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ASYMPTOTIC EXPANSIONS IN ANSCOMBE'S THEOREM FOR REPEATED SIGNIFICANCE TESTS AND ESTIMATION AFTER SEQUENTIAL TESTING[1]

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Let $x_1, x_2, \dots$ be independent and normally distributed with unknown mean θ and variance 1. Let $\tau = \inf\{n \geq 1: |s_n| \geq \sqrt{2a(n+c)}\}$. Then a repeated significance test for a normal mean rejects the hypothesis $\theta = 0$ if and only if $\tau \leq N_0$ for some positive integer N_0. The problem we consider is estimation of θ based on the data $x_1, \dots, x_T$, $T = \min\{\tau, N_0\}$. We shall solve this problem by obtaining the asymptotic expansion of the distribution of $(s_\tau - \tau\theta)/\sqrt{\tau}$ as $a \to \infty$, and then constructing the confidence intervals for θ.

1. Introduction. Let $x_1, x_2, \dots$ be independent and normally distributed random variables with unknown mean θ and variance 1. For $a > 0$ and $c \geq 0$, let us define a stopping time

$$\tau = \tau_{a,c} = \inf\{n \geq 1: |s_n| \geq c_n\}, \tag{1.1}$$

where $s_n = x_1 + \cdots + x_n$ and $c_n = \sqrt{2a(n+c)}$, $n \geq 1$. Then for a given positive integer N_0, the repeated significance test for a normal mean rejects the null hypothesis H_0: $\theta = 0$ in favor of the alternative H_1: $\theta \neq 0$ if and only if

$$\tau \leq N_0. \tag{1.2}$$

Using the method of numerical integration, Armitage, McPherson and Rowe [3] and McPherson and Armitage [6] studied the error probabilities and the expected sample sizes of the test under the null and the alternative hypotheses, respectively; some of their results are also found in Armitage [2], page 105. (It may be worth noting that the application of the methods in [3] and [6] to group sequential methods is found in Pocock [9].) Applying the nonlinear renewal theorem of Woodroofe [14], [15] and Lai and Siegmund [5], Siegmund [10] has calculated the limit of error probabilities and expected sample sizes of the Armitage repeated significance test. The results of [10] as a method of numerical analysis are satisfactory in most of the cases, except for the type I error probability for large $c > 0$. The latter problem has stimulated the research of developing an asymptotic expansion in the nonlinear renewal theorem. The first result in this direction was obtained by Siegmund [10], which, however, is valid only for a Brownian motion. Although it is limited to the normal case, the result for the random walk was obtained by Takahashi and Woodroofe [13]; their

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result gives very accurate numerical approximations for the error probabilities when $c > 0$ is large (Woodroofe and Takahashi [17]).

Now let $T = \min\{\tau, N_0\}$; then $x_1, \dots, x_T$ are the available data after the repeated significance test. In this paper, we shall consider methods of constructing confidence intervals of θ from $x_1, \dots, x_T$. Note that Anscombe's [1] central limit theorem for randomly stopped sums asserts

$$(1.3) \qquad (s_\tau - \theta\tau)/\sqrt{\tau} \to_d \Phi(x) \quad \text{as } a \to \infty,$$

where Φ denotes the standard normal distribution function. It follows that the naive $(1-\alpha)100\%$ confidence interval for θ would be

$$(1.4) \qquad \left[\bar{x}_T - z_{\alpha/2}/\sqrt{T}, \bar{x}_T + z_{\alpha/2}/\sqrt{T}\right],$$

where $\bar{x}_T = s_T/T$ and $1 - \Phi(z_{\alpha/2}) = \frac{1}{2}\alpha$. But this interval suffers from a bias; if $\theta > 0$ is large, then $\bar{x}_\tau$ tends to have a large upward bias. To correct the bias, Siegmund [11] utilizes our intuitive feelings that small values of the stopping rule τ are evidence in favor of large values of $|\theta|$. He nicely quantifies this heuristic argument to construct a confidence interval of θ, which, however, disregards the overshoot $s_\tau - c_\tau$. As indicated in Tables 3 and 4, Siegmund's intervals are wider than those of (1.4). The differences are slight for Armitage's test, but they are serious for Pocock's test where $|\theta|$ is large and N_0 is small. Hence, in what follows, using the asymptotic expansion in the nonlinear renewal theorem of Takahashi and Woodroofe [13], we shall obtain an asymptotic expansion in Anscombe's theorem with remainder $o(a^{-1})$ as $a \to \infty$. Asymptotic expansions in the distributions of stopping time and the overshoots will be considered in Section 2. Section 3 gives the asymptotic expansion in Anscombe's theorem. Confidence intervals for θ and their numerical accuracy will be discussed in Sections 4 and 5. Other problems in sequential analysis are considered in Section 6, among which is an asymptotic expansion for the power of the repeated significance test for these θ in the neighborhood of $\pm\theta_0$, where $\theta_0 = \sqrt{2a/N_0}$.

The recent paper by Woodroofe and Keener [16] is closely related to this one. It obtains more general, but less detailed results.

2. Asymptotic expansions. Let P_θ, $\theta \in (-\infty, \infty)$ denote the probability measure under which $x_1, x_2, \dots$ are independent and normally distributed with mean θ and variance 1. It is convenient to consider the one-sided version of τ. We let

$$(2.1) \qquad t = t_{a,c} = \inf\{n \geq 1 \colon s_n \geq c_n\}$$

and $R = R_{a,c} = s_t - c_t$. We shall consider the asymptotic expansions for the joint distribution of (t, R) and the marginal distribution of t as $a \to \infty$. Let

$$\psi_a(n, r) = P_0\{t \geq n | s_n = c_n + r\},$$

$$\psi(\varepsilon, r) = P_0\{s_k \leq \tfrac{1}{2}\varepsilon k - r, \forall k \geq 1\}, \qquad -\infty < r < \infty, \varepsilon > 0.$$

Then,

$$\psi_a(n, r) = P_0\{s_{nk} \le \beta_{nk}^a, k \le n-1\},$$

where $s_{nk} = s_k - (k/n)s_n$, $\beta_{n,k}^a = c_{n-k} - [1-(k/n)](c_n + r)$, $1 \le k \le n-1$. Now as $n = n(a) \to \infty$, $a \to \infty$ with $\sqrt{2a/n} \to \varepsilon \in (0, \infty)$, $\beta_{nk}^a \to \frac{1}{2}\varepsilon k - r$ and $s_{nk} \to s_k$ (a.e.) for each fixed $k \ge 1$. It follows that

$$\psi_a(n, r) \to \psi(\varepsilon, r) \quad \text{as } a \to \infty \tag{2.2}$$

(cf. [14], [15]). Now there are two major sources of error in approximating $\psi_a(n, r)$ by $\psi(\varepsilon, r)$: β_{nk}^a is not exactly equal to $\frac{1}{2}\varepsilon k - r$, and the joint distribution of $\{s_{nk}, 1 \le k \le n-1\}$ is not quite equal to that of $\{s_k, k \ge 1\}$. Obtaining the asymptotic expansions of these errors, we have the following refinement of (2.2).

LEMMA 2.1. *Let* $\varepsilon_n = \varepsilon_n(a) = \sqrt{2a/n}$, $\varepsilon_n^* = \varepsilon_n^*(a, r) = \varepsilon_n + (2r/n)$ *and*

$$D_a(n, r) = n[\psi_a(n, r) - \psi(\varepsilon_n^*, r)] + \gamma_c(\varepsilon_n, r), \tag{2.3}$$

for $-\infty < r < \infty$, $n \ge 1$ *and* $a > 0$, *where*

$$\gamma_c(\varepsilon, r) = \{[\psi'(\varepsilon,0)/\psi(\varepsilon,0)] + \tfrac{1}{2}r - \tfrac{3}{2}c\varepsilon\}\psi'(\varepsilon, r) + \psi''(\varepsilon, r)$$

and the prime denotes differentiation with respect to ε. *If* $n = n(a) \to \infty$ *as* $a \to \infty$ *with* ε_n *bounded away from* 0 *and bounded above by* $O(a^{1/8})$, *then* $D_a(n, r) \to 0$ *uniformly on compacta in* $-\infty < r < \infty$; *if* $\lambda_0 > 0$ *and* $0 < \lambda_n = o(\exp\{\sqrt{\log n}\})$, *then there are positive constants* C *and* η *for which*

$$\begin{aligned} &|D_a(n, r)| \le C\varepsilon_n, \qquad -\infty < r \le \lambda_n, \\ &|D_a(n, r)| \le C\exp\{-\eta\varepsilon_n^2\} + \bar{o}(n^{-\infty}), \qquad -\infty < r \le \lambda_0, \end{aligned} \tag{2.4}$$

where $\bar{o}(n^{-\infty})$ *denotes a term of smaller order of magnitude than* $n^{-\alpha}$ *for all* $\alpha \ge 1$.

The proof of Lemma 2.1 when $c = 0$ is given by Takahashi and Woodroofe [13]. The extension to $c > 0$ is straightforward and is omitted.

Now for any fixed $\theta > 0$, let $t^* = \frac{1}{2}\theta(t - N)/\sqrt{N}$, where $N = 2a/\theta^2$. It is not difficult to see that the joint distribution of t^* and R,

$$H_a(x, r) = P_\theta\{t^* < x, R \le r\},$$

has a density in R: $h_a(x, r) = (\partial/\partial r)H_a(x, r)$. By the same reasoning leading to (2.2), it follows that

$$h_a(x, r) \to \frac{2}{\theta}\psi(\theta, r)\Phi(x) \quad \text{as } a \to \infty, \tag{2.5}$$

uniformly on compacta in $r \ge 0$ and $x \in (-\infty, \infty)$, where $\Phi(x) = \int_{-\infty}^{x}\phi(x)\,dx$, $\phi(x) = (1/\sqrt{2\pi})\exp\{-\frac{1}{2}x^2\}$ ([14], [15]). Moreover, it can be shown that

$$P_\theta\{t^* < x, R > r_0\} \to \frac{2}{\theta}\int_{r_0}^{\infty}\psi(\theta, r)\,dr\,\Phi(x) \quad \text{as } a \to \infty, \tag{2.6}$$

for all $r_0 \ge 0$. The next theorem is the refinement of (2.6).

THEOREM 2.1. *Let $\theta > 0$ be fixed and t^* be defined as before. Then, for any x such that $K_x = N + (2x/\theta)\sqrt{N}$ is a positive integer, and for any $r_0 \geq 0$,*

$$(2.7)\quad \begin{aligned} &P_\theta\{t^* < x, R > r_0\} \\ &= \frac{2}{\theta}\int_{r_0}^{\infty}\Big\{\psi(\theta, r)\Phi(x) + \frac{1}{\sqrt{N}}\rho_{1c}(\theta, r, x)\phi(x) \\ &\qquad + \frac{1}{N}\left[\rho_{2c}(\theta, r, x)x\phi(x) + \rho_{3c}(\theta, r)\Phi(x)\right]\Big\}\,dr + o(a^{-1}) \end{aligned}$$

as $a \to \infty$, where

$$\rho_{1c}(\theta, r, x) = -\left(\frac{\theta}{4} + \frac{x^2}{2\theta}\right)\psi(\theta, r) + \psi'(\theta, r) - r\psi(\theta, r) - \frac{c\theta}{2}\psi(\theta, r),$$

$$\begin{aligned} \rho_{2c}(\theta, r, x) = &\Bigg[\left\{\frac{1}{\theta}\left(\frac{\theta}{4} + \frac{x^2}{2\theta}\right) - \frac{1}{2}\left(\frac{\theta}{4} + \frac{x^2}{2\theta}\right)^2 + \frac{\theta^2}{96}\right\}\psi(\theta, r) \\ &+ \left\{\left(\frac{\theta}{4} + \frac{x^2}{2\theta}\right) - \frac{1}{\theta}\right\}\{\psi'(\theta, r) - r\psi(\theta, r)\} \\ &+ r\psi'(\theta, r) - \frac{r^2}{2}\psi(\theta, r) - \frac{1}{2}\psi''(\theta, r)\Bigg] \\ &+ c\Bigg[\left\{1 - \frac{\theta}{2}\left(\frac{\theta}{4} + \frac{x^2}{2\theta}\right)\right\}\psi(\theta, r) \\ &\qquad + \frac{\theta}{2}\{\psi'(\theta, r) - r\psi(\theta, r)\} - \frac{c\theta^2}{8}\psi(\theta, r)\Bigg], \end{aligned}$$

and

$$\begin{aligned} \rho_{3c}(\theta, r) = &-\frac{r}{\theta}\psi(\theta, r) + \left\{\frac{1}{\theta} + \frac{r}{2} - \left[\frac{\psi'(\theta,0)}{\psi(\theta,0)}\right]\right\}\psi'(\theta, r) \\ &- \frac{1}{2}\psi''(\theta, r) + c\{\theta\psi'(\theta, r) - \psi(\theta, r)\}. \end{aligned}$$

The proof of Theorem 2.1 is long and complicated; we defer it to the Appendix. By letting $r_0 = 0$, we have an asymptotic expansion for $P_\theta\{t^* < x\}$. The following notation will be used throughout the rest of this paper:

$$(2.8)\quad \nu_{jl}^{(i)}(\theta) = \sum_{n=1}^{\infty}\frac{2}{\theta}\int_{(\theta/2)(n-1)}^{(\theta/2)n} r^j\left(\frac{\theta}{2}n\right)^l\frac{\partial^i}{\partial\theta^i}\psi(\theta, r)\,dr, \qquad i, j, l = 0, 1, 2.$$

We shall sometimes write $\nu_{jl}(\theta)$ for $\nu_{jl}^{(0)}(\theta)$ if there is no confusion. Note that $\nu_{10}(\theta)$ and $\nu_{20}(\theta)$ are the asymptotic means of R and R^2, respectively.

Theorem 2.2. *Under the same conditions as in Theorem* 2.1,

$$
\begin{aligned}
P_\theta\{t^* < x\} = \Phi(x) &+ \frac{1}{\sqrt{N}}\left\{\frac{1}{\theta} - \nu_{10}(\theta) - \left(\frac{\theta}{4} + \frac{x^2}{2\theta}\right) - \frac{c\theta}{2}\right\}\phi(x) \\
&+ \frac{1}{N}\left\{\left[\frac{1}{2} + \frac{x^2}{\theta^2} - \frac{1}{2}\left(\frac{\theta}{4} + \frac{x^2}{2\theta}\right)^2 - \frac{1}{\theta^2} + \frac{\theta^2}{96}\right.\right. \\
&\qquad \left. - \left(\frac{\theta}{4} + \frac{x^2}{2\theta} - \frac{1}{\theta}\right)\nu_{10}(\theta) + \nu_{10}^{(1)}(\theta) - \frac{1}{2}\nu_{20}(\theta)\right] \qquad (2.9) \\
&\qquad \left. + c\left[-\frac{\theta}{2}\left(\frac{\theta}{4} + \frac{x^2}{2\theta}\right) + \frac{3}{2} - \frac{\theta}{2}\nu_{10}(\theta) - \frac{c\theta^2}{8}\right]\right\}x\phi(x) \\
&+ o(a^{-1}) \quad \text{as } a \to \infty.
\end{aligned}
$$

Proof. Since $(2/\theta)\int_0^\infty \psi(\theta, r)\,dr = 1$, by interchanging the order of integration and differentiation we have

$$
\nu_{00}^{(1)}(\theta) = \frac{1}{\theta}, \qquad \nu_{00}^{(2)}(\theta) = 0, \qquad \nu_{10}^{(1)}(\theta) = \frac{1}{\theta}\nu_{10}(\theta) + \frac{\partial}{\partial\theta}\nu_{10}(\theta)
$$

(cf. [13], Section 2). Now, from [15] we have

$$
\frac{2}{\theta}\int_0^\infty \rho_{3c}(\theta, r)\,dr = -\frac{1}{2\theta}\nu_{10}(\theta) + \frac{1}{2}\frac{\partial}{\partial\theta}\nu_{10}(\theta) - \frac{1}{\theta}\left[\frac{\psi'(\theta,0)}{\psi(\theta,0)}\right] + \frac{1}{\theta^2},
$$

$$
\nu_{10}(\theta) = \frac{1}{\theta} + \frac{\theta}{4} - \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\phi\left(\frac{\theta}{2}\sqrt{n}\right) + \frac{\theta}{2}\sum_{n=1}^{\infty}\Phi\left(\frac{-\theta}{2}\sqrt{n}\right).
$$

It follows that

$$
\frac{2}{\theta}\int_0^\infty \rho_{3c}(\theta, r)\,dr = 0. \qquad (2.10)
$$

The theorem follows from (2.7) with $r_0 = 0$ and (2.10). □

Remark 2.1. Since t has a lattice distribution with span 1, we have an alternative expression for (2.9),

$$
\begin{aligned}
P_\theta\{t^* \le x\} = \Phi(x) &+ \frac{1}{\sqrt{N}}\left\{\frac{1}{\theta} - \nu_{10}(\theta) + \left(\frac{\theta}{4} - \frac{x^2}{2\theta}\right) - \frac{c\theta}{2}\right\}\phi(x) \\
&+ \frac{1}{N}\left\{\left[-\frac{1}{2} + \frac{x^2}{\theta^2} - \frac{1}{2}\left(\frac{\theta}{4} - \frac{x^2}{2\theta}\right)^2 - \frac{1}{\theta^2} + \frac{\theta^2}{96}\right.\right. \\
&\qquad \left. + \left(\frac{\theta}{4} - \frac{x^2}{2\theta} + \frac{1}{\theta}\right)\nu_{10}(\theta) + \nu_{10}^{(1)}(\theta) - \frac{1}{2}\nu_{20}(\theta)\right] \qquad (2.11) \\
&\qquad \left. + c\left[\frac{\theta}{2}\left(\frac{\theta}{4} - \frac{x^2}{2\theta}\right) + \frac{3}{2} - \frac{\theta}{2}\nu_{10}(\theta) - \frac{c\theta^2}{8}\right]\right\}x\phi(x) \\
&+ o(a^{-1}) \quad \text{as } a \to \infty.
\end{aligned}
$$

REMARK 2.2. We may use a polygonal approximation $F_a^\#(x)$ to the distribution of t^*. If we set

$$\begin{aligned} F_a^\#(x) &= \frac{1}{2}\left[P_\theta\{t^* < x\} + P_\theta\{t^* \le x\}\right] \\ &= \Phi(x) + \frac{1}{\sqrt{N}}\left\{\frac{1}{\theta} - \nu_{10}(\theta) - \frac{x^2}{2\theta} - \frac{c\theta}{2}\right\}\phi(x) \\ &\quad + \frac{1}{N}\left\{\left[\frac{x^2}{\theta^2} - \frac{1}{2}\left(\frac{\theta^2}{24} + \frac{x^4}{4\theta^2}\right) - \frac{1}{\theta^2}\right.\right. \\ &\qquad \left. - \left(\frac{x^2}{2\theta} - \frac{1}{\theta}\right)\nu_{10}(\theta) + \nu_{10}^{(1)}(\theta) - \frac{1}{2}\nu_{20}(\theta)\right] \\ &\qquad \left. + c\left[-\frac{x^2}{4} + \frac{3}{2} - \frac{\theta}{2}\nu_{10}(\theta) - \frac{c\theta^2}{8}\right]\right\} x\phi(x)(1 + o(1)), \end{aligned} \tag{2.12}$$

then formal integration yields

$$\int_{-\infty}^{\infty} x\, dF_a^\#(x) = E_\theta\{t^*\} + o(N^{-1}). \tag{2.13}$$

It follows that

$$E_\theta\{t\} = \frac{2a-1}{\theta^2} + \frac{2}{\theta}\nu_{10}(\theta) + c + o(a^{-1/2}) \tag{2.14}$$

as $a \to \infty$ (cf. [10]). The justification of (2.14) is straightforward and we shall omit the proof.

Now, the proof of Lemma 2.1 indicates that the remainder term is of the order $o(a^{-3/2})$ as $a \to \infty$. We may obtain a refinement of Theorem 2.1 up to terms involving $a^{-3/2}$ as $a \to \infty$. It follows that the $o(a^{-1/2})$ term in the right-hand side of (2.14) is of the order a^{-1} and the exact constant for this term will be calculated by the method of this paper. Unfortunately, the algebra is very complicated; we shall address the problem in a separate paper.

3. Refinement of Anscombe's theorem. In this section we shall develop an asymptotic expansion in Anscombe's central limit theorem for the randomly stopped sums [1]. Let $\theta > 0$ be fixed and let $a \to \infty$ through the integer multiples of $\theta^2/2$. Let $B = B(a, \theta) = \{|t - N| \le N^{3/5}\}$; then it is easy to see that $1 - P_\theta\{B\} = \bar{o}(a^{-\infty})$ as $a \to \infty$. We shall expand $\sqrt{t}$ and $1/\sqrt{t}$ into Taylor series about $t = N$ on the set B. It follows that as $a \to \infty$,

$$\sqrt{t} = \sqrt{N} + \frac{1}{2}\frac{t-N}{\sqrt{N}} - \frac{1}{8}\frac{(t-N)^2}{\sqrt{N^3}} + \frac{1}{16}\frac{(t-N)^3}{\sqrt{N^5}} + O(N^{-11/10}) \quad \text{a.e. on } B,$$

$$1/\sqrt{t} = \frac{1}{\sqrt{N}} - \frac{1}{2}\frac{t-N}{\sqrt{N^3}} + O(N^{-13/10}) \quad \text{a.e. on } B.$$

Remember that $t^* = \frac{1}{2}\theta(t-N)/\sqrt{N}$ and $R = s_t - c_t$. It is easy to see that as $a \to \infty$,

$$\frac{s_t - \theta t}{\sqrt{t}} = -t^* + \frac{1}{\sqrt{N}}\left\{\frac{1}{2\theta}t^{*2} + R + \frac{c\theta}{2}\right\} \tag{3.1}$$
$$+ \frac{1}{N}\left\{\frac{-1}{2\theta^2}t^{*3} - \frac{1}{\theta}Rt^* + ct^*\right\} + \tilde{o}(a^{-1}),$$

where $\tilde{o}(a^{-1})$ is a random variable such that

$$P_\theta\left\{|\tilde{o}(a^{-1})| > \varepsilon a^{-1}\right\} = o(a^{-1}) \quad \text{as } a \to \infty.$$

It follows that for all $x \in (-\infty, \infty)$,

$$P_\theta\left\{\frac{s_t - \theta t}{\sqrt{t}} \le x\right\}$$
$$= P_\theta\left\{-t^* + \frac{1}{\sqrt{N}}\left\{\frac{1}{2\theta}t^{*2} + R + \frac{c\theta}{2}\right\}\right.$$
$$\left. + \frac{1}{N}\left\{\frac{-1}{2\theta^2}t^{*3} - \frac{1}{\theta}Rt^* + ct^*\right\} \le x\right\} + o(a^{-1}) \tag{3.2}$$
$$= P_\theta\left\{t^* \ge -x + \frac{1}{\sqrt{N}}\left\{\frac{x^2}{2\theta} + R + \frac{\theta c}{2}\right\} - \frac{3cx}{2N}\right\} + o(a^{-1}) \quad \text{as } a \to \infty.$$

Here the last equality is obtained by solving the inequality inside the probability sign of the second equation with respect to t^*. We shall apply Theorem 2.1 to the right-most side of (3.2), and for this purpose we shall write

$$K_x' = K_x'(a, \theta, c) = N - \frac{2x}{\theta}\sqrt{N} + \left(\frac{x^2}{\theta^2} + c\right) - \frac{3cx}{\theta\sqrt{N}}.$$

Now for all x such that K_x' is a positive integer, we have

$$P_\theta\left\{\frac{s_t - \theta t}{\sqrt{t}} \le x\right\}$$
$$= 1 - \sum_{n=1}^{\infty} \int_{(\theta/2)(n-1)}^{(\theta/2)n} P_\theta\left\{t^* < -x + \frac{1}{\sqrt{N}}\left\{\frac{x^2}{2\theta} + \frac{\theta}{2}n + \frac{\theta c}{2}\right\}\right. \tag{3.3}$$
$$\left. - \frac{3cx}{2N}, R \in dr\right\} + o(a^{-1}) \quad \text{as } a \to \infty.$$

Hence, to obtain an informal expansion, we may apply Theorem 2.1 and (2.10) to the probability inside the integral sign for each $n \ge 1$. By Taylor's theorem and by some straightforward but tedious algebra, it follows that (3.3) becomes

$$
\begin{aligned}
1 - \Bigg[& \sum_{n=1}^{\infty} \frac{2}{\theta} \int_{(\theta/2)(n-1)}^{(\theta/2)n} \psi(\theta, r)\, dr\, \Phi(-x) \\
& + \frac{1}{\sqrt{N}} \left\{ \sum_{n=1}^{\infty} \frac{2}{\theta} \int_{(\theta/2)(n-1)}^{(\theta/2)n} \left\{ -\frac{\theta}{4}\psi(\theta, r) + \psi'(\theta, r) \right.\right. \\
& \qquad \left.\left. + \left(\frac{\theta}{2}n - r\right)\psi(\theta, r) \right\} dr \right\} \phi(x) \\
(3.4) \quad & + \frac{1}{N} \left\{ \sum_{n=1}^{\infty} \frac{2}{\theta} \int_{(\theta/2)(n-1)}^{(\theta/2)n} \left\{ \left(\frac{\theta^2}{48} - \frac{1}{4} - 2c \right) \psi(\theta, r) \right.\right. \\
& \qquad \left. + \left(\frac{1}{\theta} - \frac{\theta}{4} \right) \psi'(\theta, r) + \frac{1}{2}\psi''(\theta, r) \right\} dr \\
& + \sum_{n=1}^{\infty} \frac{2}{\theta} \int_{(\theta/2)(n-1)}^{(\theta/2)n} \left\{ \left[\frac{1}{2}\left(\frac{\theta}{2}n - r\right)^2 + \left(\frac{1}{\theta} - \frac{\theta}{4}\right)\left(\frac{\theta}{2}n - r\right) \right] \psi(\theta, r) \right. \\
& \qquad \left.\left. + \left(\frac{\theta}{2}n - r\right)\psi'(\theta, r) \right\} dr \right\} x\phi(x) \Bigg] + o(a^{-1})
\end{aligned}
$$

as $a \to \infty$. Indeed this heuristic argument can be justified as in Woodroofe and Takahashi [17] (also see the Appendix). We have thus obtained the main result of this paper.

THEOREM 3.1. *Let $\theta > 0$ be fixed and $\nu_{jl}^{(i)}(\theta)$ be as in* (2.8). *Then for each $x \in (-\infty, \infty)$ for which K_x' is a positive integer*

$$
(3.5) \quad
\begin{aligned}
P_\theta\left\{ \frac{s_t - \theta t}{\sqrt{t}} \le x \right\} &= \Phi(x) + \frac{1}{\sqrt{2a}} Q_1(\theta)\phi(x) \\
&\quad + \frac{1}{2a} Q_2(\theta) x\phi(x) + o(a^{-1})
\end{aligned}
$$

as $a \to \infty$, where

$$
\begin{aligned}
Q_1(\theta) &= -\left\{1 - \tfrac{1}{4}\theta^2 + \theta\left(\nu_{01}(\theta) - \nu_{10}(\theta)\right)\right\}, \\
Q_2(\theta) &= -\left\{ \tfrac{1}{48}\theta^4 - \tfrac{1}{2}\theta^2 + 1 - 2c\theta^2 + \theta^2\left(\tfrac{1}{2}\nu_{02}(\theta) + \tfrac{1}{2}\nu_{20}(\theta) - \nu_{11}(\theta)\right) \right. \\
&\qquad \left. + \left(\theta - \tfrac{1}{4}\theta^3\right)\left(\nu_{01}(\theta) - \nu_{10}(\theta)\right) + \theta^2\left(\nu_{01}^{(1)}(\theta) - \nu_{10}^{(1)}(\theta)\right) \right\}.
\end{aligned}
$$

We shall next present the Cornish–Fisher-type expansion for the distribution function of $(s_t - \theta t)/\sqrt{t}$, and for this we shall let

$$
(3.6) \qquad Q_1^*(\theta) = -Q_1(\theta), \qquad Q_2^*(\theta) = -\left(\tfrac{1}{2}Q_1(\theta)^2 + Q_2(\theta)\right).
$$

COROLLARY 3.1. *Let* $z' = z'(z, a, \theta) = z + Q_1^*(\theta)/\sqrt{2a} + Q_2^*(\theta)z/2a$. *If* $K_{z'}'$ *is a positive integer, then*

$$P_\theta\left\{\frac{s_t - \theta t}{\sqrt{t}} \le z + \frac{1}{\sqrt{2a}}Q_1^*(\theta) + \frac{z}{2a}Q_2^*(\theta)\right\} = \Phi(z) + o(a^{-1}) \tag{3.7}$$

as $a \to \infty$.

PROOF. The proof is a straightforward application of Theorem 3.1. We shall omit the proof. □

In order to calculate the constants $Q_1(\theta)$ and $Q_2(\theta)$, it is necessary to evaluate the integrals involving $\psi(\theta, r)$ and $\psi'(\theta, r)$ in the intervals $\frac{1}{2}\theta(n-1) \le r < \frac{1}{2}\theta n$, $n \ge 1$. Although we do not know any bona fide method performing these integrations, we may apply the diffusion approximation of Doob [4] and its modification by Siegmund [12]. Let $\{W(t), t \ge 0\}$ denote a standard Wiener process; then for any positive constants b and d,

$$P\{W(t) \le bt + d, \forall t \ge 0\} = 1 - \exp\{-2bd\}. \tag{3.8}$$

(See [4].) When we approximate $P_0\{s_k \le bk + d, \forall k \ge 0\}$ by the left-hand side of (3.8), Siegmund [12] suggests changing d to $d^* = d + 0.583$. Here the constant 0.583 is the asymptotic mean of the overshoot when the standard normal random walk crosses the straight line boundary as the height of the boundary goes up to infinity. The numerical accuracy of this method has also been demonstrated in [12]. Hence, we have

$$\begin{aligned} \psi(\theta, r) &= P_0\{s_k \le \tfrac{1}{2}\theta k - r, \forall k \ge 1\} \\ &= \int_{-\infty}^{(\theta/2)-r} P_0\{s_k - s_1 \le \tfrac{1}{2}\theta(k-1) + \tfrac{1}{2}\theta - x - r, \forall k \ge 1\}\phi(x)\,dx \\ &\simeq \int_r^{\infty}[1 - \exp\{-\theta(x - r^*)\}]\phi(x - \tfrac{1}{2}\theta)\,dx, \end{aligned} \tag{3.9}$$

where $r^* = r - 0.583$. We shall approximate $\psi'(\theta, r)$ by the derivative of the right-most side of (3.9). After some algebra,

$$\begin{aligned} \psi(\theta, r) &\simeq \Phi(\tfrac{1}{2}\theta - r) - e^{\theta r^*}\Phi(-\tfrac{1}{2}\theta - r), \\ \psi'(\theta, r) &\simeq \tfrac{1}{2}\phi(\tfrac{1}{2}\theta - r) - e^{\theta r^*}\{r^*\Phi(-\tfrac{1}{2}\theta - r) - \tfrac{1}{2}\phi(\tfrac{1}{2}\theta + r)\}. \end{aligned} \tag{3.10}$$

Thus, we have approximations for $\nu_{jl}^{(i)}(\theta)$, which are amenable for numerical calculation. For example,

$$\begin{aligned} \nu_{01}(\theta) - \nu_{10}(\theta) \simeq 2/\theta\Bigg\{&\sum_{n=1}^{\infty}\int_{(\theta/2)(n-1)}^{(\theta/2)n}(\tfrac{1}{2}\theta n - r) \\ &\times\{\Phi(\tfrac{1}{2}\theta - r) - e^{\theta r^*}\Phi(-\tfrac{1}{2}\theta - r)\}\,dr\Bigg\}. \end{aligned} \tag{3.11}$$

The selected values of the approximations $\hat{Q}_1(\theta)$ and $\hat{Q}_2(\theta)$ of $Q_1(\theta)$ and $Q_2(\theta)$

are listed in Table 1. The approximations for $Q_1^*(\theta)$ and $Q_2^*(\theta)$ may be available via (3.6).

We shall next consider the numerical accuracy of (3.5). In Table 2, the "Simulation" values are obtained from the average of 100,000 simulations of $P_\theta\{(s_t - \theta t)/\sqrt{t} \leq x\}$ using the subroutine RANN 2 in SSL2 (supplied by Fujitsu) and the system Facom M360 at Toyama University. The ith order

TABLE 1

Approximate values of $Q_1(\theta)$ and $Q_2(\theta)$

θ	$\hat{Q}_1(\theta)$	$\hat{Q}_2(\theta)$
0.05	−1.0000	−1.0000
0.10	−0.9999	−1.0000
0.15	−0.9999	−1.0002
0.20	−0.9999	−1.0006
0.25	−0.9999	−1.0013
0.30	−0.9999	−1.0023
0.35	−1.0000	−1.0040
0.40	−1.0002	−1.0062
0.45	−1.0005	−1.0091
0.50	−1.0010	−1.0128
0.55	−1.0015	−1.0175
0.60	−1.0021	−1.0232
0.65	−1.0030	−1.0300
0.70	−1.0040	−1.0381
0.75	−1.0051	−1.0475
0.80	−1.0067	−1.0584
0.85	−1.0079	−1.0709
0.90	−1.0096	−1.0850
0.95	−1.0115	−1.1008
1.00	−1.0137	−1.1185
1.05	−1.0160	−1.1382
1.10	−1.0186	−1.1599
1.15	−1.0214	−1.1837
1.20	−1.0245	−1.2097
1.25	−1.0278	−1.2380
1.30	−1.0313	−1.2686
1.35	−1.0351	−1.3015
1.40	−1.0392	−1.3369
1.45	−1.0435	−1.3748
1.50	−1.0481	−1.4151
1.55	−1.0530	−1.4580
1.60	−1.0582	−1.5034
1.65	−1.0636	−1.5512
1.70	−1.0693	−1.6016
1.75	−1.0753	−1.6545
1.80	−1.0816	−1.7098
1.85	−1.0881	−1.7674
1.90	−1.0950	−1.8275
1.95	−1.1021	−1.8897
2.00	−1.1095	−1.9542

TABLE 2
Numerical accuracy of (3.5)

x	First order	Second order	Third order	Simulation
		$\sqrt{2a} = 3.45,\ \theta = 0.6,\ c = 0$		
0.1	0.540	0.425	0.421	0.419
0.5	0.691	0.589	0.574	0.572
1.0	0.841	0.771	0.750	0.746
1.5	0.933	0.896	0.879	0.874
2.0	0.979	0.962	0.952	0.949
		$\sqrt{2a} = 2.986,\ \theta = 1.6,\ c = 0$		
0.1	0.540	0.399	0.392	0.431
0.5	0.691	0.567	0.537	0.594
1.0	0.841	0.756	0.713	0.773
1.5	0.933	0.887	0.855	0.892
2.0	0.977	0.958	0.940	0.968

approximation is the first i term on the right-hand side of (3.5) ($i = 1$, 2 and 3). On the whole, the second and third order approximations give us substantially better estimates for the simulated values than the first order approximation. For large a and small θ ($\sqrt{2a} = 3.45$, $\theta = 0.6$, $c = 0$), the third order approximation is better than the second order, but the relation is reversed for small a and large θ ($\sqrt{2a} = 2.986$, $\theta = 1.6$, $c = 0$). This may be caused by the diffusion approximation of $\psi(\theta, r)$ and the approximation formula for $\psi'(\theta, r)$, which we do not know yet.

To close this section, we refer the reader to Woodroofe and Keener ([16], Section 4) for a new method of calculating Q_i, $i = 1, 2$.

4. Confidence intervals. Let $\bar{x}_t = s_t/t$ and construct confidence intervals of θ after the sequential test. If z is fixed, then it follows from (3.7) that for all $\theta > 0$ such that $K'_{z'}(a, \theta, c)$ is a positive integer,

$$P_\theta\left\{\theta \geq \bar{x}_t - \frac{z}{\sqrt{t}} - \frac{1}{\sqrt{2at}} Q_1^*(\theta) - \frac{z}{2a\sqrt{t}} Q_2^*(\theta)\right\} = \Phi(z) + o(a^{-1}) \tag{4.1}$$

as $a \to \infty$. If we solve the inequality inside the probability sign with respect to θ, then we find $\theta^*(z, \bar{x}_t) = \theta^*(z, \bar{x}_t, a, c)$, for which $[\theta^*(z, \bar{x}_t), \infty)$ is a third order one-sided confidence interval of θ with the coverage probability $\Phi(z)$ up to the terms involving a^{-1} as $a \to \infty$. Hence, a third order two-sided $(1 - \alpha)100\%$ confidence interval of θ would be given by

$$\left[\theta^*(z_{\alpha/2}, \bar{x}_t), \theta^*(-z_{\alpha/2}, \bar{x}_t)\right], \tag{4.2}$$

where $1 - \Phi(z_{\alpha/2}) = \frac{1}{2}\alpha$. By approximating $Q_i^*(\theta)$ by $\hat{Q}_i^*(\theta)$, $i = 1, 2$, we have the approximations $\hat{\theta}^*(\pm z_{\alpha/2}, \bar{x}_t)$ for $\theta^*(\pm z_{\alpha/2}, \bar{x}_t)$, which are amenable for numerical calculations. (The careful reader will have observed that (4.1) holds only these θ's such that $K'_{z'}$ are positive integers. Mathematically the interval (4.2) contains only these θ's. We keep this comment in our mind all the way.)

Remember that $\tau = \inf\{n \geq 1: |s_n| \geq \sqrt{2a(n+c)}\}$, and the repeated significance test for a normal mean rejects the null hypothesis $\theta = 0$ in favor of the alternative $\theta \neq 0$ if and only if $\tau \leq N_0 = [\![2a/\theta_0^2]\!]$, for some $\theta_0 > 0$. Clearly, $\tau \leq t$ holds a.e. Moreover, if $\theta > \theta_0$, then

$$P_\theta\{\tau < t\} = P_\theta\{s_\tau \leq 0\} = o(a^{-1}) \quad \text{as } a \to \infty. \tag{4.3}$$

Treating the case when $\theta < -\theta_0$ in a similar fashion, we can construct a $(1-\alpha)100\%$ confidence interval for θ based on $x_1, \ldots, x_T$, $T = \min\{\tau, N_0\}$. Let

$$\begin{aligned} \bar{\theta}^* = &\left\{\theta^*(-z_{\alpha/2}, \bar{x}_\tau) I_{\{s_\tau \geq 0\}} - \theta^*(z_{\alpha/2}, |\bar{x}_\tau|) I_{\{s_\tau \leq 0\}}\right\} I_{\{\tau \leq N_0\}} \\ &+ \left\{\bar{x}_{N_0} + z_{\alpha/2}/\sqrt{N_0}\right\} I_{\{\tau > N_0\}} \end{aligned} \tag{4.4a}$$

and

$$\begin{aligned} \underline{\theta}^* = &\left\{\theta^*(z_{\alpha/2}, \bar{x}_\tau) I_{\{s_\tau \geq 0\}} - \theta^*(-z_{\alpha/2}, |\bar{x}_\tau|) I_{\{s_\tau \leq 0\}}\right\} I_{\{\tau \leq N_0\}} \\ &+ \left\{\bar{x}_{N_0} - z_{\alpha/2}/\sqrt{N_0}\right\} I_{\{\tau > N_0\}}. \end{aligned} \tag{4.4b}$$

For all $\theta > \theta_0$, $P_\theta\{s_\tau \leq 0\} + P_\theta\{\tau > N_0\} = o(a^{-1})$ as $a \to \infty$, it follows that the third order confidence interval $(\underline{\theta}^*, \bar{\theta}^*)$ has a desired coverage probability up to the terms involving a^{-1} as $a \to \infty$ for all $|\theta| \neq \theta_0$.

In order to simplify the calculations involved in (4.4), we may consider the second order confidence interval. Note that $P_\theta\{|\bar{x}_t - \theta| > a^{-1/2}\log a\} = o(a^{-1})$ as $a \to \infty$. It follows that as $a \to \infty$,

$$P_\theta\left\{\theta \geq \bar{x}_t - \frac{z}{\sqrt{t}} - \frac{1}{\sqrt{2at}} Q_1^*(\bar{x}_t)\right\} = \Phi(z) + o(a^{-1/2}). \tag{4.5}$$

It follows that

$$\left[\bar{x}_T - z_{\alpha/2}/\sqrt{T}, \bar{x}_T + z_{\alpha/2}/\sqrt{T}\right] - \operatorname{sgn}(s_\tau) Q_1^*(|\bar{x}_\tau|) I_{[\tau \leq N_0]} \tag{4.6}$$

is a second order $(1-\alpha)100\%$ confidence interval for θ with the remainder of the type $o(a^{-1/2})$ as $a \to \infty$, where $\operatorname{sgn}(x) = -1, 0, 1$ if $x \lesseqqgtr 0$ and I_A denotes the indicator function of the set A.

5. Numerical examples. In this section we shall compare the numerical accuracy of naive confidence interval (1.4), Siegmund's interval, and our second order (4.6) and third order (cf. (4.4)) intervals. In this study the average of 32,000 independent replications of s_τ for several values of θ and τ are used, where the values of τ are selected to correspond roughly to $E_\theta\{\tau\}$. The first example is to construct 90% confidence intervals after the repeated significance test of Armitage (see Table 3). The test parameters are $\sqrt{2a} = 3.45$, $N_0 = 148$ and $c = 0$, which gives a significance level $\alpha = 0.01$ and type II error probability $\gamma_a(\theta) = 0.05$ at $\theta = 0.4$. The row (4.6′) is the second order confidence interval from $s_\tau = \sqrt{2a\tau}$. Siegmund's intervals were obtained from Equations (13) and (14) of Siegmund [10]. We have used the approximation formulae of Nabeya [7] and its modification [8] to calculate the infinite series in (14) of [10]. Table 4 shows the 90% confidence intervals after Pocock's group sequential test [9]. Here the test parameters are $\sqrt{2a} = 2.986$, $N_0 = 5$ and $c = 0$, which also gives us a significance level $\alpha = 0.01$.

TABLE 3

90% *confidence intervals* (*Armitage test*), $\sqrt{2a} = 3.45$, $N_0 = 148$

	Confidence interval	Center of the interval	Width of the interval
	$\theta = 0.4$, $\tau = 69$, $s_\tau = 29.28$		
(1.4)	0.226–0.622	0.424	0.396
Siegmund	0.173–0.596	0.385	0.423
(4.6)	0.191–0.588	0.390	0.396
(4.6′)	0.182–0.583	0.380	0.396
Third order	0.183–0.596	0.390	0.413
	$\theta = 0.6$, $\tau = 32$, $s_\tau = 20.17$		
(1.4)	0.339–0.921	0.630	0.582
Siegmund	0.261–0.887	0.574	0.626
(4.6)	0.288–0.870	0.579	0.582
(4.6′)	0.268–0.849	0.559	0.582
Third order	0.276–0.883	0.580	0.607
	$\theta = 0.8$, $\tau = 19$, $s_\tau = 15.74$		
(1.4)	0.451–1.206	0.829	0.755
Siegmund	0.348–1.169	0.759	0.821
(4.6)	0.384–1.139	0.762	0.755
(4.6′)	0.347–1.102	0.725	0.755
Third order	0.369–1.159	0.764	0.791

In Armitage's test, where τ is large and θ is small, Siegmund's intervals are $6 \sim 8\%$ wider than that of (4.6). Note that in each of the cases in Table 3, third order looks slightly better than (4.6) in bias correction, but the width of the interval of third order is wider than that of (4.6). Compared with Siegmund's interval, third order performs better both in bias correction and the width of the intervals. Although we have pointed out some differences, the performance of

TABLE 4

90% *confidence intervals* (*Pocock test*), $\sqrt{2a} = 2.986$, $N_0 = 5$

	Confidence interval	Center of the interval	Width of the interval
	$\theta = 1.895$, $\tau = 3$, $s_\tau = 6.052$		
(1.4)	1.068–2.967	2.017	1.899
Siegmund	0.682–3.240	1.916	2.558
(4.6)	0.853–2.752	1.803	1.899
(4.6′)	0.567–2.466	1.517	1.899
Third order	0.814–2.991	1.903	2.177
	$\theta = 1.382$, $\tau = 4$, $s_\tau = 6.713$		
(1.4)	0.856–2.501	1.678	1.645
Siegmund	0.549–2.633	1.591	2.084
(4.6)	0.677–2.322	1.500	1.645
(4.6′)	0.495–2.140	1.318	1.645
Third order	0.639–2.480	1.560	1.841

third order, (4.6) and Siegmund's are about the same in the practical point of view. On the other hand, the differences are bigger in Pocock's test. Table 4 indicates that Siegmund's interval and third order are about 30% and 10 ~ 20% wider than (4.6), respectively. The bias correction of third order and (4.6) are not as good as in Armitage's test, but they are in the right direction. Moreover, (4.6) may be obtained easily from Table 1; we would say that (4.6) outperforms the others.

It is interesting to compare Siegmund's interval and (4.6′). Since both disregard the overshoots, we can compare Siegmund's and our method more directly. On the whole, Siegmund's method performs better in bias correction, but usually it suffers from the wider intervals.

Finally we note that both Siegmund's and third order fail to give us the upper bound when $\tau = 1$. Siegmund's method fails conceptually, whereas third order fails numerically (it may be conceptual also).

6. Concluding remarks. Let $\gamma_a(\theta)$ denote the type II error probability of the repeated significance test defined in Section 1. For $|\theta| > \theta_0$, Takahashi and Woodroofe [13] considered the asymptotic expansion of $\gamma_a(\theta)$ for $c = 0$ with remainder $o(a^{-3/2}\exp\{-ka\})$ as $a \to \infty$ for some constant $k > 0$. Their expansion is valid for these θ such that $(|\theta| - \theta_0)\sqrt{N_0} \to \infty$ as $a \to \infty$. When $|\theta|$ is in the neighborhood of θ_0, i.e., $(|\theta| - \theta)\sqrt{N_0}$ is bounded above by some constant as $a \to \infty$, we may apply Theorem 2.2 to derive the asymptotic expansion of $\gamma_a(\theta)$. As before, we shall let t be the one-sided version of τ and we shall suppose $\theta > \theta_0$. Let $\gamma_a^+(\theta) = P_\theta\{t > N_0\}$; then it is not difficult to see that

$$\gamma_a(\theta) = \gamma_a^+(\theta)(1 + o(a^{-1})) \quad \text{as } a \to \infty \tag{6.1}$$

(cf. [13], Corollary 1). Now

$$\gamma_a^+(\theta) = P_\theta\left\{t^* > \tfrac{1}{2}\theta(N_0 - N)/\sqrt{N_0}\right\} \tag{6.2}$$

and $x = \frac{1}{2}\theta(N_0 - N)/\sqrt{N_0}$ is bounded by the assumption. It follows that the asymptotic expansion of $\gamma_a(\theta)$ is given by 1 minus the right-hand side of (2.11) with x as previously defined.

The method of this paper (cf. Remark 2.2) may be used to obtain the asymptotic expansion of $E_\theta\{\bar{x}_\tau\}$ up to the terms involving a^{-2} as $a \to \infty$. Another problem of interest is to calculate the coverage probabilities of the second and third order confidence intervals for these θ in the neighborhood of $\pm\theta_0$ with width $O(a^{-1/2})$ as $a \to \infty$. These problems are now under investigation and will be published elsewhere.

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APPENDIX

The proof of Theorem 2.1. is outlined. We shall start with some technical results. Let $[\![U]\!]$ denote the largest integer less than or equal to U. For fixed $\theta > 0$, we let $N_1 = [\![N - N^{3/5}]\!]$, $N = [\![2a/\theta^2]\!]$ and $z_n = \sqrt{2a} - \theta\sqrt{n}$, $n \geq 1$.

LEMMA A.1. *For all* $x \in (-\infty, \infty)$ *such that* $K_x = N + (2x/\theta)\sqrt{N}$ *is a positive integer,*

$$\sum_{n=N_1}^{K_x - 1} \phi(z_n)\frac{1}{\sqrt{n}} = \frac{2}{\theta}\left\{\Phi(x) - \frac{1}{\sqrt{N}}\left[\frac{\theta}{4} + \frac{x^2}{2\theta}\right]\phi(x) + \frac{1}{N}\left[\frac{1}{\theta}\left(\frac{\theta}{4} + \frac{x^2}{2\theta}\right) - \frac{1}{2}\left(\frac{\theta}{4} + \frac{x^2}{2\theta}\right)^2 + \frac{\theta^2}{96}\right]x\phi(x)\right\} + o(N^{-1}),$$

$$\sum_{n=N_1}^{K_x - 1} z_n\phi(z_n)\frac{1}{\sqrt{n}} = \frac{2}{\theta}\left\{\phi(x) + \frac{1}{\sqrt{N}}\left[\frac{\theta}{4} + \frac{x^2}{2\theta}\right]x\phi(x)\right\} + o(N^{-1/2})$$

and

$$\sum_{n=N_1}^{K_x - 1} z_n^2\phi(z_n)\frac{1}{\sqrt{n}} = \frac{2}{\theta}\{\Phi(x) - x\phi(x)\} + o(1)$$

as $a \to \infty$.

PROOF. Let $z_n' = \sqrt{2a} - \theta\sqrt{n - \frac{1}{2}}$. Then by Taylor's theorem

$$\sum_{n=N_1}^{K_x - 1} \int_{z_{n+1}'}^{z_n'} \phi(x)\,dx = 1 - \Phi(z_{K_x}') + \bar{o}(a^{-\infty})$$
$$= \Phi(x) - \frac{1}{\sqrt{N}}\left[\frac{\theta}{4} + \frac{x^2}{2\theta}\right]\phi(x) + \frac{1}{N}\left[\frac{1}{\theta}\left(\frac{\theta}{4} + \frac{x^2}{2\theta}\right) - \frac{1}{2}\left(\frac{\theta}{4} + \frac{x^2}{2\theta}\right)^2\right]x\phi(x)(1 + o(1)) + \bar{o}(a^{-\infty}).$$

On the other hand, expanding $\phi(x)$ about $x = z_n$ for each $x \in (z_{n+1}', z_n')$,

$$\sum_{n=N_1}^{K_x - 1} \int_{z_{n+1}'}^{z_n'} \phi(x)\,dx = \frac{\theta}{2}\sum_{n=N_1}^{K_x - 1} \phi(z_n)\frac{1}{\sqrt{n}}\left[1 + \frac{\theta^2}{96N}(z_n^2 - 1) + o(a^{-1})\right].$$

The first assertion follows easily, and the rest of the lemma is proved in a similar fashion. □

LEMMA A.2. *If $N_1 \le n \le K_x - 1$, then*

$$\phi\left[\frac{c_n + r - \theta n}{\sqrt{n}}\right] = \phi(z_n) - \frac{1}{\sqrt{N}}\left(r + \frac{1}{2}c\theta\right)z_n\phi(z_n) + \frac{1}{N}\left[\frac{1}{2}\left(r + \frac{1}{2}c\theta\right)^2(z_n^2 - 1) - \left(c + \frac{r}{\theta}\right)z_n^2\right]\phi(z_n)(1 + o(1))$$

as $a \to \infty$, uniformly in $0 \le r \le d_a$, $d_a = O(\log^k a)$ for some $k \ge 1$ as $a \to \infty$.

The proof of Lemma A.2 is accomplished by straightforward application of Taylor's theorem. We shall thus omit the proof.

Now to prove Theorem 2.1, we let $\varepsilon_n = \sqrt{2a/n}$ and $\varepsilon_n^* = \varepsilon_n + (2r/n)$. It follows from the method of Woodroofe [14], [15] that

$$\begin{aligned} P_\theta\{t^* < x, R > r_0\} &= \sum_{n=1}^{K_x - 1} \int_{r_0}^{\infty} P_\theta\{t = n, R \in dr\} \\ &= \left\{\sum_{n=1}^{N_1 - 1} + \sum_{n=N_1}^{K_x - 1}\right\} \int_{r_0}^{\infty} \psi_a(n, r)\frac{1}{\sqrt{n}}\phi\left[\frac{c_n + r - \theta n}{\sqrt{n}}\right] dr \qquad \text{(A.1)} \\ &= \mathrm{I} + \mathrm{II}, \quad \text{say.} \end{aligned}$$

To estimate I, we let $m = m(a) = [\![a^{3/4}]\!]$. Then for all $n \le m$, there is a constant $C^* > 0$ such that for sufficiently large a,

$$\phi\left[(c_n + r - \theta n)/\sqrt{n}\right] \le C^* e^{-a} e^{-r(\varepsilon_m - \theta)}. \qquad \text{(A.2)}$$

For all $m < n \le N_1 - 1$, it is easily seen that

$$\psi_a(n, r) \le \Phi\left[\frac{1}{2}\varepsilon_n\sqrt{n} - \frac{4r}{\sqrt{n}}\right],$$

and hence, for these n,

$$\int_{r_0}^{\infty} \psi_a(n, r)\, dr = O(\sqrt{n}) \quad \text{as } n \to \infty. \qquad \text{(A.3)}$$

Moreover, it is not difficult to see that

$$\sum_{n < N_1} \frac{1}{\sqrt{n}}\phi\left[\frac{c_n - \theta n}{\sqrt{n}}\right] = \bar{o}(a^{-\infty}) \quad \text{as } a \to \infty. \qquad \text{(A.4)}$$

If $n \le N$, then $\phi[(c_n + r - \theta n)/\sqrt{n}] \le \phi[(c_n - \theta n)/\sqrt{n}]$ for all $r \ge 0$. It follows from (A.2)–(A.4) that

$$\mathrm{I} = \bar{o}(a^{-\infty}) \quad \text{as } a \to \infty. \qquad \text{(A.5)}$$

To analyse II, we write

$$\text{(A.6)}\quad \begin{aligned} \mathrm{II} &= \sum_{n=N_1}^{K_x-1}\int_{r_0}^{\infty}[\psi_a(n,r)-\psi(\varepsilon_n^*,r)]\frac{1}{\sqrt{n}}\phi\left[\frac{c_n+r-\theta n}{\sqrt{n}}\right]dr \\ &\quad + \sum_{n=N_1}^{K_x-1}\int_{r_0}^{\infty}\psi(\varepsilon_n^*,r)\frac{1}{\sqrt{n}}\phi\left[\frac{c_n+r-\theta n}{\sqrt{n}}\right]dr \\ &= \mathrm{II}_1+\mathrm{II}_2, \quad \text{say.} \end{aligned}$$

Now, by Lemma 2.1,

$$\text{(A.7)}\quad \begin{aligned} \mathrm{II}_1 &= \sum_{n=N_1}^{K_x-1}\int_{r_0}^{\infty} -\frac{1}{n}\gamma_c(\varepsilon_n,r)\frac{1}{\sqrt{n}}\phi\left[\frac{c_n+r-\theta n}{\sqrt{n}}\right]dr \\ &\quad + \sum_{n=N_1}^{K_x-1}\int_{r_0}^{\infty}\frac{1}{n}D_a(n,r)\frac{1}{\sqrt{n}}\phi\left[\frac{c_n+r-\theta n}{\sqrt{n}}\right]dr \\ &= \mathrm{II}_{11}+\mathrm{II}_{12}, \quad \text{say.} \end{aligned}$$

Since $r/\sqrt{n}\to 0$, $c/n\to 0$, and $\varepsilon_n\to\theta$ as $a\to\infty$ for all $N_1\le n\le K_x-1$, the dominated convergence theorem and Lemma A.1 yield

$$\text{(A.8)}\quad \frac{\theta}{2}\mathrm{II}_{11}=\frac{1}{N}\int_{r_0}^{\infty}\gamma_c(\theta,r)\,dr\,\Phi(x)+o(a^{-1})$$

as $a\to\infty$. To control II_{12}, we divide the range of integration into three subintervals $r_0\le r\le\lambda_0$, $\lambda_0<r\le\lambda_n$ and $r>\lambda_n$, where $\lambda_0>r_0$ and $\lambda_n=o(\exp\sqrt{\log n})$. Then by Lemmas 2.1 and A.2, we have

$$\text{(A.9)}\quad \mathrm{II}_{12}=o(a^{-1}) \quad \text{as } a\to\infty,$$

See Woodroofe and Takahashi [17] for related calculations.

It remains to evaluate II_2. We shall divide the range of integration into $r_0\le r\le d_a$ and $r\ge d_a=O(\log^k a)$, $k\ge 1$. The contribution of the latter interval is easily seen to be of the order $o(a^{-1})$ as $a\to\infty$. For $r_0\le r\le d_a$, we expand $\psi(\varepsilon_n^*,r)$ into Taylor series about $\varepsilon_n^*=\theta$. Note that $\varepsilon_n^*=\theta+z_n/\sqrt{N}+(z_n^2/\theta+2r)/N+o(a^{-1})$ uniformly in $r\le d_a$ and all $N_1\le n<K_x$. By Lemma A.2, it follows that

$$\text{(A.10)}\quad \begin{aligned} \mathrm{II}_2 = \sum_{n=N_1}^{K_x-1}\int_{r_0}^{\infty}\Bigg[&\psi(\theta,r)+\frac{1}{\sqrt{N}}\left\{-\left(r+\frac{c\theta}{2}\right)\psi(\theta,r)+\psi'(\theta,r)\right\}z_n \\ &+\frac{1}{N}\Bigg\{-\left(r+\frac{c\theta}{2}\right)\psi'(\theta,r)z_n^2 \\ &+\left\{\frac{1}{2}\left(r+\frac{c\theta}{2}\right)^2-\left(c+\frac{r}{\theta}\right)\right\}\psi(\theta,r)z_n^2 \\ &-\frac{1}{2}\left(r+\frac{c\theta}{2}\right)^2\psi(\theta,r)+\frac{1}{\theta}\psi'(\theta,r)z_n^2+2r\psi'(\theta,r) \\ &+\frac{1}{2}\psi''(\theta,r)z_n^2\Bigg\}\Bigg]dr\frac{\phi(z_n)}{\sqrt{n}}+o(a^{-1}) \end{aligned}$$

as $a \to \infty$. By Lemma A.1, the main part of the theorem follows from (A.8) and (A.10), and the theorem follows easily from substitution.

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