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USING THE STEPWISE BAYES TECHNIQUE TO CHOOSE BETWEEN EXPERIMENTS

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In Meeden and Ghosh (1983) a theory was developed for choosing, possibly at random, from a group of experiments the one to be observed. Here we consider the problem when the class of possible designs is restricted to a subclass of all designs. A theorem which identifies some admissible decision procedures and generalizes the early work is proved. Some applications to finite population sampling are discussed.

1. Introduction. Suppose that θ, the true but unknown state of nature, is known to belong to some finite set Θ and the statistician is faced with the decision problem specified by the decision space D and the loss function $L(\theta, d)$, $d \in D$. Before making the decision, however, the statistician may choose, possibly at random, to observe one of k different experiments. Let

$$\Gamma = \left\{ \gamma = (\gamma_1, \dots, \gamma_k) \colon \gamma_i \geq 0 \text{ for } i = 1, \dots, k \text{ and } \sum_{i=1}^{k} \gamma_i = 1 \right\}.$$

We call $\gamma = (\gamma_1, \dots, \gamma_k)$ a design and if the statistician uses γ then he observes the ith experiment with probability γ_i. The problem for the statistician is to choose a γ and then a decision rule for each possible experiment. In Meeden and Ghosh (1983), the admissible decision procedures were characterized.

It sometimes happens that the class, Γ, of all possible designs is not available to the statistician. For example, there may be a different cost associated with observing each of the experiments and the statistician can only consider designs whose expected cost is no larger than some predetermined constant. More generally let Γ^* denote a subclass of Γ. The theorem of Section 2 identifies some admissible decision procedures when the statistician's choice of designs is restricted to Γ^*. This is a mild generalization of a theorem of Meeden and Ghosh (1983). In Section 3 the theorem is applied to some examples in finite population sampling.

2. Admissibility when choosing an experiment. In what follows $L(\theta, d)$ is a nonnegative loss function. Assume that $L(\cdot, \cdot)$ is such that for any prior distribution λ on Θ, $\sum_\theta L(\theta, d)\lambda(\theta)$, as a function of d, is uniquely minimized by a member of D. Convenient conditions which guarantee this are that D be compact and convex and that $L(\theta, \cdot)$ be convex.

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Let $\chi_1, \dots, \chi_k$ be the finite sample spaces of the k (≥ 2) experiments available to the statistician. For $i = 1, \dots, k$ let X_i be a random variable taking values in χ_i with $F_i = \{f_i(\cdot|\theta): \theta \in \Theta\}$ a family of possible probability functions. For each i assume that for each $x_i \in \chi_i$ there exists a $\theta \in \Theta$ such that $f(x_i|\theta) > 0$. Finally let δ_i denote a typical decision rule (possibly randomized) from χ_i to D with risk function $r_i(\theta; \delta_i)$. We note that both L and D may also be allowed to depend on i with no change in the results to follow.

Let $\boldsymbol{\delta} = (\delta_1, \dots, \delta_k)$. For the statistician a decision procedure for this problem is a pair $(\gamma, \boldsymbol{\delta})$. For such a pair its risk function is

$$r(\theta; \gamma, \boldsymbol{\delta}) = \sum_{i=1}^{k} \gamma_i r_i(\theta; \delta_i).$$

A pair $(\gamma, \boldsymbol{\delta})$ is said to be admissible relative to Γ^* if $\gamma \in \Gamma^*$ and if there does not exist another pair $(\gamma', \boldsymbol{\delta}')$ with $\gamma' \in \Gamma^*$ and $r(\theta; \gamma', \boldsymbol{\delta}') \leq r(\theta; \gamma, \boldsymbol{\delta})$ for all $\theta \in \Theta$ with strict inequality for at least one θ.

Note if γ is such that $\gamma_i = 0$ for some i, then we will only consider pairs $(\gamma, \boldsymbol{\delta})$ where the corresponding member of $\boldsymbol{\delta}$ is unspecified, i.e., for a given design there is no need to consider decision rules for experiments which are impossible to observe.

Theorem 1 will exhibit the nature of some admissible pairs $(\gamma, \boldsymbol{\delta})$ when the class of possible designs is restricted to Γ^*. Before stating the theorem some additional notation is needed.

If λ is a prior distribution over Θ, then

$$R(\gamma, \boldsymbol{\delta}; \lambda) = \sum_{i=1}^{k} \gamma_i R_i(\delta_i; \lambda)$$

is the Bayes risk of the pair $(\gamma, \boldsymbol{\delta})$ against λ where $R_i(\delta_i; \lambda)$ is the Bayes risk of δ_i against λ.

Let $g_i(x_i; \lambda) = \sum_\theta f_i(x_i|\theta)\lambda(\theta)$, $i = 1, \dots, k$, be the marginal probability function of X_i under the prior λ. Two priors λ^i and λ^j $(i \neq j)$ are said to be orthogonal if $\Theta(\lambda^i) \cap \Theta(\lambda^j)$ is empty where $\Theta(\lambda^r) = \{\theta: \lambda^r(\theta) > 0\}$ $r = i, j$. For a set of priors $\lambda^1, \dots, \lambda^n$ define a sequence of sets associated with the ith experiment,

$$\Lambda_i^1 = \{x_i: g_i(x_i; \lambda^1) > 0\}$$

and

$$\Lambda_i^r = \left\{x_i: x_i \notin \bigcup_{j=1}^{r-1} \Lambda_i^j \text{ and } g_i(x_i; \lambda^r) > 0\right\}$$

for $r = 2, \dots, n$.

A decision rule δ_i defined on χ_i is said to be stepwise Bayes against $\lambda^1, \dots, \lambda^n$ if $\delta_i(x_i) = \delta_i^r(x_i)$ for all $x_i \in \Lambda_i^r$ for $r = 1, \dots, n$ where δ_i^r is Bayes against λ^r. δ_i is said to be unique stepwise Bayes against $\lambda^1, \dots, \lambda^n$ if it is stepwise Bayes against $\lambda^1, \dots, \lambda^n$ and $\bigcup_{r=1}^{n} \Lambda_i^r = \chi_i$. Finally the pair $(\gamma, \boldsymbol{\delta})$ is (unique) stepwise

Bayes against $\lambda^1, \dots, \lambda^n$ if δ_i is (unique) stepwise Bayes against $\lambda^1, \dots, \lambda^n$ for each i with $\gamma_i > 0$.

In the following example we will compute a pair of stepwise Bayes decision rules.

EXAMPLE. Consider just two experiments with $\chi_1 = \chi_2 = \{0, 1, 2\}$ and with $\Theta = \{0, \frac{1}{2}, 1\}$. Let $f_1(0|0) = 1$, $f_1(0|\frac{1}{2}) = f_1(0|1) = \frac{2}{10}$ and $f_1(2|\frac{1}{2}) = f_1(1|1) = \frac{1}{10}$. Let $f_2(0|0) = f_2(1|\frac{1}{2}) = 1$ and $f_2(1|1) = f_2(2|1) = \frac{1}{10}$. The problem is to estimate θ with squared-error loss with $D = [0, 2]$. Let λ^1 put mass one on $\theta = 0$ and λ^2 put mass one-half on $\theta = \frac{1}{2}$ and $\theta = 1$. It is easy to see that $\boldsymbol{\delta} = (\delta_1, \delta_2)$, the unique stepwise decision rule is given by $\delta_1(0) = 0$, $\delta_1(1) = \frac{9}{16}$ and $\delta_1(2) = \frac{15}{16}$ and $\delta_2(0) = 0$, $\delta_2(1) = \frac{1}{2}$ and $\delta_2(2) = \frac{3}{4}$.

Returning now to the general situation suppose $\boldsymbol{\delta} = (\delta_1, \dots, \delta_k)$ is such that δ_i is stepwise Bayes for each i against the sequence of priors $\lambda^1, \dots, \lambda^n$ and Γ^* is the class of designs available to the statistician. How should the statistician choose a $\gamma \in \Gamma^*$ such that the pair $(\gamma, \boldsymbol{\delta})$ is admissible relative to Γ^*? A partial answer to this question is given in the following theorem which is a mild generalization of part (a) of Theorem 1 of Meeden and Ghosh (1983) where the class of possible designs has been restricted to Γ^* a subset of Γ, the class of all possible designs.

THEOREM 1. *Let Γ^* be a class of designs and $\lambda^1, \dots, \lambda^n$ be a set of mutually orthogonal priors such that*

(i) *$\bigcup_{i=1}^k \Lambda_i^r$ is nonempty for $r = 1, 2, \dots, n$ and*

(ii) *δ_i is stepwise Bayes against $\lambda^1, \dots, \lambda^n$ for the ith problem for $i = 1, 2, \dots, k$.*

Let $\boldsymbol{\delta} = (\delta_1, \dots, \delta_k)$ and for $j = 1, 2, \dots, n$ define the following subsets of Γ^*:

$$\Phi_j = \Phi(\lambda^1, \dots, \lambda^j) = \left\{\gamma \in \Phi_{j-1}\colon R(\gamma, \boldsymbol{\delta}; \lambda^j) = \inf\left\{R(\gamma', \boldsymbol{\delta}; \lambda^j)\colon \gamma' \in \Phi_{j-1}\right\}\right\}$$

where $\Phi_0 = \Gamma^*$. Let

$$N_n = N(\lambda^1, \dots, \lambda^n) = \{i\colon \gamma_i > 0 \text{ for some } \gamma \in \Phi_n\}.$$

If δ_i is unique stepwise Bayes for each $i \in N_n$, then for any $\gamma \in \Phi_n$ the pair $(\gamma, \boldsymbol{\delta})$ is admissible relative to Γ^* if and only if $(\gamma, \boldsymbol{\delta})$ is admissible relative to Φ_n.

The proof is very similar to that given in Meeden and Ghosh (1983) and will be omitted.

Suppose we have a $\boldsymbol{\delta}$ which is unique stepwise Bayes against $\lambda^1, \dots, \lambda^n$ and we wish to find a $\gamma \in \Gamma^*$ such that the pair $(\gamma, \boldsymbol{\delta})$ is admissible relative to Γ^*. One possible conjecture is that $\gamma \in \Phi_n$ is enough to guarantee the admissibility of the pair $(\gamma, \boldsymbol{\delta})$. To see that this cannot generally be true consider the following example.

Suppose there are just two possible experiments with

$$r_1(\theta, \delta_1) = r_2(\theta, \delta_2) \quad \text{for } \theta \neq \theta_0$$

and

$$r_1(\theta_0, \delta_1) < r_2(\theta_0, \delta_2).$$

Let $\Gamma^* = \Gamma$, the class of all possible designs. Note that the pair (γ', δ) where $\gamma' = (1, 0)$ dominates (γ, δ) for every $\gamma \neq \gamma'$. If $\theta_0 \notin \cup_{i=1}^{n} \Theta(\lambda^i)$, then $\Phi_n = \Gamma^* = \Gamma$ and not every member of Φ_n leads to an admissible pair.

The following corollary gives a useful condition that guarantees that every member of Φ_n yields an admissible pair.

COROLLARY 1. *If* $\cup_{i=1}^{n} \Theta(\lambda^i) = \Theta$ *and if* $\gamma \in \Phi_n$ *then* (γ, δ) *is admissible relative to* Γ^*.

The proof of the corollary is straightforward and will be omitted.

Theorem 1 and Corollary 1 are essentially a generalizaton of part (a) of Theorem 1 of Meeden and Ghosh (1983) where the class of all possible designs Γ has been replaced by Γ^*. The earlier theorem was stated for the case with just two experiments, i.e., $k = 2$. In addition, it was assumed that neither $r_1(\theta, \delta_1) \leq r_2(\theta, \delta_2)$ for all θ with strict inequality for at least one θ nor vice versa. This assumption, in the $k = 2$ case, that neither risk vector dominates the other also guarantees that every design $\gamma \in \Phi_n$ yields a pair which is admissible relative to Γ^*, i.e., no convex combination of the two vectors can dominate any other convex combination of the vectors.

Note that for the $k > 2$ case the assumption that none of the k risks vectors of $\delta_1, \dots, \delta_k$ is dominated by any other one is not sufficient to guarantee that every design $\gamma \in \Phi_n$ yields a pair which is admissible relative to Γ^*. This is because for $k > 2$ the condition:

(C.1) none of the risk vectors of $\delta_1, \dots, \delta_k$ is dominated by any other one,

does not imply the condition:

(C.2) no convex combination of the risk vectors of $\delta_1, \dots, \delta_k$ is dominated by any other convex combination of the vectors,

as it does when $k = 2$.

We note in passing that condition (C.2) always guarantees that every design belonging to Φ_n will yield an admissible pair. However, it will usually not be as easy to verify as the condition of Corollary 1.

In the next corollary we give a partial converse to Theorem 1.

COROLLARY 2. *If* (γ, δ) *is admissible relative to* Γ^* *and if* Γ^* *is convex then there exists a sequence of mutually orthogonal priors* $\lambda^1, \dots, \lambda^n$ *such that* (i) *and* (ii) *of Theorem* 1 *are satisfied. In addition* $\gamma \in \Phi_1$.

PROOF. Since (γ, δ) is admissible relative to Γ^* there exists a prior, say λ^1, against which (γ, δ) is Bayes. Hence δ_i restricted to Λ_i^1 is Bayes for $i = 1, \dots, k$ and $\gamma \in \Phi(\lambda^1)$. If $\Lambda_i^1 = \chi_i$ for $i = 1, \dots, k$ the corollary is proved so assume this is not the case. Consider now the decision problem when $\theta \in \Theta - \Theta(\lambda^1)$ and we only consider pairs of the type (γ, δ') where $\delta_i' = \delta_i$ on Λ_i^1 for $i = 1, \dots, k$. For this restricted problem (γ, δ) is admissible and hence Bayes against some prior, say λ^2. As before δ_i is stepwise Bayes against λ^1 and λ^2. We continue in this way until we get a set of mutually orthogonal priors, say $\lambda^1, \dots, \lambda^n$, such that δ is unique stepwise Bayes. (Note: We remove from the set of priors any prior, say λ^r, for which $\cup_{i=1}^k \Lambda_i^r$ is empty.) This completes the proof. □

As we remarked earlier, Theorem 1 gives a method of finding designs to use with a unique stepwise Bayes decision rule δ such that the pair is admissible relative to Γ^*. Corollary 2 suggests that not all such "admissible" designs can be found using Theorem 1 since an "admissible" γ must belong only to Φ_1 not Φ_n. By returning to the example given just before Theorem 1 we see that this is indeed true.

For the example it is easy to find the risk functions $r_1(\theta, \delta_1)$ and $r_2(\theta, \delta_2)$ and check that neither one dominates the other. If $\Gamma^* = \Gamma$, the class of all possible designs, then by our earlier remarks for any $\gamma \in \Gamma$ the pair (γ, δ) is admissible relative to Γ.

We also note that the above choice of λ^1 and λ^2 is the only sequence of mutually orthogonal priors against which δ is unique stepwise Bayes. To see this, note that since (γ, δ) is admissible the pair is Bayes against some prior. This prior can only be λ^1, the prior which puts mass one on $\theta = 0$ since $\delta_1(0) = \delta_2(0)$. The next prior can only be λ^2, the prior which puts mass $\frac{1}{2}$ on $\theta = \frac{1}{2}$ and $\theta = 1$. Hence λ^1 and λ^2 is the unique sequence which makes δ a unique stepwise Bayes estimator.

Now it is easy to check that $R_1(\delta_1, \lambda^1) = R_2(\delta_2, \lambda^1) = 0$ and $R_1(\delta_1, \lambda^2) < R_2(\delta_2, \lambda^2)$ and so Φ_2 contains just one design, which puts all its mass on the first experiment. However, as we have seen for every design γ the pair (γ, δ) is admissible relative to Γ. This example also shows that part (b) of Theorem 1 of Meeden and Ghosh (1983) is false.

3. Applications. We will now show how Corollary 1 can be used to prove uniform admissibility in finite population sampling. [For a more detailed discussion see Section 3 of Meeden and Ghosh (1983).]

Consider a finite population U with units labeled $1, 2, \dots, N$. Let y_i be the value of a single characteristic attached to the unit i. The vector $y = (y_1, \dots, y_N)$ is the unknown state of nature and is assumed to belong to $\Theta \subset R^N$. A subset s of $\{1, 2, \dots, N\}$ is called a sample. Let $n(s)$ denote the number of elements belonging to s. Let S denote the set of all possible samples. A design is a probability measure defined on S. Given $y \in \Theta$ and $s = \{i_1, \dots, i_n\}$ where $1 \le i_1 < i_2 < \cdots < i_n \le N$ let $y(s) = (y_{i_1}, \dots, y_{i_n})$. Suppose we wish to estimate $\gamma(y) = \sum_{i=1}^N y_i$, the population total, with squared-error loss. Let $e(s, y)$ denote an estimator where $e(s, y)$ depends on y only through $y(s)$. Typically the class of possible designs is restricted to some family and one wishes to find a pair (p, e) which is admissible relative to the given family of designs. In finite population sampling this is known as uniform admissibility. Two such families which are often considered for a given positive integer n ($n < N$) are $\Gamma_1 = \{p: \sum_{s \in S} n(s)p(s) \le n\}$, the class of designs of expected sample size less than or equal to n and $\Gamma_2 = \{p: p(s) = 0 \text{ if } n(s) \neq n\}$, the class of designs of fixed sample size n.

We will consider the following estimator:

$$e_1(y, s) = \sum_{i \in s} y_i + \{n(s)\}^{-1}\Big\{\sum_{i \in s}(y_i/m_i)\Big\}\Big(\sum_{i \notin s} m_i\Big),$$

proposed by Basu (1971), where $(m_1, \dots, m_N)$ is a vector of positive constants which are not all equal.

Let S_n denote all samples of size n,

$$S(\max) = \{s\colon s \in S_n \text{ and } \sum_{i \in s} m_i = \max_{s' \in S_n} \sum_{i \in s'} m_i)$$

and

$$\Gamma_2(\max) = \{p\colon p(s) = 0 \text{ if } s \notin S(\max)\}.$$

If $\alpha_1, \dots, \alpha_r$ are r ($1 \le r \le N$) distinct real numbers, let

$$\bar{Y}_m(\alpha_1, \dots, \alpha_r) = \{y\colon y_i/m_i = \alpha_j \text{ for some } j = 1, \dots, r, \text{ for all } i = 1, \dots, N\}.$$

THEOREM 2. *If $p \in \Gamma_2(\max)$, then the pair (p, e_1) is uniformly admissible relative to Γ_1 when the parameter space is assumed to be $\bar{Y}_m(\alpha_1, \dots, \alpha_r)$ and hence the pair is uniformly admissible relative to Γ_1 when the parameter space is R^N.*

PROOF. This theorem is just Theorem 4 of Meeden and Ghosh (1983) with Γ_2 replaced by Γ_1. The proof given below closely follows the earlier argument.

There it was shown that $e_1(y, s)$ is unique stepwise Bayes for any $s \in S_n$ against a certain sequence of priors when y was assumed to belong to $\bar{Y}_m(\alpha_1, \dots, \alpha_r)$. It is easy to check that this is true for all $s \in S$, not just those belonging to S_n. For this sequence of priors, say $\lambda^1, \dots, \lambda^n$, we next identify some designs which belong to Φ_n.

For a design p and the estimator e_1 the pair (p, e_1) has risk function

$$r(y; p, e_1) = \sum_{s \in S} p(s)\left[n^{-1}(s)\Big(\sum_{i \in s} z_i\Big)\Big(\sum_{i \notin s} m_i\Big) - \Big(\sum_{i \notin s} z_i m_i\Big)\right]^2,$$

where $z_i = y_i/m_i$ for $i = 1, \dots, N$. For a given s let $a_{i,s} = n^{-1}(s)(\sum_{i \notin s} m_i)$ for $i \in s$ and $a_{i,s} = -m_i$ for $i \notin s$. Hence

$$r(y; p, e_1) = \sum_{s \in S} p(s)\left[\sum_{i=1}^{N} a_{i,s} z_i\right]^2.$$

It follows as before that for any prior λ^i in the sequence

$$R(p, e_1; \lambda^i) = \left[E(Z_1)^2 - E(Z_1 Z_2)\right]\sum_{s \in S} p(s)\Big(\sum_{i=1}^{N} a_{i,s}^2\Big).$$

Since $E(Z_1^2) - E(Z_1 Z_2) \geq 0$, by the Schwarz inequality, a design belonging to Γ_1 will belong to Φ_n if for each $i = 1, \ldots, n$ it attains the following infimum:

$$\inf_{p \in \Gamma_1} R(p, e_1; \lambda^i)$$

$$(3.1) \qquad = \left[E(Z_1)^2 - E(Z_1 Z_2)\right] \inf_{p \in \Gamma_1} \sum_{s \in S} p(s) \left(\sum_{i=1}^{N} a_{i,s}^2 \right)$$

$$= \left[E(Z_1^2) - E(Z_1 Z_2)\right] \inf_{p \in \Gamma_1} \sum_{s \in S} p(s) \left[n^{-1}(s) \left(\sum_{i \notin s} m_i \right)^2 + \sum_{i \notin s} m_i^2 \right].$$

We may assume that the population is labeled so that $m_1 \geq m_2 \geq \cdots \geq m_N$. Let $\bar{S}_k = \{s\colon s \in S_k \text{ and } \sum_{i \in s} m_i = \sum_{j=1}^{k} m_j\}$, $\bar{S} = \bigcup_{k=1}^{N} \bar{S}_k$ and $\bar{\Gamma}_1 = \{p\colon p \in \Gamma_1$ and $p(s) = 0$ if $s \notin \bar{S}\}$. Clearly any design p which attains the infimum of (3.1) must belong to $\bar{\Gamma}_1$. Therefore

$$(3.2) \qquad \inf_{p \in \Gamma_1} R(p, e_1; \lambda^i) = \left[E(Z_1^2) - E(Z_1 Z_2)\right] \inf_{p \in \bar{\Gamma}_1} \sum_{i=1}^{N} p(i) \psi(i),$$

where $p(i)$ is the probability, under design p, of selecting the sample of size i that has the i largest m_j's and

$$\psi(i) = i^{-1} \left(\sum_{j=i+1}^{N} m_j \right)^2 + \sum_{j=i+1}^{N} m_j^2 \quad \text{for } i = 1, \ldots, N.$$

[Note $\psi(N) = 0$.]

Let $\tilde{\psi}$ be a function defined on the interval $[1, N]$ which is obtained from ψ by connecting the points $(i, \psi(i))$ and $(i + 1, \psi(i + 1))$ with straight line segments for $i = 1, \ldots, N - 1$.

We will be able to complete the proof of this theorem once the following lemma is proved.

Lemma. *$\tilde{\psi}$ is a strictly decreasing convex function on* $[1, N]$.

Proof. It is easy to check that $\tilde{\psi}$ is strictly decreasing.

Let i_0 be an integer satisfying $2 < i_0 < N - 2$. Let L_k denote the slope of the line connecting the two points $(i_0, \tilde{\psi}(i_0)$ and $(i_0 + k, \tilde{\psi}(i_0 + k))$. To prove the lemma it suffices to show that $L_2 \geq L_1$ and $L_{-1} \geq L_{-2}$. We will just show that $L_2 \geq L_1$ since the proof of the other is similar.

Now

$$L_2 - L_1 = \frac{1}{2(i_0 + 2)} \left(\sum_{j=i_0+3}^{M} m_j \right)^2 + \frac{1}{2i_0} \left(\sum_{j=i_0+1}^{N} m_j \right)^2 + \frac{1}{2} m_{i_0+1}^2$$

$$- \frac{1}{2} m_{i_0+2}^2 - (i_0 + 1)^{-1} \left(\sum_{j=i_0+2}^{N} m_j \right)^2.$$

Let $d = \sum_{j=i_0+3}^{N} m_j$. Then we have

$$\begin{aligned} L_2 - L_1 = &\left[i_0(i_0+1)(i_0+2)\right]^{-1} d^2 \\ &+ \left\{ i_0^{-1} m_{i_0+1} - (i_0+1)^{-1} m_{i_0+2} + \left[i_0^{-1} - (i_0+1)^{-1}\right] m_{i_0+2} \right\} d \\ &+ (2i_0)^{-1} m_{i_0+1}^2 + (2i_0)^{-1} m_{i_0+2}^2 + i_0^{-1} m_{i_0+1} m_{i_0+2} \\ &- (i_0+1)^{-1} m_{i_0+2}^2 + 2^{-1}\left(m_{i_0+1}^2 - m_{i_0+2}^2\right), \end{aligned}$$

which is greater than zero and the lemma is proved. □

Returning to the proof of the theorem we note for any $p \in \bar{\Gamma}_1$

$$\sum_{i=1}^{N} p(i)\psi(i) = \sum_{i=1}^{N} p(i)\tilde{\psi}(i) \geq \tilde{\psi}\left(\sum_{i=1}^{N} ip(i)\right) \geq \tilde{\psi}(n)$$

by the lemma and Jensen's inequality. Since for any design $p \in \Gamma_2(\max)$, $\sum_s n(s)p(s) = n$, it follows that such a design must belong to Φ_n as well. To show the admissibility of e_1 and such a design relative to Φ_n, and hence to Γ_1 as well, we will invoke Corollary 1.

Since Γ_1 contains designs which put positive mass on all possible samples the sequence of priors against which e_1 is Bayes when the parameter space is $\bar{Y}_m(\alpha_1, \dots, \alpha_r)$ for $1 \leq r \leq N$ is of length r. This sequence of priors puts positive mass on each member of $\bar{Y}_m(\alpha_1, \dots, \alpha_r)$ and by Corollary 1 the result follows.

Theorem 4 of Meeden and Ghosh (1983) claimed to prove that e_1 along with any design belonging to $\Gamma_2(\max)$ was uniformly admissible relative to Γ_2. Since Γ_1 contains Γ_2, Theorem 2 of this paper implies that the earlier result is true. However, the argument given earlier was incomplete. This is because when the set of designs is assumed to be Γ_2 and the parameter space is $\bar{Y}_m(\alpha_1, \dots, \alpha_r)$ and $r > n$; then e_1 is stepwise Bayes against a sequence of priors which does not put mass on every point in the parameter space. In this case one cannot use Corollary 1 to prove admissibility for the designs belonging to Φ_n. In the earlier work, in an attempt to overcome this difficulty, it was shown that for $s \in S_k$ none of the risk functions of e_1 was dominated by any other one. However, as was remarked in Section 2 this is not enough to guarantee admissibility.

It is possible to give a complete proof of Theorem 4 of the earlier paper when the class of possible designs is Γ_2, without using Theorem 2 of this paper. To see this, consider the parameter space $\bar{Y}_m(\alpha_1, \dots, \alpha_r)$ where $r > n$. As was remarked above, by Corollary 1, which was used implicitly in the earlier paper, for any design $p \in \Gamma_2(\max)$ the pair (p, e_1) is admissible relative to Γ_2 when the parameter space is $\bar{Y}_m(\beta_1, \dots, \beta_n)$ for every $\{\beta_1, \dots, \beta_n\} \subset \{\alpha_1, \dots, \alpha_r\}$. By reading carefully through this earlier argument one finds that if $r(y; p', e') \leq r(y; p, e_1)$ for all $y \in \bar{Y}_m(\beta_1, \dots, \beta_n)$ for some $p' \in \Gamma_2$ and e', then $p' \in \Gamma_2(\max)$ and $e' = e_1$. Now suppose there exists a $p' \in \Gamma_2(\max)$ and e' such that (p', e') dominates (p_1, e_1) on $\bar{Y}_m(\alpha_1, \dots, \alpha_r)$. Then it must do so on every $\bar{Y}_m(\beta_1, \dots, \beta_n)$ with $\{\beta_1, \dots, \beta_n\} \subset \{\alpha_1, \dots, \alpha_r\}$ so that $p' \in \Gamma_2(\max)$ and $e' = e_1$ for every sample possible under some parameter point of $\bar{Y}_m(\beta_1, \ldots, \beta_n)$ for some $\{\beta_1, \ldots, \beta_n\} \subset \{\alpha_1, \ldots, \alpha_r\}$. Since every sample possible under $\bar{Y}_m(\alpha_1, \ldots, \alpha_r)$ is also possible under some $\bar{Y}_m(\beta_1, \ldots, \beta_n)$ this shows that $e' = e_1$ when the parameter space is $\bar{Y}_m(\alpha_1, \ldots, \alpha_r)$ and completes the proof. □

We note that the convexity argument used to prove Theorem 2 is a generalization of an argument given in Joshi (1966) where uniform admissibility results for the sample mean were given.

Finally we note that the results of Section 2 are in some sense related to the work of Scott (1975). It is easy to see that his admissibility and uniform admissibility results hold for the problem discussed in Section 2 and not just for finite population sampling.

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