{even}}$ and $\frac{1}{2}(J_{\text{odd}} + 1)$ are geometrically distributed random variables. Since $J = \min\{J_{\text{odd}}, J_{\text{even}}\} \le J_{\text{odd}} + J_{\text{even}}$,

$$
\begin{aligned}
E_\infty J &\le E_\infty J_{\text{odd}} + E_\infty J_{\text{even}} \le 4/P_\infty\{V(1, M_1) \ge A\} \qquad (14)\\
&\le 4/[P_\infty(H_1 < \infty) - P_\infty(L_2 \le H_1 < \infty)].
\end{aligned}
$$

Therefore, by (3), Lemma 1 and Theorem 1(i), there exists $C^\#$ such that if one fixes $C > C^\#$, then for large enough A

$$E_\infty J \le 5\gamma(y)A/E_\infty Q(1, L_1) = CO(1), \qquad (15)$$

where $O(1)$ is bounded as $A \to \infty$. Now (6), (12), (15) and Theorem 1(i) prove that the last expression on the right side of (5) is $O(A/C)$.

To complete the proof of Lemma 4, it remains to analyze the first expression on the right side of (5). So

$$
\begin{aligned}
&\sum_{j=1}^{\infty} \int_{\{V(i, M_i)<A,\ i=-1,0,\dots,j-2;\ V(j,M_j)\ge A\}} V(j, M_j)\, dP_\infty \\
&\quad = \sum_{j=1}^{\infty} E_\infty\{V(j, M_j) | V(j, M_j) \ge A\} \\
&\qquad \times P_\infty\{V(i, M_i) < A,\ i = -1,0,\dots, j-2;\ V(j, M_j) \ge A\} \\
&\quad = E_\infty\{V(1, M_1) | V(1, M_1) \ge A\} \\
&\qquad \times \Big[1 + \sum_{j=2}^{\infty} P_\infty\{V(i, M_i) < A,\ i = -1,0,\dots, j-2; \\
&\qquad\qquad V(j-1, M_{j-1}) \ge A,\ V(j, M_j) \ge A\}\Big]
\end{aligned} \tag{16}
$$

and

$$
\begin{aligned}
&\sum_{j=2}^{\infty} P_\infty\{V(i, M_i) < A,\ i = -1,0,\dots, j-2;\ V(j-1, M_{j-1}) \ge A,\ V(j, M_j) \ge A\} \\
&\quad \le A^{-1} \sum_{j=2}^{\infty} \int_{\{V(i,M_i)<A,\ i=-1,0,\dots,j-2;\ V(j-1,M_{j-1})\ge A,\ V(j,M_j\ge A)\}} V(j, M_j)\, dP_\infty \\
&\quad = \left[\frac{C \log A}{A} + \frac{1}{C}\right] O(1),
\end{aligned}
$$

by virtue of (6), (12), (15) and Theorem 1(i). Now (16) and Lemma 3 complete the proof of Lemma 4. □

LEMMA 5. *Let* $0 < \eta < 1$. *There exist* $C_\eta^{(5)}$ *and* $A_{\eta, c}^{(5)}$ *such that if* $C \ge C_\eta^{(5)}$ *is fixed and* $A \ge A_{\eta, C}^{(5)}$, *then*

$$E_\infty Q(J + 1, M_j) = O(\log A),$$

where $O(\log A)/(\log A)$ *is bounded as* $A \to \infty$ [*and* $Q(J+1, M_J)$ *is understood to be zero if* $M_J = L_J$].

PROOF. Let J_{odd} and J_{even} be as in (13). Then

$$
\begin{aligned}
&E_\infty Q(J + 1, M_J) \\
&\quad \le E_\infty Q(J_{\text{odd}} + 1, M_{J_{\text{odd}}}) + E_\infty Q(J_{\text{even}} + 1, M_{J_{\text{even}}}) \\
&\quad = 2E_\infty\{Q(2, M_1) | J = 1\} \\
&\quad \le 2\int_{\{L_1 < H_1 < \infty\}} Q(2, H_1)\, dP_\infty / [P_\infty(H_1 < \infty) - P_\infty\{L_2 < H_1 < \infty\}] \\
&\quad \le \{[4 \log(A/C)]/[\psi(y) - \eta] + D(1)\}(1 - \eta)^{-1}\gamma(y)C,
\end{aligned}
$$

where the last inequality follows from (12), (3) and Lemma 1. □

LEMMA 6. *Let $\lambda > 0$. There exist $C_\lambda^{(6)}$ and $A_{\lambda,C}^{(6)}$ such that if one fixes $C \ge C_\lambda^{(6)}$, then for any $A \ge A_{\lambda,C}^{(6)}$*

$$0 \le E_\infty R(M_J) - E_\infty V(J, M_J) \le \lambda A.$$

PROOF. Clearly,

$$R(M_J) = V(J, M_J) + Q(J+1, M_J) + \sum_{j=1}^{J-1} V(j, M_J). \tag{17}$$

Therefore, because of Lemma 5, it suffices to show that $E_\infty \sum_{j=1}^{J-1} V(j, M_J) = o(A)$ for appropriately chosen C. Let J_{odd} and J_{even} be as in (13). Then

$$\begin{aligned}
E_\infty \sum_{j=1}^{J-1} V(j, M_J) &= \sum_{j=1}^{\infty} E_\infty\{V(j, M_J);\ j \le J-1\} \\
&= \sum_{j=1}^{\infty} E_\infty\{V(j, M_j);\ j \le J-1\} \\
&= E_\infty \sum_{j=1}^{J-1} V(j, M_j) \\
&= E_\infty \sum_{\substack{j=1\\ \text{odd } j}}^{J-1} V(j, M_j) + E_\infty \sum_{\substack{j=2\\ \text{even } j}}^{J-1} V(j, M_j) \\
&\le E_\infty \sum_{\substack{j=1\\ \text{odd } j}}^{J_{\text{odd}}-2} V(j, M_J) + E_\infty \sum_{\substack{j=2\\ \text{even } j}}^{J_{\text{even}}-2} V(j, M_j) \\
&= E_\infty\{V(1, M_1) | V(1, M_1) < A\} \\
&\quad \times E_\infty\{(J_{\text{odd}} - 1)/2 + (J_{\text{even}} - 2)/2\} \\
&\le E_\infty\{V(1, M_1) | V(1, M_1) < A\}[E_\infty J_{\text{odd}} + E_\infty J_{\text{even}}]/2 \\
&\le 2E_\infty\{V(1, M_1) | V(1, M_1) < A\}/P_\infty\{V(1, M_1) \ge A\}.
\end{aligned} \tag{18}$$

Now

$$\begin{aligned}
E\{&V(1, M_1);\ V(1, M_1) < A\} \\
&= E_\infty V(1, M_1) - E_\infty\{V(1, H_1);\ H_1 < \infty\} \\
&\quad + E_\infty\{V(1, H_1);\ H_1 < \infty\} - E_\infty\{V(1, M_1);\ V(1, M_1) \ge A\} \\
&= E_\infty R(L_1) - E_\infty R(L_1) + E_\infty\{V(1, H_1);\ H_1 < \infty\} \\
&\quad \times \left[1 - \frac{E_\infty\{V(1, M_1);\ V(1, M_1) \ge A\}}{E_\infty\{V(1, H_1);\ H_1 < \infty\}}\right] \\
&= [E_\infty R(L_1)]\, o(1),
\end{aligned} \tag{19}$$

where $o(1) \to 0$ as $A \to \infty$ because of considerations as in the proof of Lemma 3.

For large enough C, $P_\infty\{V(1, M_1) < A\}$ is arbitrarily close to 1. Also, for large enough C and A, by (3) and by Lemma 1, $P_\infty\{V(1, M_1) \geq A\} \geq [2C\gamma(y)]^{-1}$. It therefore follows from (18) and (19) that for such C

$$E_\infty \sum_{j=1}^{J-1} V(j, M_J) \leq Ao(1), \tag{20}$$

which completes the proof of Lemma 6. □

PROOF OF THEOREM 1(ii) (continued). Since $R(M_J) \geq V(J, M_J) \geq A$, it follows that $M_J \geq N_A$ and so $E_\infty N_A \leq E_\infty M_J = E_\infty R(M_J)$. Hence, by Lemmas 4 and 6

$$\limsup_{A \to \infty} E_\infty N_A / A \leq \gamma(y).$$

It therefore only remains to be shown that $\liminf_{A \to \infty} E_\infty N_A / A \geq \gamma(y)$. Denote

$$J^* = \max\left\{ j | L_j = N_A \text{ and } V(j-1, L_j) < A, \text{ or } L_j < N_A \right\}.$$

Now

$$R(N_A) = V(J^*, N_A) + Q(J^* + 1, N_A) + \sum_{j=1}^{J^*-1} V(j, N_A). \tag{21}$$

Since $V(j, M_j) < A$ for $j \leq J^* - 1$, it follows that $J^* \leq J$. Also, since

$$E_\infty \left(\sum_{j=1}^{J^*-1} V(j, N_A);\ M_J > N_A \right) = E_\infty \left(\sum_{j=1}^{J^*-1} V(j, M_J);\ M_J > N_A \right),$$

it follows that

$$E_\infty \sum_{j=1}^{J^*-1} V(j, N_A) = E_\infty \sum_{j=1}^{J^*-1} V(j, M_J) \leq E_\infty \sum_{j=1}^{J-1} V(j, M_J),$$

which by (20) is $Ao(1)$. Also,

$$E_\infty Q(J^* + 1, N_A) = E_\infty\{Q(J^* + 1, N_A);\ N_A < M_J\} + E_\infty\{Q(J^* + 1, N_A);\ N_A = M_J\};$$

note that

$$E_\infty\{Q(J^* + 1, N_A);\ N_A < M_J\} < A/C$$

and

$$E_\infty\{Q(J^* + 1, N_A);\ N_A = M_J\} = E_\infty\{Q(J + 1, M_J);\ N_A = M_J\} \leq E_\infty Q(J + 1, M_J),$$

which is $O(\log A)$ as in Lemma 5. Therefore, for a proper (large) choice of C and all large enough A, it follows from (21) that

$$E_\infty[R(N_A) - V(J^*, N_A)] \leq 2A/C. \tag{22}$$

Let $\varepsilon = 2/(C\lambda)$. Since by (21) $R(N_A) - V(J^*, N_A) \geq 0$, it follows from (22) that $P_\infty\{R(N_A) - V(J^*, N_A) \geq \lambda A\} \leq 2/(\lambda C) = \varepsilon$, and so

$$P_\infty\{V(J^*, N_A) > (1 - \lambda)A\} \geq 1 - \varepsilon. \tag{23}$$

Let $N^*_{(1-\lambda)A}$ denote the stopping time of the type M_J when A is replaced by $(1-\lambda)A$ and C is replaced by $(1-\lambda)C$. Thus, (23) implies that $P_\infty\{N^*_{(1-\lambda)A} \le N_A\} \ge 1-\varepsilon$. Therefore,

$$
\begin{aligned}
E_\infty N_A &= E_\infty\big(N_A;\, N_A \ge N^*_{(1-\lambda)A}\big) + E_\infty\big(N_A;\, N_A < N^*_{(1-\lambda)A}\big)\\
&\ge E_\infty\big(N^*_{(1-\lambda)A};\, N_A \ge N^*_{(1-\lambda)A}\big) + E_\infty\big(N_A;\, N_A < N^*_{(1-\lambda)A}\big)\\
&= E_\infty N^*_{(1-\lambda)A} - E_\infty\big(N^*_{(1-\lambda)A} - N_A;\, N_A < N^*_{(1-\lambda)A}\big) \qquad (24)\\
&\ge E_\infty N^*_{(1-\lambda)A} - \varepsilon E_\infty\big[Q(I, L_1) + N^*_{(1-\lambda)A}\big]\\
&\ge (1-2\varepsilon)E_\infty N^*_{(1-\lambda)A}.
\end{aligned}
$$

Since $E_\infty N^*_{(1-\lambda)A} = E_\infty R(N^*_{(1-\lambda)A})$, it follows from (24) and Lemmas 6 and 4 that $\liminf_{A\to\infty} E_\infty N_A/A \ge (1-2\varepsilon)(1-\lambda)\gamma(y)$. By choosing C to be large enough, one can get ε and λ to be arbitrarily small, completing the proof of Theorem 1. □

4. Proof of Theorem 2. The idea of the proof of Theorem 2 is similar to that of Theorem 1. It is technically more difficult: $R(n, F)$ is not a Markov process and there is no simple analog to (1). Nevertheless, the idea of the proof of Theorem 1 can be salvaged. Under P_∞, $R(n, F) - n$ is a zero expectation martingale, so again

$$E_\infty N(A, F) = E_\infty R(N(A, F), F) = A E_\infty R(N(A, F), F)/A,$$

and again the proof becomes an analysis of the asymptotic behavior of $E_\infty R(N(A, F), F)/A$ as $A \to \infty$. Instead of (1) regard

$$
\begin{aligned}
R(n+r, F) &= \int \exp\left\{\sum_{i=n+1}^{n+r} Z_i^y\right\} R(n, y)\, dF(y) \qquad (25)\\
&\quad + \int \sum_{k=n+1}^{n+r} \exp\left\{\sum_{i=k}^{n+r} Z_i^y\right\} dF(y).
\end{aligned}
$$

Here too when $R(n, F)$ is large, the second expression on the right side of (25) is relatively small for low values of r. Therefore,

$$R(n+r, F) \approx \int \exp\left\{\sum_{i=n+1}^{n+r} Z_i^y\right\} R(n, y)\, dF(y) \overset{\text{def}}{=} U^*(n, r),$$

for low values of r. Since Z_i^y have negative drift (under P_∞), the process $U^*(N(A/C, F), r)$, $r = 1, 2, \dots$, will either exceed A "soon" or not at all. So again, one can approximate $N(A, F)$ by the first time that $R(n, F)$ crosses A/C and "soon" thereafter crosses A.

Let N be a stopping time. Denote $d\varphi_N(y) = R(N, y)\, dF(y)/R(N, F)$. Note that

$$U^*(N, r) = R(N, F) \int \exp\left\{\sum_{i=N+1}^{N+r} Z_i^y\right\} d\varphi_N(y). \qquad (26)$$

The integral on the right side of (26) can be regarded as a φ_N-mixture statistic of a power one test of H_0: $\theta = 0$ versus H_1: $\theta \neq 0$, which stops and rejects H_0 as soon as this mixture crosses a level $\Delta = A/R(N, F)$. Let $N + T$ denote the first time this mixture crosses Δ. If $T < \infty$ and $N = N(A/C, F)$, then $N + T \approx N(A, F)$. Since $\int \exp\{\sum_{i=N+1}^{N+r} Z_i^y\} \, d\varphi_N(y)$ is a P_∞-martingale with unit expectation,

$$
\begin{aligned}
E_\infty R(N(A,F),F) &\approx E_\infty\{U^*(N,T) | T < \infty\} \\
&= E_\infty\left\{R(N,F)\int \exp\left\{\sum_{i=N+1}^{N+T} Z_i^y\right\} d\varphi_N(y) | T < \infty\right\} \\
&= \frac{E_\infty\{R(N,F)\int \exp\{\sum_{i=N+1}^{N+T} Z_i^y\}\, d\varphi_N(y);\, T < \infty\}}{P_\infty(T < \infty)} \qquad (27)\\
&= \frac{E_\infty R(N,F)}{P_\infty(T < \infty)} \\
&= \frac{E_\infty N}{P_\infty(T < \infty)}.
\end{aligned}
$$

Now $P_\infty\{T < \infty | \mathscr{F}(X_1, \dots, X_N)\}$ is the significance level of the aforementioned test [conditional on $\mathscr{F}(X_1, \dots, X_N)$]. By Lai and Siegmund (1979) [see also Pollak (1986)], letting $\gamma(\cdot)$ be as in Section 2, if Δ is large

$$
\begin{aligned}
P_\infty(T < \infty) &= E_\infty P_\infty\{T < \infty | \mathscr{F}(X_1, \dots, X_N)\} \\
&= E_\infty(1/\Delta)1/[\gamma(\varphi_N)](1 + o(1)) \\
&\approx E_\infty(1/\Delta)\int [1/\gamma(y)]\, d\varphi_N(y) \\
&= E_\infty(1/A)\int [1/\gamma(y)]\, R(N, y)\, dF(y) \qquad (28)\\
&= (1/A)\int [1/\gamma(y)]\, E_\infty R(N, y)\, dF(y) \\
&= (1/A) E_\infty N \int [1/\gamma(y)]\, dF(y) \\
&= (1/A) E_\infty N(1/[\gamma(F)]).
\end{aligned}
$$

Putting (27) and (28) together yields $E_\infty R(N(A, F), F)/A \approx \gamma(F)$, which is the heart of Theorem 2. The formal details are now given.

Proof of Theorem 2(i). The proof of the first part of Theorem 2(i) is analogous to the proof of the first part of Theorem 1(i) with Π_A replaced by

$$\Pi_A^F = \min\left\{n \,\middle|\, \max_{k=1,\dots,n} \int \exp\left\{\sum_{i=k}^{n} Z_i^y\right\} dF(y) \geq A\right\}.$$

To prove the second part of Theorem 2(i), choose $0 \neq \omega_1, \omega_2$ in the interior of Ω and in the support of F, so that $F([\omega_1, \omega_2]) > 0$ and $I(\omega_1) < \infty$, $I(\omega_2) < \infty$.

Note that

$$N(A, F) \le \min\left\{n \,\middle|\, \max_{1\le k\le n} \int_{\omega_1}^{\omega_2} \exp\left\{\sum_{i=k}^{n} Z_i^y\right\} dF(y) \le A\right\}.$$

Denote

$$\Gamma_0 = 0,$$

$$\Gamma_i = \min\left\{n \,\middle|\, \sum_{j=\Gamma_{i-1}+1}^{n} Z_j^{\omega_1} \le 0 \text{ or } \int_{\omega_1}^{\omega_2} \exp\left\{\sum_{j=\Gamma_{i-1}+1}^{n} Z_j^y\right\} dF(y) \ge A\right\},$$

$$Y = \min\left\{i \,\middle|\, \int_{\omega_1}^{\omega_2} \exp\left\{\sum_{j=\Gamma_{i-1}+1}^{\Gamma_i} Z_j^y\right\} dF(y) \ge A\right\}.$$

Clearly, $N(A, F) \le \sum_{i=1}^{Y} \Gamma_i$. Hence,

$$E_\infty R(N(A, F), F) = E_\infty N(A, F) \le E_\infty \Gamma_1 E_\infty Y. \tag{29}$$

Now

$$E_\infty \Gamma_1 \le E_\infty \min\left\{n \,\middle|\, \sum_{j=1}^{n} Z_j^{\omega_1} \le 0\right\} < \infty. \tag{30}$$

In a manner similar to Theorem 1 of Pollak (1986), one can show that

$$AP_\infty(Y = 1) \to_{A\to\infty} \int_{\omega_1}^{\omega_2} [1/\gamma(y)]\, P_1^y\left(\sum_{i=1}^{n} Z_i^{\omega_1} > 0,\ n = 1, 2, \dots\right) dF(y). \tag{31}$$

Therefore, for given A_0 there exists a constant $\Delta = \Delta(A_0, F)$ such that if $A \ge A_0$, then

$$P_\infty(Y = 1) \ge \Delta/A. \tag{32}$$

Note that

$$E_\infty Y = 1/P_\infty(Y = 1). \tag{33}$$

Now (29), (30), (31), (32) and (33) compete the proof of Theorem 2(i). □

Note that if $\Theta \subset \Omega$ is a compact set bounded away from zero and interior to Ω, then the bound in (30) and the convergence in (31) can be made to hold uniformly for all F with support contained in Θ, so that there exists a constant χ (dependent on Θ) such that

$$E_\infty N(A, F) \le \chi A, \quad \text{for all } A \ge 1, \text{uniformly in } F \in \{G \mid \operatorname{supp}(G) \subseteq \Theta\}. \tag{34}$$

Proof of Theorem 2(ii). In order to emphasize the analogy to the proof of Theorem 1(ii), we will redefine the notation of Section 3 into terms of the mixture analog. Let $A > C > 1$. Define $L_0 = 0$. For $j = 1, 2, \ldots$ define

$$L_j = \min\left\{n | n > L_{j-1}, \int \sum_{k=L_{j-1}+1}^{n} \exp\left\{\sum_{i=k}^{n} Z_i^y\right\} dF(y) \geq A/C\right\},$$

$$\begin{aligned} Q(j,n) &= \int \sum_{k=L_{j-1}+1}^{n} \exp\left\{\sum_{i=k}^{n} Z_i^y\right\} dF(y), \quad \text{for } n > L_{j-1}, \\ &= 0, \quad \text{for other } n, \\ V(j,n) &= \int \exp\left\{\sum_{i=L_j+1}^{n} Z_i^y\right\} \exp\left\{\sum_{k=L_{j-1}+1}^{L_j} Z_i^y\right\} dF(y), \quad \text{if } n > L_j, \\ &= Q(j, L_j), \quad \text{if } n = L_j, \\ H_j &= \min\{n | n \geq L_j, V(j,n) \geq A\}, \quad = \infty, \quad \text{if no such } n \text{ exists}, \\ M_j &= H_j \wedge L_{j+1}, \\ J &= \min\{j | V(j, M_j) \geq A\}, \\ T(B,F) &= \min\left\{n \Big| \int \exp\left\{\sum_{i=1}^{n} Z_i^y\right\} dF(y) \geq B\right\}, \\ &= \infty, \quad \text{if no such } n \text{ exists}, \\ dF_j(y) &= \sum_{k=L_{j-1}+1}^{L_j} \exp\left\{\sum_{i=k}^{L_j} Z_i^y\right\} dF(y)/Q(j, L_j). \end{aligned}$$

By Theorem 1 of Pollak (1986), $BP_\infty\{T(B,F) < \infty\} \to_{B\to\infty} 1/\gamma(F)$.

Until further notice, we will assume that the support of F is contained in a set $\Theta = [a_1, b_1] \cup [a_2, b_2]$ where $-\infty < a_1 < b_1 < 0 < a_2 < b_2 < \infty$ and a_1, b_2 are in the interior of Ω. Thus (34) holds.

LEMMA 1*. *Let* $0 < \eta < 1$. *There exist* A_η^1 *and* C_η^1 *such that if* $A > C \geq C_\eta^1$ *and* $A \geq A_\eta^1$, *then*

$$\begin{aligned} (1-\eta)\left[E_\infty Q(j, L_j)/A\right]/\gamma(F) &\leq P_\infty\{H_j < \infty\} \\ &\leq (1+\eta)\left[E_\infty Q(j, L_j)/A\right]/\gamma(F). \end{aligned}$$

PROOF.

$$\begin{aligned} P_\infty(H_j < \infty) &= E_\infty P_\infty\{H_j < \infty | \mathscr{F}(L_j)\} \\ &= E_\infty P_\infty\{T(A/Q(j, L_j), F_j) < \infty | \mathscr{F}(L_j)\}. \end{aligned} \tag{35}$$

By virtue of (34), C can be chosen to be large enough so as to make $P_\infty\{Q(j, L_j) > A/\sqrt{C}\}$ arbitrarily small. If C is large enough and $Q(j, L_j) \leq$

$A/\sqrt{C}$, then by Theorem 2 of Pollak (1986)

$$\begin{aligned}P_\infty\{T(A/Q(j,L_j),F_j)<\infty|\mathscr{F}(L_j)\}\\ &=\{[Q(j,L_j)/A]/\gamma(F_j)\}[1+o_p(1)]\\ &=[1/A]\int[1/\gamma(y)]\exp\left\{\sum_{k=L_{j-1}+1}^{L_j} Z_i^y\right\}dF(y)[1+o_p(1)],\end{aligned}$$

where $o_p(1)\to_P 0$ as $C\to\infty$ uniformly in F_j. Therefore, it follows from (35) that

$$\begin{aligned}AP_\infty(H_j<\infty)&=[1+o(1)]\int[1/\gamma(y)]E_\infty(L_j-L_{j-1})\,dF(y)\\ &=[1+o(1)]E_\infty Q(j,L_j)/\gamma(F),\end{aligned}$$

where $o(1)\to 0$ as $C\to\infty$. This accounts for Lemma 1*. □

LEMMA 2*. *Let $0<\eta<1$. There exist A_η^2 and C_η^2 such that if $A>C\ge C_\eta^2$ and $A\ge A_\eta^2$, then*

$$(1-\eta)A\gamma(F)\le E_\infty\{V(j,H_j)|H_j<\infty\}\le(1+\eta)A\gamma(F).$$

PROOF. The same as that of Lemma 2, with Lemma 1* replacing Lemma 1. □

LEMMA 3*. *Let $0<\eta<1$. There exist $A_{\eta,C}^3$ and C_η^3 such that if $C>C_\eta^3$ is fixed and $A\ge A_{\eta,C}^3$, then*

$$(1-\eta)A\gamma(F)\le E_\infty\{V(j,M_j)|V(j,M_j)\ge A\}\le(1+\eta)A\gamma(F).$$

PROOF. Denote $\Phi_G(\cdot)=\int_\Theta P_1^y(\cdot)\,dG(y)$. Note that for $r\ge A/C$,

$$\sup_{\{G|\operatorname{supp}(G)\subseteq\Theta\}}\Phi_G\{T(A/r,G)\ge x\}\to_{x\to\infty}0.$$

Therefore, when C is fixed, insert F_j instead of G and $Q(j,L_j)$ instead of r in the inequality (36) to get

$$\begin{aligned}P_\infty\{x\le H_j-L_j<\infty\}&=E_\infty P_\infty\{x\le H_j-L_j<\infty|\mathscr{F}(L_j)\}\\ &\le E_\infty\Phi_G\{T(A/r,G)\ge x\}r/A|_{G=F_j,\,r=Q(j,L_j)}\\ &\to_{x\to\infty}0.\end{aligned}\tag{36}$$

The proof of Lemma 3* is now the same as that of Lemma 3 with (34) replacing Theorem 1(i) and Lemma 2* replacing Lemma 2. □

LEMMA 4*. *Let $0<\eta<1$. There exist $A_{\eta,c}^4$ and C_η^4 such that if $C\ge C_\eta^4$ is fixed and $A\ge A_{\eta,c}^4$, then*

$$1-\eta\le E_\infty V(J,M_J)/[A\gamma(F)]\le 1+\eta.$$

PROOF. Note that if (12) were shown to hold, then the proof of Lemma 4 would carry through for Lemma 4* [with the obvious changes of replacing Theorem 1(i) by (34). Lemmas 1 and 3 by Lemmas 1* and 3* and $\gamma(y)$ by $\gamma(F)$].

Therefore, all that is left to show is that (12) is satisfied. We will now sketch a proof.

Note that the proof of Lemmas 2 and 3 of Pollak and Siegmund (1975) can be carried through uniformly for $\theta \in \Theta$, so that there exists $\lambda > 0$ such that for $k = 1, 2, \dots,$ all $y \in \Theta$ and all G with support contained in Θ (when $\alpha > 1$ is fixed)

$$P_1^y\{|\overline{X}_k - \psi'(y)| > \eta\} < \exp\{-\lambda k\},$$
$$P_\infty\{|\overline{X}_k| > \eta\} < \exp\{-\lambda k\},$$
$$P_1^y\{T(B,G) < (\log B)/[\alpha I(y)]\} < \exp\{-\lambda \log A\}.$$

Let G be a probability measure whose support is contained in Θ. Then in a manner analogous to the derivation (8) one gets that

$$E_\infty\{R(T(B,G),F);\ R(T(B,G)-1,F) < B,\ T(B,G) < \infty\}$$
$$= \sum_{k=1}^{\infty} \int_\Theta E_\infty P_k^y\{k \le T(B,G),\ R(T(B,G)-1,F) < B | \mathscr{F}(k-1)\}\, dF(y).$$

Suppose first that $\Theta = [a, b]$ where $0 < a < b$ and $b\psi'(a) - \psi(b) > 0$. Then the idea of (9)–(12) carries through if one replaces τ_C by $T(B,G)$, one replaces $|\overline{X}_{k-1}| > \eta/|y|$ by $|\overline{X}_{k-1}| > \delta$ and one replaces $\Sigma_{i=k}^{\tau_C - 1} Z_i^y$ in (10) and its sequel by $\Sigma_{i=k}^{T(B,G)-1}[bX_i - \psi(b)]$. The details are omitted.

For general Θ, note that Θ can be represented as a finite union of intervals $[\alpha_i, \beta_i]$, where α_i, β_i are such that $\beta_i\psi'(\alpha_i) - \psi(\beta_i) > 0$ if $\beta_i > 0$ and $\alpha_i\psi'(\beta_i) - \psi(\alpha_i) > 0$ if $\beta_i < 0$. Clearly, $\{R(n, F) < B\}$ implies

$$\left\{\int_{\alpha_i}^{\beta_i} \exp\left\{\sum_{i=1}^{n} Z_i^y\right\} dF(y)/F\{[\alpha_i, \beta_i]\}\right\} < B/F\{[\alpha_i, \beta_i]\}.$$

Therefore, if $y \in [\alpha_i, \beta_i]$ and $F\{[\alpha_i, \beta_i]\} > 0$, then the analog of (10)–(11) carries through with B replaced by $B/F\{[\alpha_i, \beta_i]\}$ and $R(n, F)$ replaced by

$$\left\{\int_{\alpha_i}^{\beta_i} \sum_{k=1}^{n} \exp\left\{\sum_{i=1}^{n} Z_i^y\right\} dF(y)/F\{[\alpha_i, \beta_i]\}\right\}. \qquad \square$$

LEMMA 5*. *Let $0 < \eta < 1$. There exist C_η^5 and $A_{\eta,c}^5$ such that if $C \ge C_\eta^5$ is fixed and $A \ge A_{\eta,c}^5$, then*

$$E_\infty Q(J+1, M_J) = O(\log A),$$

where $O(\log A)/(\log A)$ is bounded as $A \to \infty$ [and where $Q(J+1, M_J)$ is understood to be zero if $M_J = L_J$].

PROOF. Analogous to that of Lemma 5. $\square$

LEMMA 6*. *Let $\lambda > 0$. There exist C_λ^6 and $A_{\lambda,c}^6$ such that if one fixes $C \ge C_\lambda^6$, then for any $A \ge A_{\lambda,c}^6$*

$$0 \le E_\infty R(M_J, F) - E_\infty V(J, M_J) \le \lambda A.$$

PROOF. Analogous to that of Lemma 6. $\square$

PROOF OF THEOREM 2(ii) (continued). For Θ of the form assumed above, the proof is analogous to the proof of Theorem 1(ii).

For the general case, consider $\Theta_n = [a_n, -1/n] \cup [1/n, b_n]$, where a_n, b_n are interior points of Ω and $a_n \to_{n\to\infty} \inf\{x|x \in \Omega\}$, $b_n \to_{n\to\infty} \sup\{x|x \in \Omega\}$. Let F_n^* be defined by $dF_n^*(x) = \Xi(x \in \Theta_n)\, dF(x)$. For arbitrary $\alpha > 0$, define $F_n^\alpha = (1+\alpha)F_n^*$. We will let $N(A, F_n^*)$, $N(A, F_n^\alpha)$ have the obvious meaning, despite F_n^* and F_n^α not being probability measures. Clearly, $N(A, F) \le N(A, F_n^*)$. Therefore,

$$\begin{aligned}&\limsup_{A\to\infty} E_\infty N(A,F)/A \\ &\quad \le \lim_{A\to\infty} E_\infty N(A, F_n^*)/A \\ &\quad = \lim_{A\to\infty} E_\infty\{N(A/F(\Theta_n), F_n^*/F(\Theta_n))/[A/F(\Theta_n)]\}/F(\Theta_n).\end{aligned}$$

Setting $n \to \infty$ yields that $\limsup_{A\to\infty} E_\infty N(A, F)/A \le \gamma(F)$.

Clearly,

$$E_\infty \int_{\Omega-\Theta_n} R(N(A,F), y)\, dF(y) = E_\infty N(A,F)[1 - F(\Theta_n)].$$

Therefore,

$$P_\infty\left\{\int_{\Omega-\Theta_n} R(N(A,F), y)\, dF(y) \ge \lambda A\right\} \le \frac{1-F(\Theta_n)}{\lambda}\frac{E_\infty N(A,F)}{A},$$

which, for any $\lambda > 0$, can be made arbitrarily small by taking n to be large enough. Now for $\lambda < \alpha/(1+\alpha)$

$$\begin{aligned}&P_\infty\{N(A,F) < N(A,F_n^\alpha)\} \\ &\quad \le P_\infty\left\{N(A,F) < N(A,F_n^\alpha);\ \int_{\Omega-\Theta_n} R(N(A,F), y)\, dF(y) \ge \lambda A\right\} \\ &\quad \le P_\infty\left\{\int_{\Omega-\Theta_n} R(N(A,F), y)\, dF(y) \ge \lambda A\right\}.\end{aligned}$$

In other words, for arbitrary $\varepsilon > 0$, $P_\infty\{N(A,F) < N(A,F_n^\alpha)\} < \varepsilon$ for large enough n. It is easy to see that

$$E_\infty\{N(A,F_n^\alpha) - N(A,F) | N(A,F_n^\alpha) > N(A,F)\} \le E_\infty N(A,F_n^\alpha).$$

Hence, $E_\infty N(A,F) \ge E_\infty N(A,F_n^\alpha)(1-\varepsilon)$. Letting $\varepsilon \to 0$, $\alpha \to 0$ yields that $\liminf_{A\to\infty} E_\infty N(A,F)/A \ge \gamma(F)$, completing the proof of Theorem 2. □

5. The average run length when $\nu = 1$. Define

$$\begin{aligned} C_2^{y,\theta} &= E_1^\theta \log\left[1 + \sum_{k=1}^{\infty} \exp\left\{-\sum_{i=1}^{k} Z_i^y\right\}\right], \\ C_3^{y,\theta} &= \lim_{B\to\infty} E_1^\theta \sum_{i=1}^{M(B,y)} Z_i^y - B, \\ C_1^{y,\theta} &= C_3^{y,\theta} - C_2^{y,\theta}, \\ C_2^{\theta,F} &= -\tfrac{1}{2}\log\left[2\pi(F'(\theta))^2/\psi''(\theta)\right], \\ C_4^\theta &= \tfrac{1}{2}\log I(\theta) - \tfrac{1}{2}, \\ C_1^{\theta,F} &= C_2^{\theta,F} + C_3^{\theta,\theta} - C_2^{\theta,\theta} - C_4^\theta. \end{aligned}$$

The computation of $C_3^{y,\theta}$ is an application of renewal theory. The calculation of $C_2^{y,\theta}$ seems to be feasible only with the aid of Monte Carlo.

THEOREM 3. *If $y, \theta \in \Omega$, $0 < y\psi'(\theta) - \psi(y) < \infty$, and the P_1^θ-distribution of Z_i^y is nonlattice, then*

$$E_1^\theta N(A, y) = \frac{1}{y\psi'(\theta) - \psi(y)}\left[\log A + C_1^{y,\theta} + o(1)\right],$$

where $o(1) \to 0$ as $A \to \infty$.

THEOREM 4. *Suppose $F'(y) = dF(y)/dy$ exists, is positive and is continuous in an open neighborhood of $\theta \in \Omega$. Then*

$$E_1^\theta N(A, F) = \left[\log A + \tfrac{1}{2}\log\log A + C_1^{\theta,F} + o(1)\right]/I(\theta),$$

where $o(1) \to 0$ as $A \to \infty$.

PROOF OF THEOREMS 3 AND 4. For the proof of Theorem 4, assume (without loss of generality) that $\theta > 0$. Consider first the case where F is concentrated on $[\theta_0, \theta_1]$, where $0 < \theta_0 < \theta < \theta_1 < \infty$ are such that $y\psi'(\theta) - \psi(y) > 0$ for $\theta_0 \le y \le \theta_1$ and F has a derivative F' which is positive and continuous on $[\theta_0, \theta_1]$. For $\theta_0 \le y \le \theta_1$ denote

$$W^{n,y} = 1 + \sum_{k=2}^{n} \exp\left\{-\sum_{i=1}^{k-1} Z_1^y\right\}.$$

Note that $W^{n,y}$ converges a.s. P_1^θ as $n \to \infty$ to a random variable $W_{y,\theta}$. Since

$$\sum_{n=m}^{\infty} (W^{n+1,y} - W^{n,y}) = \sum_{n=m}^{\infty} \exp\left\{-\left[y\sum_{i=1}^{n} X_i - n\psi(y)\right]\right\} \to_{m\to\infty} 0 \quad \text{a.s. } P_1^\theta,$$

uniformly in $y \in [\theta_0, \theta_1]$, it follows that $W_{y,\theta}$ is a.s. P_1^θ continuous in $y \in [\theta_0, \theta_1]$ and $W^{n,y} \to_{n\to\infty} W_{y,\theta}$ a.s. P_1^θ uniformly in $y \in [\theta_0, \theta_1]$. Note that

$$R(n, y) = \exp\left\{\sum_{i=1}^{n} Z_1^y\right\} W^{n,y},$$

$$R(n, F) = \int_{\theta_0}^{\theta_1} \exp\left\{\sum_{i=1}^{n} Z_i^y\right\} W^{n,y}\, dF(y).$$

The proof of Theorem 3 is therefore a direct application of the nonlinear renewal theorem. The proof of Theorem 4 follows the proof of the asymptotic formula for the expected sample size of power one tests, based on nonlinear renewal theory [cf. Lai and Siegmund (1977)]. The details presented here follow the proof presented in Woodroofe (1982), Section 6.3. With minor modifications, the proof is the same.

One difference is that Woodroofe's $u_n(\overline{Y}_n)$ now has $\pi(ds)$ replaced by $W^{n,s}\pi(ds)$. Note that the upper bound on the newly defined $u_n(\overline{Y}_n)$ is not uniform in $W^{n,s}$. One must show that (13) and (14) of Section 4.4 of Woodroofe

(1982) are nevertheless satisfied. One can dispense with (14) by noting that $W^{n,s} \geq 1$. To show that (13) is satisfied, it more than suffices to prove the existence of a constant $\alpha > 0$ such that

$$E_1^\theta\left(\int_{\theta_0}^{\theta_1} W_{y,\theta}\, dF(y)\right)^\alpha < \infty. \tag{37}$$

Let $\varepsilon > 0$, $\Lambda = \min\{n \| \bar{X}_m - \psi'(\theta)| \leq \varepsilon \text{ for all } m \geq n\}$. Suppose that ε is small enough so that there exists $\beta > 0$ such that $\sum_{i=1}^n Z_i^y \geq \beta n$ if $n \geq \Lambda$ for all $\theta_0 \leq y \leq \theta_1$. There exists a constant $\gamma > 0$ such that

$$|\psi(\theta - y) + \psi(y) - \psi(\theta)| < \gamma \quad \text{for all } \theta_0 \leq y \leq \theta_1.$$

There exists a constant $\delta > 0$ such that $P_1^\theta(\Lambda = \lambda) \leq \exp\{-\delta\lambda\}$. Choose $1 > \alpha > 0$ such that $\alpha\gamma - \delta(1 - \alpha) < 0$. Now

$$\begin{aligned}
\int_{\theta_0}^{\theta_1} W_{y,\theta}\, dF(y) &= \int_{\theta_0}^{\theta_1}\left[1 + \sum_{k=2}^{\Lambda} \exp\left(-\sum_{i=1}^{k-1} Z_i^y\right) + \sum_{k=\Lambda+1}^{\infty} \exp\left(-\sum_{i=1}^{k-1} Z_i^y\right)\right] dF(y)\\
&\leq \int_{\theta_0}^{\theta_1}\left[1 + \sum_{k=2}^{\Lambda} \exp\left(-\sum_{i=1}^{k-1} Z_i^y\right) + \frac{1}{1 - e^{-\beta}}\right] dF(y),
\end{aligned}$$

$$\begin{aligned}
&E_1^\theta\left(\int_{\theta_0}^{\theta_1} \sum_{k=2}^{\Lambda} \exp\left(-\sum_{i=1}^{k-1} Z_i^y\right) dF(y) \,\Big|\, \Lambda = \lambda\right)\\
&\quad\leq \frac{E_1^\theta \int_{\theta_0}^{\theta_1} \sum_{k=2}^{\lambda} \exp\left(-\sum_{i=1}^{k-1} Z_i^y\right) dF(y)}{P_1^\theta(\Lambda = \lambda)}\\
&\quad= \frac{1}{P_1^\theta(\Lambda = \lambda)} \int_{\theta_0}^{\theta_1} \sum_{k=2}^{\lambda} e^{[\psi(\theta - y) + \psi(y) - \psi(\theta)](k-1)}\, dF(y)\\
&\quad\leq \frac{1}{P_1^\theta(\Lambda = \lambda)} \frac{1}{\gamma} e^{\gamma\lambda}.
\end{aligned}$$

By Jensen's inequality,

$$\begin{aligned}
E_1^\theta\left(\int_{\theta_0}^{\theta_1} W_{y,\theta}\, dF(y)\right)^\alpha &= E_1^\theta E_1^\theta\left[\left(\int_{\theta_0}^{\theta_1} W_{y,\theta}\, dF(y)\right)^\alpha \Big| \Lambda\right]\\
&\leq \sum_{\lambda=1}^{\infty}\left(\frac{1}{P_1^\theta(\Lambda = \lambda)} \frac{1}{\gamma} e^{\gamma\lambda} + \frac{2 - e^{-\beta}}{1 - e^{-\beta}}\right)^\alpha P_1^\theta(\Lambda = \lambda).
\end{aligned} \tag{38}$$

The inequality (37) now follows because

$$\begin{aligned}
\sum_{\lambda=1}^{\infty}\left(\frac{1}{P_1^\theta(\Lambda = \lambda)} \frac{1}{\gamma} e^{\gamma\lambda}\right)^\alpha P_1^\theta(\Lambda = \lambda) &= \frac{1}{\gamma^\alpha} \sum_{\lambda=1}^{\infty} e^{\alpha\gamma\lambda}\left[P_1^\theta(\Lambda = \lambda)\right]^{1-\alpha}\\
&\leq \frac{1}{\gamma^\alpha} \sum_{\lambda=1}^{\infty} e^{\lambda(\alpha\gamma - (1-\alpha)\delta)}\\
&< \infty.
\end{aligned}$$

To complete the proof of Theorem 4 for the case that F is concentrated on $[\theta_0, \theta_1]$ as above, one need only show that (16) of Woodroofe (1982), Section 4.4, holds. For this, following Woodroofe's (1982, Section 6.3) proof, it suffices to note that (for large A)

$$P_\infty\left(N(A,F) \le \frac{\log A}{2I(\theta)}\right) = \sum_{i=1}^{(\log A)/(2I(\theta))} P_\infty(R(i,F) \ge A) \le \sum_{i=1}^{(\log A)/(2I(\theta))} \frac{i}{A} \le \frac{1}{(I(\theta))^2}\frac{(\log A)^2}{A}.$$

Since

$$P_1^y\left\{\max_{1 \le n \le (\log A)/(2I(\theta))} \sum_{i=1}^n Z_i^y \ge \tfrac{3}{4}\log A\right\} = P_1^y\{M(\tfrac{3}{4}\log A, y) \le (\log A)/(2I(\theta))\} \le \exp\{-\omega \log A\},$$

for some $\omega > 0$,

$$P_1^y(N(A,F) \le (\log A)/(2I(\theta))) \le \frac{\exp\{\frac{3}{4}\log A\}}{(I(\theta))^2}\frac{(\log A)^2}{A} + \exp\{-\omega\log A\} = o\left(\frac{1}{\log A}\right),$$

which is equivalent to (16) of Woodroofe (1982), Section 4.4.

For the general proof of Theorem 4, let F be a measure on the real line. There exist constants $0 < \xi < I(\theta)/2$, $\omega > 0$ and $0 < \theta_0 < \theta < \theta_1 < \infty$ such that $y\psi'(\theta) - \psi(y) > 0$ for $y \in [\theta_0, \theta_1]$, $\max\{y\psi'(\theta - \omega) - \psi(y),\ y\psi'(\theta + \omega) - \psi(y)\} < \xi$ for $y \in [\theta_0, \theta_1]$ and $F(y)$ has a derivative $F'(y)$ for $\theta_0 \le y \le \theta_1$, which is positive and continuous for $\theta_0 \le y \le \theta_1$. Since $P_1^\theta(N(A,F) \ge (2\log A)/I(\theta))$ is arbitrarily small when A is large enough and since for all $x > 0$, $E_1^\theta(N(A,F)|N(A,F) > x) \le x + (2\log A)/I(\theta)$ for large enough A, it suffices to show that

$$(\log A)P_1^\theta\left\{\max_{n=1,\dots,(2\log A)/I(\theta)} \int_{\mathbf{R}-[\theta_0,\theta_1]} \times \sum_{k=1}^n \exp\left(y\sum_{i=k}^n X_i - (n-k+1)\psi(y)\right) dF(y) \ge \frac{4A}{\log A}\right\} \to_{A\to\infty} 0. \tag{39}$$

The remainder of the proof is therefore an analysis of this expression.

Let $-\infty < \theta_0^* < \theta_0 < \theta_1 < \theta_1^* < \infty$ be such that

$$-\zeta = \max\{y\psi'(\theta) - \psi(y) | y \notin (\theta_0^*, \theta_1^*)\} < 0.$$

In the same manner which lead to (37) above, it can be shown that there exists a constant $\alpha > 0$ such that

$$\Gamma = E_1^\theta\left(\int_{\mathbf{R}-[\theta_0^*,\theta_1^*]} \sum_{k=1}^{\infty} \exp\left(y\sum_{i=1}^{k} X_i - k\psi(y)\right) dF(y)\right)^\alpha < \infty,$$

and hence by Jensen's inequality

$$(40)\quad \begin{aligned}&(\log A) P_1^\theta\left(\max_{n=1,\dots,(2\log A)/I(\theta)} \int_{\mathbf{R}-[\theta_0^*,\theta_1^*]} \times \sum_{k=1}^{n} \exp\left(y\sum_{i=k}^{n} X_i - (n-k+1)\psi(y)\right) dF(y) \geq \frac{A}{\log A}\right)\\ &\leq (\log A) \sum_{n=1}^{(2\log A)/I(\theta)} P_1^\theta\left(\left(\int \sum_{k=1}^{n} \exp\left(y\sum_{i=k}^{n} X_i - (n-k+1)\psi(y)\right) dF(y)\right)^\alpha \geq \left(\frac{A}{\log A}\right)^\alpha\right)\\ &\leq \frac{2(\log A)^2}{I(\theta)} \frac{\Gamma}{(A/\log A)^\alpha}\\ &\to_{A\to\infty} 0.\end{aligned}$$

For large enough A

$$(41)\quad \begin{aligned}&\max_{n=1,\dots,(2\log A)/I(\theta)} \int_{[\theta_0^*,\theta_1^*]-[\theta_0,\theta_1]} \sum_{k=1}^{n} \exp\left(y\sum_{i=k}^{n} X_i - (n-k+1)\psi(y)\right)\\ &\qquad\times \Xi\left(\frac{1}{n-k+1}\sum_{i=k}^{n} X_i \in (\psi'(\theta-\omega), \psi'(\theta+\omega))\right) dF(y)\\ &< \max_{n=1,\dots,(2\log A)/I(\theta)} \int \sum_{k=1}^{n} e^{\xi(n-k+1)} dF(y)\\ &\leq \max_{n=1,\dots,(2\log A)/I(\theta)} \frac{e^{\xi(n+1)}}{\xi}\\ &= \frac{e^\xi}{\xi} A^{2\xi/I(\theta)}\\ &< \frac{A}{\log A}.\end{aligned}$$

Let $\eta > 0$ be such that $P_1^\theta(\sum_{i=1}^k X_i/k \notin (\psi'(\theta-\omega), \psi'(\theta+\omega))) \leq \exp\{-\eta k\}$ for all k. Let $\lambda > 0$ be such that $E_1^\theta[\sum_{i=1}^k (X_i - \psi'(\theta))]^4 \leq \lambda k^2$ for all k. For large enough A and for $n \le (2\log A)/I(\theta)$

$$P_1^\theta\left\{\int_{\theta_1}^{\theta_1^*}\sum_{k=1}^{n}\exp\left(y\sum_{i=k}^{n}X_i-(n-k+1)\psi(y)\right)\right.$$

$$\left.\times\,\Xi\left(\frac{1}{n-k+1}\sum_{i=k}^{n}X_i\notin(\psi'(\theta-\omega),\psi'(\theta+\omega))\right)dF(y)>\frac{A}{\log A}\right\}$$

$$\le P_1^\theta\left\{\sum_{k=1}^{n}\exp\left(\theta_1^*\sum_{i=k}^{n}X_i\right)\right.$$

$$\left.\times\,\Xi\left(\frac{1}{n-k+1}\sum_{i=k}^{n}X_i\notin(\psi'(\theta-\omega),\psi'(\theta+\omega))\right)>\frac{A}{\log A}\right\}$$

$$\le P_1^\theta\left\{\max_{k=1,\dots,n}\exp\left(\theta_1^*\sum_{i=1}^{k}X_i\right)\right.$$

$$\left.\times\,\Xi\left(\frac{1}{k}\sum_{i=1}^{k}X_i\notin(\psi'(\theta-\omega),\psi'(\theta+\omega))\right)>\frac{A/n}{\log A}\right\}$$

$$\le\sum_{k=1}^{n}P_1^\theta\left\{\sum_{i=1}^{k}(X_i-\psi'(\theta))\Xi\left(\frac{1}{k}\sum_{i=1}^{k}X_i\notin(\psi'(\theta-\omega),\psi'(\theta+\omega))\right)\right.$$

$$\left.>\frac{1}{\theta_1^*}\log A-\frac{2}{\theta_1^*}\log\log A-k\psi'(\theta)+\frac{1}{\theta_1^*}\log(I(\theta)/2)\right\}$$

$$\le\sum_{k=1}^{n\wedge(\log A)^{1/4}}\frac{\lambda k^2}{\left[\frac{1}{\theta_1^*}\log A-\frac{2}{\theta_1^*}\log\log A-k\psi'(\theta)+\frac{1}{\theta_1^*}\log\left(\frac{I(\theta)}{2}\right)\right]^4}$$

$$+\sum_{k=(\log A)^{1/4}}^{n}P_1^\theta\left\{\frac{1}{k}\sum_{i=1}^{k}X_i\notin(\psi'(\theta-\omega),\psi'(\theta+\omega))\right\}$$

$$\le\frac{\lambda(\log A)^{3/4}}{\left[\frac{1}{\theta_1^*}\log A-\frac{2}{\theta_1^*}\log\log A-(\log A)^{3/4}\psi'(\theta)+\frac{1}{\theta_1^*}\log(I(\theta)/2)\right]^4}$$

$$+e^{-\eta(\log A)^{1/4}}\frac{1}{1-e^{-\eta}}.$$

It follows that

$$(\log A)P_1^{\theta}\left\{\max_{k=1,\dots,(2\log A)/I(\theta)}\int_{\theta_1}^{\theta_1^*}\sum_{k=1}^{n}\exp\left(y\sum_{i=k}^{n}X_i-(n-k+1)\psi(y)\right)\right.$$

(42)
$$\left.\times\Xi\left(\frac{1}{n-k+1}\sum_{i=k}^{n}X_i\notin(\psi'(\theta-\omega),\psi'(\theta+\omega))\right)dF(y)>\frac{A}{\log A}\right\}$$

$$\to_{A\to\infty}0.$$

In a similar fashion one gets that

$$(\log A)P_1^{\theta}\left\{\max_{k=1,\dots,(2\log A)/I(\theta)}\int_{\theta_0^*}^{\theta_0}\sum_{k=1}^{n}\exp\left(y\sum_{i=k}^{n}X_i-(n-k+1)\psi(y)\right)\right.$$

(43)
$$\left.\times\Xi\left(\frac{1}{n-k+1}\sum_{i=k}^{n}X_i\notin(\psi'(\theta-\omega),\psi'(\theta+\omega))\right)dF(y)>\frac{A}{\log A}\right\}$$

$$\to_{A\to\infty}0.$$

Formulas (40)–(43) account for (39) and so the proof of Theorem 4 is complete. □

6. Monte Carlo. A Monte Carlo study was made for the normal model with unit variance. Letting f_θ denote the density of the $N(\theta,1)$ distribution, simulations of $N(A,\theta)$, $R(N(A,\theta),\theta)$ were made for $\theta = 0.4, 0.8, 1.0, 1.2, 1.6, 2.0, 2.5, 3.0, 4.0$ and $A = 10, 20, 30, 100$, using $X_i \sim N(0,1)$ random numbers. For each of the 36 combinations of θ and A, 10,000 realizations were obtained. The results show that the asymptotic formulas (derived in the previous sections) give a very good picture of $E_\infty N(A,\theta)$ even for surprisingly low values of A.

As expected, the Monte Carlo estimate of $E_\infty(R(N(A,\theta),\theta) - N(A,\theta))$ was zero: In only one of the 36 cases did the estimate exceed two of its (Monte Carlo) standard deviations. The results of Lai and Siegmund (1977) lead one to conjecture that the linear correlation coefficient between $N(A,\theta)$ and $R(N(A,\theta),\theta)$ is asymptotically ($A \to \infty$) zero. The Monte Carlo results support this conjecture; the highest Monte Carlo correlation between $N(A,\theta)$ and $R(N(A,\theta),\theta)$ was 0.0234. [In 28 of the 36 cases the correlation between $N(A,\theta)$ and $R(N(A,\theta)$, was not significantly different from zero at a 5% level of significance, and in all of the 36 cases this correlation was not significantly different from zero at a 1% level of significance.] Therefore, estimates of $E_\infty N(A,\theta)$ were made using a linear combination $\alpha_{A,\theta}N(A,\theta) + (1-\alpha_{A,\theta})R(N(A,\theta),\theta)$, where $\alpha_{A,\theta}$ was chosen to minimize $\alpha^2\mathrm{Var}\,N(A,\theta) + (1-\alpha)^2\mathrm{Var}\,R(N(A,\theta),\theta)$ (the variances being Monte Carlo variances). The results are presented in Table 1.

Table 2 presents the ratio between the theoretical value TH and the Monte Carlo estimate MC.

The results show a surprisingly good fit, even for low values of A (as long as θ is not too large). It seems clear that for most practical purposes the asymptotic

TABLE 1

Values of $E_\infty N(A, \theta)$ predicted by asymptotic theory (TM) and estimated by Monte Carlo (MC)

θ \ A		10		20		30		100	
		$E_\infty N(A,\theta)$	S.D. of MC	$E_\infty N(A,\theta)$	S.D. of MC	$E_\infty N(A,\theta)$	S.D. of MC	$E_\infty N(A,\theta)$	S.D. of MC
0.4	TH	12.62		25.24		37.86		126.21	*
	MC	13.01	0.03	25.57	0.05	38.20	0.08	126.44	0.27
0.8	TH	15.91		31.82		47.73		159.09	*
	MC	16.51	0.07	32.32	0.14	48.58	0.22	159.61	0.68
1.0	TH	17.85		35.69		53.54		178.45	*
	MC	18.44	0.09	36.23	0.21	54.53	0.30	178.25	0.95
1.2	TH	20.00		40.00		60.00	*	200.01	*
	MC	20.98	0.13	40.59	0.27	60.56	0.40	200.71	0.40
1.6	TH	25.05		50.09		75.14	*	250.47	*
	MC	26.62	0.20	52.65	0.42	76.00	0.60	248.27	1.91
2.0	TH	31.21		62.42		93.62	*	312.08	*
	MC	34.54	0.32	65.52	0.58	93.92	0.84	315.47	2.74
2.5	TH	40.72		81.44		122.16		407.20	*
	MC	48.27	0.46	89.65	0.87	128.39	1.22	406.78	3.91
3.0	TH	52.52		105.04		157.56		525.21	*
	MC	72.75	0.71	127.08	1.25	180.15	1.80	533.15	5.23
4.0	TH	83.93		167.87		251.80		839.35	
	MC	189.58	1.86	315.25	3.10	428.10	4.30	1099.02	10.96

TH represents the theoretical value one would expect for $E_\infty N(A, \theta)$, using Theorem 1(ii). MC represents the estimate based on the Monte Carlo trials. The (Monte Carlo) standard deviation of this estimate is given under the heading of "S.D. of MC." The asterisks represent those estimates where $|\mathrm{TH} - \mathrm{MC}|$ did not exceed 2 (Monte Carlo) standard deviations of TH.

TABLE 2
Ratios of asymptotic theory predictions of $E_\infty N(A, \theta)$ to Monte Carlo estimates (TH/MC)

θ \ A	10	20	30	100
0.4	0.97	0.99	0.99	1.00
0.8	0.96	0.98	0.98	1.00
1.0	0.97	0.99	0.98	1.00
1.2	0.95	0.99	0.99	1.00
1.6	0.94	0.95	0.99	1.01
2.0	0.90	0.95	1.00	0.99
2.5	0.84	0.91	0.95	1.00
3.0	0.72	0.83	0.87	0.99
4.0	0.44	0.53	0.59	0.76

formula can be safely applied. (Shewhart control charts using "3σ limits"—often used in practice—have P_∞-expected sample size of 741.)

For an indication of how well one may expect the formula of Theorem 4 to fit, see Pollak and Siegmund (1975). One would expect the formula presented here to hold as well as the formulas presented there, provided that $E_1^\theta N(A, F)$ is large enough for the distribution of $\log[1 + \sum_{k=1}^{N(A,F)} \exp\{-\sum_{i=1}^k Z_i^\theta\}]$ to have approximately reached its limiting distribution.

7. Remarks. 1. In Theorems 1, 3 and 4, if $I(\theta) = \infty$, it is possible to show that $E_\infty N_A^\theta / A \to \infty$ as $A \to \infty$ and $E_1^\theta N(A, F)/\log A \to 0$ as $A \to \infty$.

2. Using the method involved in showing the validity of Remark 1, one can show that Theorem 2 remains valid with $F(\{y | I(y) < \infty\}) > 0$.

3. The fact that $\lim_{A\to\infty} E_\infty N(A, y)/A$ exists follows from Theorem 4 of Kesten (1973). Theorem 1(ii) gives its value.

4. It seems reasonable to conjecture that Theorem 2 remains valid if the P_1^y-distribution of X_1 is just assumed to be nonlattice. The proof given above for Theorem 2 breaks down because the uniformity of the renewal–theoretic convergence used in the proof of Lemma 1* [based on Theorem 2 of Pollak (1986)] is not clear if the strongly nonlattice assumption is dropped.

5. In the lattice case, even a version of Theorem 1 seems to be difficult to formulate. Despite X_1's being lattice, R_n is not, and the proof presented here—which conditions on $\mathscr{F}(N_{A/C}^\theta)$—does not yield an expression for the nonlattice part of the asymptotic P_∞-distribution of $\log R(N(A, \theta), \theta) - \log A$.

6. The $\log\log A$ term in Theorem 4 is the price one must pay for efficiency at every alternative. [For an indication of this see Pollak (1978).]

7. The difference between the asymptotic expansion of Theorem 3 and that of its CUSUM analog is only in the constant term. To compare the two, fix (large) B as the expected number of observations until a false alarm. If the parameter value after the change is represented by a value θ, Theorem 1(i) implies that one would choose $A = A(B) = B/\gamma(\theta)$, and Theorem 3 implies that

(for y satisfying $E_1^y Z_i^\theta > 0$)

$$(44) \qquad E_1^y N(\theta, A(B)) = \left[\log B - \log \gamma(\theta) + C_1^{y,\theta}\right] / E_1^y Z_1^\theta + o(1).$$

The analogous CUSUM procedure would (for the same values of B and θ) require computing $A^*(B)$ defining by $E_\infty \Pi_{A^*(B)} = B$ and then calculating $E_1^y \Pi_{A^*(B)}$. [This can be done using the results of Siegmund (1975).] This expression would then be compared to (44). For a numerical comparison (in the case of detecting a change in the drift of a Brownian motion) see Pollak and Siegmund (1985).

8. The problem of formulating an analog of Theorem 2(ii) for mixture-type CUSUM rules Π_A^F seems to be intractable. Therefore, although use of Π_A^F is (conservatively) possible via the inequality $E_\infty \Pi_A^F \geq A$ [Lorden (1971) and Pollak and Siegmund (1975)], use of $N(A/\gamma(F), F)$ should be more appealing.

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