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SAMPLE SIZE SAVINGS FOR CURTAILED ONE-SAMPLE NONPARAMETRIC TESTS FOR LOCATION SHIFT

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Asymptotic moments of normed sample size savings are derived for one-sided curtailed nonparametric tests for symmetry for testing location shift hypotheses under the null hypothesis and contiguous alternatives. The contiguous alternative results are derived using Feller's (1943) central limit theorem result for large deviations and techniques used in Albers, Bickel and van Zwet (1976).

1. Introduction. The first step from the traditional fixed sample size test towards sequential or random sample size tests is curtailed sampling. In the curtailed procedures under consideration in this paper, sampling is stopped as soon as it is clear that the decision for the fixed sample size test has become irrevocable. Thus curtailed sampling gives us a testing procedure with the same power function as that of the fixed sample size test, while guaranteeing that the random sample size is less than or equal to the fixed sample size in all cases.

Curtailed binomial or sign tests have been frequently investigated in the area of quality control. Alling (1966), Phatak and Bhatt (1967), Craig (1968), Cohen (1970), and Shah and Phatak (1972, 1974) all do calculations of average sample size when curtailed sampling is used. Garner (1958) discussed curtailed sampling when a continuous variable is used for testing lot acceptance. Anderson and Friedman (1960) and Samuel (1970) compare the expected savings using curtailed sampling procedures versus sequential probability ratio tests. Herrmann and Szatrowski (1982, 1985) derive and evaluate the small sample properties of asymptotic formulas for the expected sample size savings for curtailed binomial tests. Eisenberg and Ghosh (1980, 1981) derive expressions for the asymptotic efficiency of curtailed tests under fixed and contiguous alternative hypotheses. The curtailed version of the t-test and Hotelling's T^2 have been described by Brown, Cohen and Strawderman (1979) and savings in sample size were investigated by Herrmann and Szatrowski (1980). Wong and Wong (1982) define and examine the behavior of a curtailed procedure for the location parameter of the Weibull distribution.

Halperin and Ware (1974), Verter (1979) and DeMets and Halperin (1982) study curtailed sampling for a two-sample problem with applications to clinical

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trials. The Halperin and Ware (1974) asymptotic result is of a different order than the result obtained here due to their assumption that the sequential observations were obtained in nondecreasing order. Pocock (1977) and O'Brien and Fleming (1979) investigate group sequential approaches to clinical trials. Lan, Simon and Halperin (1982) report on results for stochastically curtailed tests, and Pasternack (1984) discusses curtailed tests involving the probability of a reversal of a decision based on incomplete sequential data. DeMets and Lan (1984) review many of these applications to sequential trials.

In this paper, we characterize the random savings in sample size. We begin with a formal description of the problem.

2. The problem. Let $X_1, \dots, X_n$ be an independent, identically distributed sequence of random variables, observed sequentially, from a continuous distribution function with density $f(x - \theta)$, where $f(x)$ is symmetric about zero and θ is the location shift parameter. Let $a_{i,n}$, $i = 1, \dots, n$, be a set of nonnegative and nondecreasing scores in i, $a_{1,n} \leq \cdots \leq a_{n,n}$. Let $Z_i = |X_i|$ and let $Z_{(i)}$ be the ith order statistic formed from $Z_1, \dots, Z_n$. The usual nonparametric statistic for location based on this sample of size n is given by $S_n = \Sigma_1^n a_{i,n} W_i$, where $W_i = 1$ if the observed X value corresponding to $Z_{(i)}$ is positive; $W_i = 0$ otherwise. We assume (e.g., Hájek and Šidák (1967), Puri and Sen (1971)) that the scores $a_{i,n}$ can be imbedded in a nondecreasing score function J: $[0,1) \to [0, \infty)$, where J is square integrable, $\int_0^1 J(t)\,dt > 0$, and $\lim_{n\to\infty} a_{[un]+1,n} = J(u)$, $0 \leq u < 1$ ($[\cdot]$ is the greatest integer function). Also $\int_0^1 (a_{[un]+1,n} - J(u))^2\,du = o(1)$ and both $|J(u)| \leq K(1-u)^{\delta - 1/2}$ and $|dJ(u)/du| \leq K(1-u)^{\delta - 3/2}$ for some K and $0 < \delta \leq \frac{1}{2}$. Often, $a_{i,n} = J(i/(n+1))$.

For the one-sided hypothesis test for location shift, testing H_0: $\theta = 0$ versus H_A: $\theta > 0$ with significance level α, we reject the null hypothesis if $S_n \geq k_n(\alpha)$; otherwise, we accept the null hypothesis. We will assume that the significance level α has been chosen so that there exists a $k_n(\alpha)$ with the property that $\alpha = \Pr\{S_n \geq k_n(\alpha) | \theta = 0\}$ so as to avoid randomized tests, since results for randomized tests can easily be obtained from our nonrandomized test results. It is well known (e.g., Hájek and Šidák (1967), Chapter 5) that the statistic S_n is asymptotically normally distributed with mean $\Sigma a_{i,n}/2$ and variance $\Sigma(a_{i,n})^2/4 \sim n\int_0^1 J^2(t)\,dt/4$ under the null hypothesis of symmetry. Thus, for large n, we can use this result to find an approximate value for $k_n(\alpha)$ given by

$$2k_n(\alpha) \simeq 2\left\{E(S_n|\theta = 0) + z_{1-\alpha}\{\mathrm{Var}(S_n|\theta = 0)\}^{1/2}\right\} \tag{2.1}$$

$$= \sum a_{i,n} + z_{1-\alpha}\left(\sum \{a_{i,n}\}^2\right)^{1/2},$$

where $z_{1-\alpha}$ is the upper $(1-\alpha)$th percentile of the standard normal distribution.

A curtailed version of this test can be constructed with the identical power function and a random sample size less than or equal to the fixed sample size n. We define $S_{m,n} = \Sigma_1^m a_{i,n} W_{i,m}$, where $W_{i,m}$ are the W_i scores based on the order statistics of the absolute values of X_i, $i = 1, \dots, m$. Thus, $S_{m,n}$ is a test statistic with the W's based on the first m observations but using the scores for a sample of size n. Note that for any outcome, $S_{m,n}$ is nondecreasing in m.

The one-sided curtailed test procedure for testing H_0: $\theta = 0$ versus H_A: $\theta > 0$ stops at the earliest stage m of the experiment when either $S_{m,n} \geq k_n(\alpha)$ (reject the null hypothesis) or $S_{m,n} + \Sigma_{m+1}^n a_{i,n} < k_n(\alpha)$ (accept the null hypothesis). If neither condition occurs at stage m, sampling is continued. Note that this procedure terminates on or before n observations have been collected. Let N^R be the random stopping time when the curtailed procedure is used and we reject the null hypothesis, with $N^R \equiv n$ if we accept the null hypothesis. Let N^A be the random stopping time when the curtailed procedure is used and we accept the null hypothesis with $N^A \equiv n$ if we reject the null hypothesis. We note that at least one of N^R or N^A (and possibly both) are equal to n for each experiment and that N^R and N^A are defined for every outcome.

Our focus in this paper will be on characterizing the asymptotic distributions of normed versions of the sample size savings $(n - N^R)$ and $(n - N^A)$. We will investigate results for the one-sided hypothesis with H_A: $\theta > 0$ since we can transform the one-sided hypothesis testing problem with H_A: $\theta < 0$ into a one-sided hypothesis test with H_A: $\theta > 0$ by using $-X_1, \dots, -X_n$ as the observed sample. Because of the definition of N^R and N^A, the rth moment of the total sample size savings can be expressed as $E\{(n - N^R)^r|\theta\} + E\{(n - N^A)^r|\theta\}$, for $r = 1, 2, \dots$, without weighting.

3. Savings under contiguous alternatives. In this section results are presented that characterize the distribution of normed versions of the random variables $n - N^R$ and $n - N^A$ under values of θ that are close to the null value (θ is characterized in (3.1) and (3.2)). The moments of normed versions of these random savings are given in Theorems 3.1 and 3.2. In Section 4, we define the specific norming constants and limit constants in (3.3) and (3.4) for bounded score functions J and for several unbounded score functions, including the van der Waerden scores. Using these results, we show in Section 5 that the normed sample size savings have an asymptotic truncated normal distribution for many of the nonparametric tests under consideration, including the Wilcoxon and the van der Waerden tests. A brief sketch of the proofs for these results is given in this section with details deferred to the Appendix.

Let $\{\theta_i\}$ be a sequence of parameter values converging to zero with the properties

$$\theta_i = O(n^{-1/2}), \tag{3.1}$$

$$1 - \beta_n = \Pr\{S_n \geq k_n(\alpha)|\theta_n\} \to 1 - \beta, \qquad 0 < \beta < 1. \tag{3.2}$$

Define

$$b_{m,n} = \{k_n(\alpha) - E(S_{n-m,n}|\theta_n)\}/(\mathrm{Var}(S_{n-m,n}|\theta_n))^{1/2}, \tag{3.3}$$

$$c_{m,n} = b_{m,n} - \left\{\left(\sum_{n-m+1}^{n} a_{i,n}\right)/(\mathrm{Var}(S_{n-m,n}|\theta_n))^{1/2}\right\}. \tag{3.4}$$

We will assume that there exists a norming sequence $h(n) \to \infty$, $h(n) = O(n^{1/2})$ or $o(n^{1/2})$, with the property that, for $\frac{1}{2} < \eta \le 1$, and b and c positive,

$$\lim_{n\to\infty} \left(z_{1-\beta} + b_{[t_n h(n)], n}\right)/t_n^{\eta} = b, \qquad \lim_{n\to\infty} \left(bt_n^{\eta} - b_{[t_n h(n)], n}\right) = z_{1-\beta}; \tag{3.5}$$

$$\lim_{n\to\infty} \left(z_{1-\beta} + c_{[t_n h(n)], n}\right)/t_n^{\eta} = -c, \qquad \lim_{n\to\infty} \left(ct_n^{\eta} + c_{[t_n h(n)], n}\right) = -z_{1-\beta}, \tag{3.6}$$

where this convergence is uniform for all sequences $\{t_n\}$ satisfying $h^{-1}(n) < t_n < Kn^{\varepsilon}$ for some constant K and $0 < \varepsilon < \frac{1}{2}$ with $\liminf t_n > 0$.

It may be helpful to think of $h(n) = n^{1/2}$, $\eta = 1$ and $r = 1$ in a first reading of Theorems 3.1 and 3.2, where we characterize the asymptotic moments of $(n - N^R)/h(n)$ and $(n - N^A)/h(n)$. Note that with these values, $E\{((n - N^R)/h(n))^r|\theta_n\}$ becomes $n^{1/2}E\{((n - N^R)/n)|\theta_n\}$, which is a normed version of the average fraction sample size savings achieved by stopping early and rejecting the null hypothesis.

THEOREM 3.1. *For the curtailed one-sided hypothesis test described in Section* 2, *assuming* (3.1), (3.2) *and* (3.5), *for r a positive integer,*

$$\lim_{n\to\infty} E\left\{\left((n - N^R)/h(n)\right)^r|\theta_n\right\} = \int_0^{\infty} rt^{r-1}\Phi\left(z_{1-\beta} - bt^{\eta}\right) dt. \tag{3.7}$$

PROOF. A brief outline of the proof is given here. Details are deferred to the Appendix. Let n^R be the smallest sample size at which the curtailed test could stop and reject the null hypothesis, i.e., the integer value of m satisfying

$$\sum_1^{m-1} a_{i,n} < k_n(\alpha) \le \sum_1^{m} a_{i,n}. \tag{3.8}$$

Then $E\{((n - n^R)/h(n))^r|\theta_n\}$ becomes

$$h^{-r}(n) \sum_{m=n^R}^{n-1} (n - m)^r \Pr\left\{N^R = m|\theta_n\right\}. \tag{3.9}$$

We can convert these sums into sums of cumulative probabilities to yield an expression asymptotically equal to (3.9):

$$h^{-r}(n) \sum_{m=1}^{n-n^R} rm^{r-1}\Pr\left\{N^R \le n - m|\theta_n\right\}. \tag{3.10}$$

We note that the ratio of (3.9) and (3.10) converges to one unless (3.9) goes to zero, in which case (3.10) also goes to zero. Observe that we are summing from the largest to the smallest probabilities. The substitution $\{N^R \le n - m\} = \{S_{n-m,n} \ge k_n(\alpha)\}$ into (3.10) yields

$$h^{-r}(n) \sum_{m=1}^{n-n^R} rm^{r-1}\Pr\left\{S_{n-m,n} \ge k_n(\alpha)|\theta_n\right\}. \tag{3.11}$$

We then normalize $S_{n-m,n}$ to obtain

$$(3.12)\quad E\{((n - N^R)/h(n))^r|\theta_n\} = h^{-r}(n) \sum_{m=1}^{n-n^R} rm^{r-1}\Pr\{S^*_{n-m,n} \geq b_{m,n}|\theta_n\},$$

where

$$S^*_{n-m,n} = \{S_{n-m,n} - E(S_{n-m,n}|\theta_n)\}/(\mathrm{Var}(S_{n-m,n}|\theta_n))^{1/2}$$

and $b_{m,n}$ is given in (3.3).

Since $b_{n-n^R,n} \to \infty$, we need a large deviation central limit theorem for signed rank tests under contiguous alternatives to proceed further. The needed result is obtained using a central limit theorem of Feller (1943) and the techniques of Albers, Bickel and van Zwet (1976). (See Appendix A.) This large deviation result is applied to the first m_n (largest) terms of the sum, where m_n is chosen so that $m_n/h(n) \to \infty$, and the remaining terms are negligible. We then obtain

$$(3.13)\qquad h^{-r}(n) \sum_{m=1}^{m_n} rm^{r-1}\Phi(-b_{m,n}),$$

which has the property that the absolute difference between (3.12) and (3.13) is $o(1)$. (See Appendix A.) Furthermore, this sum differs from the integral (3.14) by $o(1)$:

$$(3.14)\qquad h^{-r}(n)\int_1^{m_n} rm^{r-1}\Phi(-b_{m,n})\,dm,$$

where we use asymptotic expansions (e.g., Lemmas 4.1 and 4.2) to define $b_{m,n}$ for noninteger values of m. (See Appendix B.) We then make the change of variable $m = h(n)t$ to obtain

$$(3.15)\qquad \int_0^\infty rt^{r-1}\Phi(-b_{[h(n)t],n})I(h^{-1}(n) \leq t \leq (m_n/h(n))\,dt,$$

where $I(\cdot)$ is the indicator function. Taking the limit as $n \to \infty$, using the dominated convergence theorem to take the limit under the integral sign and using assumption (3.5) yields the desired result. (See Appendix C.) □

THEOREM 3.2. *For the curtailed one-sided hypothesis test described in Section* 2, *assuming* (3.1), (3.2), (3.5) *and* (3.6), *for r a positive integer,*

$$(3.16)\quad \lim_{n\to\infty} E\{((n - n^A)/h(n))^r|\theta_n\} = \int_0^\infty rt^{r-1}\Phi(-z_{1-\beta} - ct^\eta)\,dt.$$

PROOF. A brief outline of the proof is given here. The details are very similar to the proof of Theorem 3.1 and are thus omitted. Let n^A be the smallest sample size for which the curtailed test could stop and accept the null hypothesis, i.e., the integer value of m satisfying

$$\sum_{i=m+1}^{n} a_{i,n} < k_n(\alpha) \leq \sum_{i=m}^{n} a_{i,n}.$$

Then $E\{((n - N^A)/h(n))^r|\theta_n\}$ becomes

$$h^{-r}(n) \sum_{m=n^A}^{n-1} (n-m)^r \Pr\{N^A = m|\theta_n\}. \tag{3.17}$$

The remaining details of the proof are similar to those of the proof of Theorem 3.1 using the substitution $\{N^A \le m\} = \{S_{m,n} + \Sigma_{m+1}^n a_{i,n} < k_n(\alpha)\}$ and with "$-b$" replaced by "c" in (3.13)–(3.15). □

4. Evaluation of $h(n)$ and the constants in (3.5) and (3.6).

THEOREM 4.1. *For bounded score functions*, i.e., $J(1-) \equiv \lim_{u\uparrow 1} J(u) < \infty$, (3.5) *and* (3.6) *are satisfied when* $h(n) = n^{1/2}$, $\eta \doteq 1$, *and* b *and* c *are given by*

$$b = c = J(1-)\left(\int_0^1 J^2(u)\,du\right)^{-1/2}. \tag{4.1}$$

Before proving Theorem 4.1, we state two lemmas that give us the asymptotic mean and variance of S_n and $S_{n-m,n}$ under contiguous alternatives. A well-known result (e.g., Hájek and Šidák (1967), Chapter 6) when f is the symmetric density function from Section 2 is given in:

LEMMA 4.1. *Under conditions* (3.1) *and* (3.2), S_n *is asymptotically normally distributed with*

$$2E(S_n|\theta_n) \simeq n\int_0^1 J(u)\,du + n\theta_n \int_0^1 J(u)J^*(u,f)\,du, \tag{4.2}$$

$$4\operatorname{Var}(S_n|\theta_n) \simeq n\int_0^1 J^2(u)\,du, \tag{4.3}$$

where

$$J^*(u,f) = f'(F^{-1}((u+1)/2))/f(F^{-1}((u+1)/2)), \qquad 0 < u < 1,$$

and F *is the cumulative distribution function of* f, *assuming* $f'(x)$ *exists for* $x > 0$.

LEMMA 4.2. *Under conditions* (3.1) *and* (3.2), $S_{[n(1-s)],n}$, $0 < s < 1$, *is asymptotically normally distributed with mean and variance given by*

$$\begin{aligned} 2E(S_{[n(1-s)],n}|\theta_n) &\simeq n(1-s)\int_0^1 J((1-s)u)\,du \\ &\quad + n(1-s)\theta_n \int_0^1 J((1-s)u)J^*(u,f)\,du \\ &\equiv 2\mu_n(s,\theta_n), \end{aligned} \tag{4.4}$$

$$4\operatorname{Var}(S_{[n(1-s)],n}|\theta_n) \simeq n(1-s)\int_0^1 J^2((1-s)u)\,du \equiv 4\sigma_n^2(s,\theta_n). \tag{4.5}$$

PROOF. Lemma 4.2 follows from the results of Lemma 4.1 by noting that $S_{[n(1-s)],n}$ is based on a sample size of $[n(1-s)]$ and that only the first $[n(1-s)]$ of the n scores are used. □

The proof of Theorem 4.1 will provide us with insight into how to choose $h(n)$ for unbounded score functions and how to achieve better small sample estimates using b_n and c_n rather than b or c.

PROOF OF THEOREM 4.1. Using the asymptotic normality of S_n and (3.2), we find

$$k_n(\alpha) \simeq E(S_n|\theta_n) - z_{1-\beta_n}(\text{Var}(S_n|\theta_n))^{1/2}, \tag{4.6}$$

which, when substituted into (3.3), yields

$$\begin{aligned} -b_{m,n} \simeq{} & z_{1-\beta_n}[\text{Var}(S_n|\theta_n)/\text{Var}(S_{n-m,n}|\theta_n)]^{1/2} \\ & -[E(S_n|\theta_n) - E(S_{n-m,n}|\theta_n)]/[\text{Var}(S_{n-m,n}|\theta_n)]^{1/2}. \end{aligned} \tag{4.7}$$

The proof is in three parts. First, we show that the first term on the right-hand side (RHS) of (4.7) converges to $z_{1-\beta}$. Then we show that the second term behaves asymptotically like bt_n. Finally, to establish the form of the constant c, we show the second term on the RHS of (3.4) behaves asymptotically like $-2bt_n$.

Using the asymptotic variance from (4.5), we see that the ratio of the variances in the first term of the RHS of (4.7) becomes

$$\int_0^1 J^2(u)\,du \Big/ \left\{(1 - t_nh(n)/n)\int_0^1 J^2((1 - t_nh(n)/n)u)\,du\right\}, \tag{4.8}$$

which converges to one since $t_nh(n) = o(n)$.

We next focus on the second term, first considering the normed numerator

$$n^{-1/2}\left[E(S_n|\theta_n) - E\left(S_{n(1-t_nh(n)/n),n}|\theta_n\right)\right], \tag{4.9}$$

with the $n^{-1/2}$ factor arising from the denominator, which is $O(n^{1/2})$. To satisfy (3.5), we must find an $h(n)$ so that (4.9) divided by t_n^{η} converges to a constant, say b^*. We can rewrite (4.9) using (4.4) to obtain

$$n^{-1/2}[\mu_n(0, \theta_n) - \mu_n(t_nh(n)/n, \theta_n)], \tag{4.10}$$

in the notation of Lemma 4.2. For bounded score functions, we can expand $\mu_n(s, \theta_n)$ for small s in a Taylor series yielding

$$\mu_n(s, \theta_n) \simeq \mu_n(0, \theta_n) + (\partial\mu_n(s, \theta_n)/\partial s)|_{s=0}s. \tag{4.11}$$

Using this result, (4.10) becomes asymptotically equal to

$$-n^{-1/2}(\partial\mu_n(s, \theta_n)/\partial s)|_{s=0}(t_nh(n)/n). \tag{4.12}$$

Since $\mu_n = O(n)$, we choose $h(n) = n^{1/2}$ and find the limit as $n \to \infty$ of (4.12) without the factor t_n to be

$$(4.13) \qquad -\lim_{n\to\infty} n^{-1}(\partial\mu_n(s,\theta_n)/\partial s)|_{s=0} = -n^{-1}(\partial\mu_n(s,0)/\partial s)|_{s=0},$$

since $\theta_n \to 0$. Using (4.4), the RHS of (4.13) becomes

$$(4.14) \qquad \left(\int_0^1 J(u)\,du + \int_0^1 uJ'(u)\,du\right)/2,$$

which, by using integration by parts, can be shown to equal $J(1-)/2$.

The denominator of the second term in (4.7) converges to $(\int_0^1 J^2(u)\,du)^{1/2}/2$ using (4.5), the substitutions $m = h(n)t_n$ and

$$4n^{-1}\mathrm{Var}(S_{[n(1-th(n)/n)],n}|\theta_n) \simeq (1-th(n)/n)\int_0^1 J^2((1-th(n)/n)u)\,du$$
$$\to \int_0^1 J^2(u)\,du,$$

and recalling that a factor of $n^{-1/2}$ was taken from the denominator in the evaluation of the numerator.

To evaluate (3.6), we note that with $m = [sn]$,

$$(4.15) \qquad \sum_{[n(1-s)]+1}^{n} a_{i,n} \sim n\int_{1-s}^{1} J(u)\,du = 2(\mu_n(0,0) - \mu_n(s,0)).$$

Replacing s with $t_n h(n)/n$ and using the Taylor series expansion in (4.11) with $\theta_n = 0$ yields

$$(4.16) \qquad n^{-1/2}\sum_{[n(1-t_n h(n)/n)]+1}^{n} a_{i,n} \sim -2(\partial\mu_n(s,0)/\partial s)|_{s=0}(t_n h(n)/n^{3/2})$$
$$\sim 2(J(1-)/2)t_n.$$

The desired result is obtained by combining (4.16) with the asymptotic results obtained for $b_{[th(n)],n}$ into (3.4) and then into (3.6). Note that with $m = h(n)t_n$ and for values of m for which $h^{-1}(n) < t_n < Kn^{\varepsilon}$ for some positive K and $0 < \varepsilon < \frac{1}{2}$ as in (3.5) and (3.6), we have $m/n \le Kn^{\varepsilon+1/2}/n = o(1)$ in the above, so that Lemma 4.2 is valid for large n when s satisfies $0 < Kn^{\varepsilon-1/2} < s < 1$. □

To apply the asymptotic results to small samples, it is useful to note the form of the terms in (3.5) and (3.6) for large n. For a specific sample size of n, we prefer to use $1-\beta_n$ rather than $1-\beta$. In addition, we can improve the small sample approximation by deriving, from the proof of Theorem 4.1, values for b_n and c_n that satisfy $b_n \sim b$ and $c_n \sim c$ for both bounded and unbounded score functions and which we will use in place of b and c. Further simplifications are possible for bounded score functions. These results are noted in Lemma 4.3. We can also deduce from the proof a sufficient condition for calculating $h(n)$ for unbounded score functions. This result is given in Lemma 4.4.

LEMMA 4.3. *For small sample size n applications of Theorem* 3.1 *and Theorem* 3.2, *use of b_n and c_n instead of b and c, respectively, may improve the* *approximations, where*

$$b_n = t^{-\eta}\{E(S_n|\theta_n) - E(S_{[n(1-th(n)/n)],n}|\theta_n)\} \times \{\mathrm{Var}(S_{[n(1-th(n)/n)],n}|\theta_n)\}^{-1/2}, \tag{4.17}$$

$$c_n = -b_n + t^{-\eta}\left\{\sum_{[th(n)]+1}^{n} a_{i,n}\right\}\{\mathrm{Var}(S_{[n(1-th(n)/n)],n}|\theta_n)\}^{-1/2}, \tag{4.18}$$

and it is expected that b_n and c_n do not depend upon $t > 0$. If $J(1-) < \infty$ with $h(n) = n^{1/2}$ and $\eta = 1$, then b_n and c_n as previously given simplify to

$$b_n = -n^{-1/2}(\partial\mu_n(s,\theta_n)/\partial s)|_{s=0}/\sigma_n(0,\theta_n), \tag{4.19}$$

$$c_n = -n^{-1/2}\{2(\partial\mu_n(s,0)/\partial s)|_{s=0} - (\partial\mu_n(s,\theta_n)/\partial s)|_{s=0}\}/\sigma_n(0,\theta_n), \tag{4.20}$$

where μ_n and σ_n are defined in Lemma 4.2.

PROOF. (4.17) and (4.18) follow after noting the form of the limit in (3.5), (3.6) and (4.7). (4.19) and (4.20) follow similarly using (4.13) and (4.16) also. □

LEMMA 4.4. *For contiguous alternatives, it is sufficient for* (3.5) *and* (3.6) *to choose $h(n)$ and η so that*

$$n^{1/2}\int_{1-th(n)/n}^{1} J(u)\,du \sim b^* t^{\eta}, \qquad b^* > 0,\ \tfrac{1}{2} < \eta \le 1. \tag{4.21}$$

PROOF. We note the form of (4.10) in the proof of Theorem 4.1 and that $\theta_n \to 0$ so that the limit needs to be evaluated for the term

$$n^{-1/2}(\mu_n(0,0) - \mu_n(th(n)/n,0)), \tag{4.22}$$

which, using (4.4), yields (4.21). □

We conclude this section with the results of straightforward applications of Lemma 4.4 to two examples of unbounded score functions.

LEMMA 4.5. *Let $h(n) = (n/\log n)^{1/2}$. Then, for the van der Waerden score function $J(u) = \Phi^{-1}((u+1)/2)$, $0 < u < 1$, evaluation of* (4.21) *yields $\eta = 1$ and $b^* = 1$.*

LEMMA 4.6. *Let $h(n) = n^{2\delta/(2\delta+1)}$. Then, for the family of score functions $J(u) = (1-u)^{\delta-1/2}$, $0 < \delta < \frac{1}{2}$, $0 < u < 1$, evaluation of* (4.21) *yields $\eta = \delta + \frac{1}{2}$ and $b^* = 2/(2\delta+1)$.*

5. Asymptotic convergence to truncated normal when $\eta = 1$. When $\eta = 1$, it is easy to show that $(n - N^R)/h(n)$ and $(n - N^A)/h(n)$ converge to truncated normal distributions using Theorems 3.1 and 3.2 once we have established some properties of the truncated normal distribution.

DEFINITION. Let $N^*(\mu, \sigma^2, c)$ be a random variable from a normal distribution, with mean μ and variance σ^2, which has been truncated from above at c, the mass above c being placed at c. Let $N_*(\mu, \sigma^2, c)$ be similarly defined as truncated below at c.

Note that if we know μ, c and the percentile at which c occurs, we know σ^2 and have completely specified the truncated normal distribution. It is easy to verify the following relationships involving linear transformations of truncated normal distributions.

LEMMA 5.1.

$$aN^*(\mu, \sigma^2, c) + b = \begin{cases} N^*(a\mu + b, a^2\sigma^2, ac + b) & \text{for } a > 0, \\ N_*(a\mu + b, a^2\sigma^2, ac + b) & \text{for } a < 0; \end{cases}$$

$$aN_*(\mu, \sigma^2, c) + b = \begin{cases} N_*(a\mu + b, a^2\sigma^2, ac + b) & \text{for } a > 0, \\ N^*(a\mu + b, a^2\sigma^2, ac + b) & \text{for } a < 0. \end{cases}$$

Let $L(X)$ denote the distribution function of the random variable X.

THEOREM 5.1. *Under the assumptions of Theorems* 3.1 *and* 3.2, *when* $\eta = 1$,

$$\lim_{n\to\infty} L((n - N^R)/h(n)) = L\left(N_*\left(b^{-1}z_{1-\beta}, b^{-2}, 0\right)\right), \tag{5.1}$$

$$\lim_{n\to\infty} L((n - N^A)/h(n)) = L\left(N_*\left(-c^{-1}z_{1-\beta}, c^{-2}, 0\right)\right). \tag{5.2}$$

PROOF. If we make the change of variables $u = z_{1-\beta} - bt$, (3.7) becomes

$$\lim_{n\to\infty} E\left\{\left((n - N^R)/h(n)\right)^r \middle| \theta_n\right\} = b^{-r}\int_{-\infty}^{z_{1-\beta}} r\left(z_{1-\beta} - u\right)^{r-1}\Phi(u)\,du. \tag{5.3}$$

From (5.1) and Lemma 5.1, we see that the limit of the rth moment is of the form

$$E\left\{b^{-1}\left(z_{1-\beta} - N^*(0, 1, z_{1-\beta})\right)^r\right\} = E\left\{N_*\left(b^{-1}z_{1-\beta}, b^{-2}, 0\right)^r\right\}, \tag{5.4}$$

thus establishing (5.1). (5.2) follows similarly. □

APPENDIX A

Justification of (3.13) using large deviation results. To justify going from (3.12) to (3.13), we show that (3.12) minus (3.13) is $o(1)$, i.e.,

$$h^{-r}(n)\sum_{m=1}^{n-n^R} rm^{r-1}\left(\Pr\{S^*_{n-m,n} \geq b_{m,n}\} - \Phi(-b_{m,n})\right) = o(1).$$

We do this by showing (A.1) in Section A.2 and (A.2) in Section A.3:

$$h^{-r}(n)\sum_{m=1}^{n-n^R} rm^{r-1}\Big[\Pr\{(S_{n-m,n}-E(S_{n-m,n}|Z) \geq b_{m,n}(\mathrm{Var}(S_{n-m,n}|Z)^{1/2}\} - \Phi(-b_{m,n})\Big] = o(1); \tag{A.1}$$

$$h^{-r}(n)\sum_{m=1}^{n-n^R} rm^{r-1}\Big[\Pr\{S^*_{n-m,n} \geq b_{m,n}\} - \Pr\{S_{n-m,n}-E(S_{n-m,n}|Z) \geq b_{m,n}(\mathrm{Var}(S_{n-m,n}|Z))^{1/2}\}\Big] = o(1). \tag{A.2}$$

We divide $\Sigma_1^{n-n^R}$ into $\Sigma_1^{m_n} + \Sigma_{m_n+1}^{n-n^R}$ and choose m_n so that $b_{m_n,n} = \gamma_n \to \infty$ at a rate slow enough that we can apply a large deviation central limit theorem result to the $m \leq m_n$ terms, and fast enough that the second sum is easily shown to be $o(1)$. The large deviation result that we apply to the terms in the first sum is derived in Section A.1. It is used to give us a bound for the differences in probabilities that, for carefully chosen $\gamma_n \sim n^{\varepsilon}$ and some $\varepsilon > 0$, allows us to complete the proof.

A.1. *A large deviation result.* We begin by deriving the necessary large deviation results. We start with a large deviation result of Feller (1943) for independent random variables (given in Theorem A.1 for completeness). We note that the W's defined in Section 2 ($S_n = \Sigma a_{i,n}W_i$) are not (in general) independent. However, the W's are conditionally independent given $Z \equiv Z_n = (Z_1, \dots, Z_n)$. Thus, our first large deviation result in Theorem A.2 involves applying Feller's (1943) theorem to $(S_n - E(S_n|Z))/(\mathrm{Var}(S_n|Z))^{1/2}$.

THEOREM A.1 (Feller (1943), page 363). *Let $\{Y_k\}$, $k = 1, 2, \dots,$ be independent random variables with $E(Y_k) = 0$ and $\mathrm{Var}(Y_k) = \sigma_k^2 < \infty$. Let $S_n = Y_1 + \cdots + Y_n$ and $\sigma_n^{*2} = \sigma_1^2 + \cdots + \sigma_n^2$. Finally, let $F_n(\gamma_n) = \Pr\{S_n \leq \gamma_n\}$ be the distribution function of S_n.*

Suppose that $|Y_k| \leq \lambda_n \sigma_n^*$ for $k = 1, \dots, n$. If $0 < \lambda_n\gamma_n < 1/12$, then $|\psi_n| < 9$ and $1 - F_n(\gamma_n\sigma_n^*) = \exp(-\gamma_n^2 Q_n(\gamma_n)/2)[\{1 - \Phi(\gamma_n)\} + \psi_n\lambda_n\exp(-\gamma_n^2/2)]$, where $Q_n(\gamma_n) = \Sigma_{\nu=1}^{\infty} q_{n,\nu}\gamma_n^{\nu}$ with $|q_{n,\nu}| < (12\lambda_n)^{\nu}/7$ and $\Phi(\cdot)$ is the cumulative distribution function of the standard normal.

In particular, for a uniformly bounded sequence $\{Y_k\}$, we have $\sigma_n^{*2} \sim O(n)$. Then $\lambda_n = O(n^{-1/2})$ and, for $\gamma_n = o(n^{1/6})$, $Q_n(\gamma_n) \to 0$ and $\lambda_n\gamma_n = o(n^{-1/3})$; thus $1 - F_n(\gamma_n\sigma_n^*) \sim 1 - \Phi(\gamma_n)$.

Before stating and proving our first large deviation result, we state several assumptions.

ASSUMPTION A.1. There exists a δ'', $0 < \delta'' < 1$, satisfying $\int_0^{1-\delta''} J^2(u)\,du > 0$.

ASSUMPTION A.2. For $0 < \delta' < \delta'' < 1$ and a sequence of alternatives $\{\theta_n\}$, ε_n is positive for all but a finite number of n, with $\varepsilon_n = \sup \varepsilon^*$ satisfying $0 \le \varepsilon^* \le \frac{1}{2}$ and

$$\Pr\{\varepsilon^* \le [f(Y_n - \theta_n)/(f(Y_n - \theta_n) + f(-Y_n - \theta_n))] \le 1 - \varepsilon^* | \theta_n\} \ge 1 - \delta',$$

where Y_n is a random variable with density $f(y - \theta_n)$.

THEOREM A.2. *Given the statistic S_n with conditions on the scores described in Section 2, Assumptions A.1 and A.2, and $0 < \gamma_n \lambda_n < 1/12$, then*

$$\begin{aligned} &\Pr\{S_n - E(S_n|Z) > \gamma_n \operatorname{Var}(S_n|Z)^{1/2} | \theta_n\} \\ &\quad = (1 - \omega_n)\exp(-\gamma_n^2 Q_n(\gamma_n)/2) \\ &\qquad \times [\{1 - \Phi(\gamma_n)\} + \psi_n \lambda_n \exp(-\gamma_n^2/2)] + \omega_n \xi_n, \end{aligned} \tag{A.3}$$

where

$$\begin{aligned} \lambda_n &\ge a_{n,n}\left\{\varepsilon_n(1 - \varepsilon_n) \sum_{1}^{[n(1-\delta'')]} a_{i,n}^2\right\}^{-1/2} \\ &\sim J(n/(n+1))\left\{\varepsilon_n(1 - \varepsilon_n) n \int_0^{(1-\delta'')} J^2(u)\, du\right\}^{-1/2}, \end{aligned} \tag{A.4}$$

$$\begin{aligned} &|\psi_n| < 9, \qquad Q_n(\gamma_n) = \sum_{\nu=1}^{\infty} q_{n,\nu} \gamma_n^{\nu} \quad \text{with } |q_{n,\nu}| < (12\lambda_n)^{\nu}/7, \\ &0 \le \omega_n \le \exp(-2n(\delta' - \delta'')^2) \quad \text{and} \quad 0 \le \xi_n \le 1. \end{aligned} \tag{A.5}$$

PROOF. Let $g_n(y) = f(y - \theta_n)$ and define $P_j = g_n(Z_j)/(g_n(Z_j) + g_n(-Z_j))$ for $j = 1, \ldots, n$. Using Okamoto (1958) (e.g., Albers, Bickel and van Zwet (1976), page 118) yields

$$\begin{aligned} &\Pr\{\varepsilon_n < P_j < 1 - \varepsilon_n \text{ for at least } (1 - \delta'')n \text{ indices } j\} \\ &\quad \ge 1 - \exp(-2n(\delta' - \delta'')^2). \end{aligned} \tag{A.6}$$

Let A_n be the event in $\{\ \}$ in (A.6). We next condition on Z. Define $p_j = g_n(z_j)/(g_n(z_j) + g_n(-z_j))$ and $Y_k = a_{k,n}(W_k - p_k)$ where Y_k is given in Feller's theorem. Then $E(Y_k|Z) = 0$, $\operatorname{Var}(Y_k|Z) = a_{k,n}^2 p_k q_k$ and $\operatorname{Var}(\Sigma Y_k|Z) = \Sigma a_{k,n}^2 p_k q_k$. We continue conditioning now on both Z and the event A_n to get a bound on the variance

$$\operatorname{Var}\left(\sum Y_k | Z, A_n\right) \ge \varepsilon_n(1 - \varepsilon_n) \sum_{1}^{n(1-\delta'')} a_{k,n}^2 \sim \varepsilon_n(1 - \varepsilon_n) n \int_0^{1-\delta''} J^2(u)\, du.$$

Note that by conditioning on A_n, we are sure that the variance term is going to infinity. Thus, conditioning on Z and A_n, we can apply Feller's theorem using (A.4) and (A.5). From (A.6), we note that $\omega_n = \Pr\{A_n'\} \le \exp(-2n(\delta' - \delta'')^2)$ is very small, so that when A_n' occurs we only need to use the fact that the large deviation probability is bounded by one. Unconditioning on A_n yields (A.6) for some $0 \le \xi_n \le 1$. □

Note that Assumption A.1 protects one from having all scores identically zero when looking at the first $(1-\delta'')n$ of the scores $a_{i,n}$. Assumption (A.2) is given in more generality than is necessary. It includes $\theta_n = \theta$, i.e., a fixed alternative, as well as $\theta_n \to \theta$, where $\liminf \varepsilon_n = \varepsilon > 0$ is bounded away from zero, the two cases in which we are most interested. From the proof of Theorem A.2, we note that we can allow $\{\theta_n\}$ with the property that $\liminf \varepsilon_n = 0$ and $\varepsilon_n > 0$ for all but a finite number of n.

We are most interested in the contiguous alternative cases where $\liminf \varepsilon_n = \varepsilon > 0$ and, as we shall see, $\lambda_n \gamma_n^s = o(1)$ for given $s > 0$. We choose $\lambda_n \sim \{\varepsilon_n(1-\varepsilon_n)\}^{-1/2} J(n/(n+1)) n^{-1/2}$. When $J(1-) < \infty$, this allows us to choose $\gamma_n \sim n^{\varepsilon}$, $0 < \varepsilon < (2s)^{-1}$. In other cases, we must take into account the rate at which $J(n/(n+1)) \to \infty$. Recall from Section 2 that we restricted $|J(u)| \le K(1-u)^{\delta - 1/2}$, $0 < \delta < \frac{1}{2}$. For $J(u) = (1-u)^{\delta-1/2}$, $0 < \delta \le \frac{1}{2}$, we have $J(n/(n+1)) \sim n^{1/2-\delta}$, which suggests using $\lambda_n \sim Kn^{-\delta}$ and $\gamma_n \sim n^{\varepsilon}$, $0 < \varepsilon < \delta/s$. In the case with $J(1-) < \infty$, this approach yields $\delta = \frac{1}{2}$.

A.2. *Proof of* (A.1). Divide $\Sigma_1^{n-n^R}$ in (A.1) into $\Sigma_1^{m_n} + \Sigma_{m_n+1}^{n-n^R}$ and choose $m_n \to \infty$ to satisfy $b_{m_n,n} = \gamma_n \to \infty$ and $m_n/n = o(1)$. Then the first sum (from 1 to m_n) in (A.1) in absolute value is less than or equal to

$$h^{-r}(n) r m_n^r \max_{m=1,\dots,m_n} \left|\Pr\left\{S_{n-m,n} - E(S_{n-m,n}|Z) \ge b_{m,n}\left(\operatorname{Var}(S_{n-m,n}|Z)\right)^{1/2}\right\} - \Phi(-b_{m,n})\right|. \tag{A.7}$$

Note for $m \in \{1,\dots,m_n\}$, $b_{m,n} \in [-K, \gamma_n]$, where K is a generic positive constant. The condition $m_n/n = o(1)$ allows us to apply Theorem A.2, which uses S_n rather than $S_{n-m_n,n}$, to bound the term in absolute values in (A.7) by

$$\begin{aligned}
&\left|\left[\exp\left(-b_{m,n}^2 Q_n(b_{m,n})/2\right) - 1\right]\left[1 - \Phi(b_{m,n})\right]\right| \\
&\quad + \left|\exp\left(-b_{m,n}^2 Q_n(b_{m,n})/2\right)\psi_n \lambda_n \exp\left(-b_{m,n}^2/2\right)\right| \\
&\quad + \omega_n \exp\left(-b_{m,n}^2 Q_n(b_{m,n})/2\right) \\
&\quad \times \left[\left(1 - \Phi(b_{m,n})\right) + |\psi_n| \lambda_n \exp\left(-b_{m,n}^2/2\right)\right] + \omega_n \xi_n.
\end{aligned} \tag{A.8}$$

If we choose λ_n and γ_n so that $0 < \lambda_n \gamma_n < 1/13$ (1/13 rather than 1/12 is used to bound $12\lambda_n\gamma_n$ below 1 in (A.9)), so that by Theorem A.2, $|\psi_n| < 9$ and $|q_{n,\nu}| < (12\lambda_n)^{\nu}/7$, then

$$\begin{aligned}
|Q_n(b_{m,n})| &= \sum_{\nu=1}^{\infty} |q_{n,\nu}||b_{m,n}|^{\nu} \le (1/7) \sum_{\nu=1}^{\infty} (12\lambda_n |b_{m,n}|)^{\nu} \\
&= (1/7) 12\lambda_n |b_{m,n}| / (1 - 12\lambda_n |b_{m,n}|) \le K\lambda_n |b_{m,n}|
\end{aligned} \tag{A.9}$$

for $m \in \{1,\dots,m_n\}$. Furthermore,

$$|b_{m,n}^2 Q_n(b_{m,n})| \le K\lambda_n |b_{m,n}|^3 \le K\lambda_n \gamma_n^3. \tag{A.10}$$

We shall choose $\gamma_n \to \infty$ so that $\lambda_n \gamma_n^3 = o(1)$. Thus, the last two terms of (A.8) are bounded by $K\exp\{-2n(\delta' - \delta'')^2\}$, and the second term in (A.11) is bounded by $K\lambda_n$. The second factor of the first term is bounded by one. The first factor of the first term is bounded by $K\gamma_n^3\lambda_n$. Since $\gamma_n \to \infty$, the first two terms of (A.8) substituted into (A.7) are bounded by

$$K(m_n/h(n))^r \gamma_n^3 \lambda_n. \tag{A.11}$$

Using $b_{m_n,n} = \gamma_n$ and (3.5) yields $-\gamma_n \simeq z_{1-\beta} - b(m_n/h(n))^\eta$, which can be simplified, recalling that $\frac{1}{2} < \eta \le 1$, to yield

$$(m_n/h(n))^r \simeq \left((\gamma_n + z_{1-\beta})/b\right)^{r/\eta} \sim K\gamma_n^{r/\eta} \le K\gamma_n^{2r}. \tag{A.12}$$

Thus, (A.11) is bounded by $K\gamma_n^{2r+3}\lambda_n$, which is $o(1)$ for $\gamma_n = n^\varepsilon$, $0 < \varepsilon < \delta/(2r+3)$, and r a positive integer since $\lambda_n \sim Kn^{-\delta}$, $0 < \delta \le \frac{1}{2}$, as noted at the end of Section A.1. The last two terms of (A.8) substituted into (A.7) are bounded by $K\gamma_n^{2r+3}\exp\{-2n(\delta' - \delta'')^2\}$, which is $o(1)$ for our choice of γ_n.

The sum from $m_n + 1$ to $n - n^R$ in (A.1) is bounded by $Kh^{-r}(n)n^r\Phi(-\gamma_n)$ since the probability terms are nonincreasing as m increases, there are fewer than n terms in the summation, and $m \le n$. For $\gamma_n = n^\varepsilon$, $n^r\Phi(-n^\varepsilon) \sim n^r\phi(n^\varepsilon)/n^\varepsilon = o(1)$. □

A.3. *Proof of* (A.2). As in the proof of (A.1), divide $\Sigma_1^{n-n^R}$ in (A.2) into $\Sigma_1^{m_n} + \Sigma_{m_n+1}^{n-n^R}$ and choose $m_n \to \infty$ to satisfy $b_{m_n,n} = \gamma_n \to \infty$ and $m_n/n = o(1)$. The second sum from $m_n + 1$ to $n - n^R$ can be shown to be $o(1)$ using the techniques at the end of the proof of (A.1) in Section A.2 We concentrate only on the first sum.

We shall write $\Pr\{S_n^* \le x\}$ in terms of the conditional means and variances given Z. We then restrict our attention to the case $Z \in A_n$ since $\Pr\{Z \in A_n'\}$ is negligible for our purposes. After giving some asymptotic expansions of the conditional means and variances for θ_n satisfying (3.1) and (3.2), we show that the differences in probabilities in (A.2) for $m \in \{1, \dots, m_n\}$ are negligible using Theorem A.2. We then use these results to show $\Sigma_1^{m_n}$ in (A.2) is $o(1)$.

We can rewrite $\Pr\{S_n^* \le x\}$ in terms of the conditional mean and variance given Z as

$$\begin{aligned} &\Pr\left\{S_n - E(S_n) \le x(\operatorname{Var}(S_n))^{1/2}\right\} \\ &\quad = \Pr\left\{S_n - E(S_n|Z) \le (\Delta_1(1+\Delta_3) + x(1+\Delta_2))(\operatorname{Var}(S_n|Z))^{1/2}\right\}, \end{aligned} \tag{A.13}$$

where

$$\Delta_1 = (E(S_n) - E(S_n|Z))/(E(\operatorname{Var}(S_n|Z)))^{1/2},$$

$$1 + \Delta_2 = (\operatorname{Var}(S_n)/\operatorname{Var}(S_n|Z))^{1/2},$$

and

$$1 + \Delta_3 = \{(E(\operatorname{Var}(S_n|Z)))/\operatorname{Var}(S_n|Z)\}^{1/2}.$$

We first investigate the size of the difference

$$\Pr\{S_n - E(S_n) \le x(\operatorname{Var}(S_n))^{1/2}\} \tag{A.14}$$
$$-\Pr\{S_n - E(S_n|Z) \le x(\operatorname{Var}(S_n|Z))^{1/2}\},$$

assuming $Z \in A_n$ (A_n defined in the proof of Theorem A.2). To do this, we need to characterize Δ_1, Δ_2 and Δ_3 in (A.13). Given Z, we expand $p_i = f(z_i - \theta)/(f(z_i - \theta) + f(-z_i - \theta))$ in a Taylor series in θ_n about 0 with $g(x) = f'(x)/f(x)$ to get

$$p_i = 1/2 - (\theta_n/2)g(z_i) - (\theta_n^2/2)[0] + \cdots . \tag{A.15}$$

We use (A.15) to obtain the following asymptotic expansions for θ_n satisfying (3.1) and (3.2):

$$E(S_n|Z) = \sum a_{i,n} p_i \simeq \tfrac{1}{2}\sum a_{i,n} - (\theta_n/2)\sum a_{i,n} g(Z_i) + \cdots , \tag{A.16}$$

$$E(S_n) = E[E(S_n|Z)] \simeq \tfrac{1}{2}\sum a_{i,n} - (\theta_n/2)\sum a_{i,n} E(g(Z_i)) + \cdots , \tag{A.17}$$

$$\operatorname{Var}(S_n|Z) = \sum a_{i,n}^2 p_i(1 - p_i) \tag{A.18}$$
$$\simeq \tfrac{1}{4}\sum a_{i,n}^2 - (\theta_n^2/4)\sum a_{i,n}^2 g^2(Z_i) + \cdots ,$$

$$E[\operatorname{Var}(S_n|Z)] \simeq \tfrac{1}{4}\sum a_{i,n}^2 - (\theta_n^2/4)\sum a_{i,n}^2 E(g^2(Z_i)) + \cdots , \tag{A.19}$$

$$\operatorname{Var}[E(S_n|Z)] \simeq (\theta_n^2/4)E\left[\sum a_{i,n}(g(Z_i) - E(g(Z_i))\right]^2 + \cdots . \tag{A.20}$$

We also use the well-known expression

$$\operatorname{Var}(S_n) = E[\operatorname{Var}(S_n|Z)] + \operatorname{Var}[E(S_n|Z)]. \tag{A.21}$$

We first look at Δ_1 in (A.13). By Chebyshev's inequality,

$$\Pr\{|\Delta_1| \ge d\} \le \operatorname{Var}[E(S_n|Z)]/(d^2 E(\operatorname{Var}(S_n|Z))), \qquad d > 0, \tag{A.22}$$

which, for $Z \in A_n$, is asymptotically less than or equal to $K\theta_n^2/d^2$ using (A.19) and (A.20). Since $\theta_n^2 \sim K/n$, we can choose $d = n^{-\delta}$, $0 < \delta < \frac{1}{2}$, to get

$$\Pr\{|\Delta_1| \ge n^{-\delta}|Z \in A_n\} \le K/n^{1-2\delta}. \tag{A.23}$$

We note that for $Z \in A_n$, $1 + \Delta_2 \simeq 1 + K\theta_n^2$ and $1 + \Delta_3 \simeq 1 + K\theta_n^2$, using (A.18)–(A.21). Thus, for $Z \in A_n$, $\Delta_2 \sim K/n$ and $\Delta_3 \sim K/n$. The term $x + \Delta_1 + x\Delta_2 + \Delta_1\Delta_3$ in (A.13), for $Z \in A_n$ and when $|\Delta_1| < n^{-\delta}$, $0 < \delta < \frac{1}{2}$ (which occurs asymptotically with probability greater than $1 - Kn^{2\delta - 1}$), is of the form: $x + \Delta_1 + x\Delta_2\Delta_1\Delta_3 \simeq x + \Delta$, with $\Delta = O(n^{-\delta})$ for $x/n^{1-\delta} = o(1)$, where we note that x may be $x_n \to \infty$ (i.e., $x_n \sim Kn^{\varepsilon}$ for some small positive ε) in some of our applications of these results.

We next investigate the difference in (A.14) for $Z \in A_n$ and $|\Delta_1| < n^{-\delta}$. The difference when we have $Z \in A_n'$ or $Z \in A_n$ and $|\Delta_1| \ge n^{-\delta}$ (to be shown to have negligible probability) will be less than one in absolute value times a negligible probability.

Let $h_1(x) = (1 - \Phi(x))h_3(x)$ and $h_2(x) = K\psi_n\lambda_n\phi(x)h_3(x)$, with $h_3(x) = \exp\{-x^2Q_n(x)/2\}$. We first investigate the difference in probability terms for $m = 1, \dots, m_n$ in [ ] in (A.2), rewritten as in (A.14) using the result of (A.13) for $Z \in A_n$ and $|\Delta_1| < n^{-\delta}$. If we let $\bar{h}(x) = h_1(x) + h_2(x)$, we note that the difference in probability terms in (A.14) is of the form $\bar{h}(x + \Delta) - \bar{h}(x)$, with $\Delta \to 0$ as $n \to \infty$ using Theorem A.2. The term involving $h_2(x)$ is bounded by $K\lambda_n$ for x satisfying $0 < \lambda_n x_n < 1/13$ and $12\lambda_n x_n = o(1)$. Thus,

$$|h_2(x + \Delta) - h_2(x)| \le K|h_2(x)| \le K\lambda_n.$$

The term

$$|h_1(x + \Delta) - h_1(x)| \le |h_3(x + \Delta) - h_3(x)| \simeq |\Delta||h_3'(x)|$$

for large n, where

$$h_3'(x) = -h_3(x)\left[2xQ_n(x) + x^2Q_n'(x)\right]/2.$$

We bound $|Q_n(x)| \le K\lambda_n|x|$ and $|Q_n'(x)| \le K\lambda_n$ for $\lambda_n x = o(1)$ using techniques similar to the derivation of (A.9) and (A.10). Thus, $|h_3'(x)| \le Kx^2\lambda_n$ for our sequences of x's. Using these results for the differences in probabilities in [ ] in (A.2) for $m = 1, \dots, m_n$ and the techniques used in the proof of (A.1) for these terms shows (A.2) for $Z \in A_n$ and $|\Delta_1| < n^{-\delta}$.

For the cases where $Z \in A_n'$ or $Z \in A_n$ and $|\Delta_1| \ge n^{-\delta}$, we bound the differences in probabilities in (A.2) for $m = 1, \dots, m_n$ by one times a negligible probability, i.e., the probability that $Z \in A_n'$ or $Z \in A_n$ and $|\Delta_1| \ge n^{-\delta}$. These negligible probabilities are bounded by $K\exp\{-2(n - m_n)(\delta' - \delta'')^2\}$ (from (A.6)) and $K(n - m_n)^{2\delta - 1}$ (from (A.23)), respectively. Note that $m_n/n \to 0$, so that the m_n in these two probability terms can be ignored in the following calculation. Using $(m_n/h(n))^r \le K\gamma_n^{2r}$ from (A.12) for $Z \in A_n'$ and $m = 1, \dots, m_n$, we bound the absolute value of (A.2) by $K\gamma_n^{2r}\exp\{-2n(\delta' - \delta'')^2\} = o(1)$. For $Z \in A_n$, $|\Delta_1| \ge n^{-\delta}$, and $m = 1, \dots, m_n$, we bound absolute (A.2) by $K\gamma_n^{2r}n^{2\delta - 1}$. Since $\gamma_n = O(n^{\varepsilon})$, with $\varepsilon > 0$ being chosen arbitrarily small, we can choose ε so this last term is $o(1)$. □

APPENDIX B

Justification of (3.14). The absolute difference between (3.13) and (3.14), by the triangle inequality, is bounded by

$$\text{(B.1)} \qquad \sum_{m=1}^{m_n} h^{-r}(n) r m^{r-1} \left| \Phi(-b_{m,n}) - \int_m^{m+1} \Phi(-b_{t,n})\, dt \right|.$$

We wish to show that this expression is $o(1)$. Given any $\varepsilon > 0$, choose $m_n^* < m_n$ with the property that $m_n^*/h(n) \to \varepsilon^{1/r}/r$. We divide $\Sigma_1^{m_n}$ into $\Sigma_1^{m_n^*}$ and $\Sigma_{m_n^*+1}^{m_n}$ and consider the two parts separately.

Bounding the difference in absolute values by one for $m = 1, \dots, m_n^*$, we can bound the first sum by $(m_n^*/h(n))^r r \to \varepsilon$ and thus it can be made arbitrarily small. The second sum is bounded by

$$\text{(B.2)} \qquad r(m_n/h(n))^r \max_{m \in \{m_n^*+1,\ldots,m_n\}} |\Phi(-b_{m,n}) - \Phi(-b_{m+1,n})|$$

$$\text{(B.3)} \qquad \le r(m_n/h(n))^r \phi(0) \max_{m \in \{m_n^*+1,\ldots,m_n\}} |b_{m,n} - b_{m+1,n}|,$$

since the maximum unit change in $\Phi(x)$ occurs at $x = 0$. Using (3.5), for $m \in \{m_n^* + 1, \ldots, m_n\}$, we have for large n

$$|b_{m,n} - b_{m+1,n}| \simeq K(m/h(n))^\eta((1 + 1/m)^\eta - 1) \simeq K(m/h(n))^\eta/m.$$

This last expression is bounded for any $m \in \{m_n^* + 1, \ldots, m_n\}$ by

$$K(m_n/h(n))^\eta/m_n^* \sim K(m_n/h(n))^\eta/h(n),$$

noting that $m_n^* \sim h(n)\varepsilon^{1/r}/r$. Substitution of this bound into (B.3) yields the bound

$$\text{(B.4)} \qquad K(m_n/h(n))^{r+\eta}h^{-1}(n) \le K\gamma_n^{2r}/h(n),$$

letting $\gamma_n = m_n/h(n)$. We know that for a score function $J(u)$ as constrained in Section 2, there exists a $\pi > 0$ so that $n^\pi/h(n) = o(1)$ (see, for example, Lemma 4.6). Thus (B.4) is bounded by $K\gamma_n^{2r}/n^\pi = o(1)$ for $\gamma_n = n^\varepsilon$, $0 < \varepsilon < \pi/2r$. □

APPENDIX C

Justification of moving limit under integral sign in (3.15). Let $f_n(t) = rt^{r-1}\Phi(-b_{[h(n)t],n})I\,(h^{-1}(n) \le t \le m_n/h(n))$. To apply the dominated convergence theorem, it suffices to find a $g(t) \ge 0$ with the properties that (1) $\int_0^\infty g(t)\,dt < \infty$ and (2) there exists an N such that, for all $n > N$, $|f_n(t)| \le g(t)$, $0 < t < \infty$. Since there exists $\varepsilon > 0$ such that $m_n/h(n) = o(n^{2\varepsilon})$ and we can choose m_n so this holds for any small $\varepsilon > 0$, we choose ε so 2ε is less than the ε needed in the assumptions for (3.5) and (3.6). Note that

$$-b_{[h(n)t],n} = z_{1-\beta} - bt^\eta - t^\eta\{(z_{1-\beta} + b_{[h(n)t],n})/t^\eta - b\}.$$

By the assumptions for (3.5), we know that for $n > N(b/2)$ the absolute value of the expression in { } is bounded by $b/2$ uniformly for all t satisfying $h^{-1}(n) < t < Kn^\varepsilon$. Thus $-b_{[h(n)t],n} \le z_{1-\beta} - (b - b/2)t^\eta| \le 2z_{1-\beta}| - (b/4)t^\eta$ for $n > N(b/2)$. If we choose $g(t) = rt^{r-1}\Phi(2|z_{1-\beta}| - (b/4)t^\eta)$, then (1) is satisfied and (2) is satisfied for all $n > N(b/2)$. □

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