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EFFICIENT SEQUENTIAL ESTIMATION IN EXPONENTIAL-TYPE PROCESSES

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A class of random processes whose likelihood functions are of exponential type is considered. A necessary and sufficient condition for a stopping time to be efficient (in the Cramér–Rao sense) is proved. This result is obtained after proving a characterization theorem, which asserts that after a suitable random-time transformation such processes become processes with stationary independent increments, by applying the solution of the problem of efficient sequential estimation in the case of processes with stationary independent increments.

1. Introduction. Investigations in efficient sequential estimation have been initiated by DeGroot (1959). He considered the Bernoulli process with the probability of success as an unknown parameter and found all efficient stopping times for the whole natural parametric space. Trybuła (1968) extended these investigations to processes with the continuous time-parameter (classical Poisson and Wiener processes) and also considered the problem of efficiency for subsets of the natural parametric space. Since then many papers dealing with these have appeared [see the references in Stefanov (1985)], the greater part of them dealing only with characterizing the form of efficiently estimable functions. In fact the problem of efficient sequential estimation is the problem of determining the efficient stopping times, i.e., finding a necessary and sufficient condition for a stopping time to be efficient [see also the section "Unsolved Problems" in Linnik and Romanovski (1972)]. This problem concerning processes with stationary independent increments has recently been solved [Stefanov (1985)]. Recently many authors have been interested in processes whose increments are not necessarily independent [Bai (1975), Różański (1980, 1981), Trybuła (1982), Franz (1982), Magiera (1984), Adke and Manjunath (1984) and Stefanov (1984)]. The sufficient statistics of these processes have some common features and belong to a large class of random processes, given by conditions (i)–(iii), which follow.

Let the strong Markov process (only Markov in the case when $T(t)$ are nonrandom) $\{X_1(t), \dots, X_n(t), T(t)\}_{t \geq 0}$ (t may be discrete or continuous) be defined on the probability space (Ω, B, P_θ) with values in $(R^{n+1}, B_{R^{n+1}})$, where θ is a parameter with values in the open set $\Theta \subset R^n$. Assume the Ω is sufficiently "rich" [see Shiryayev (1976), page 22]. Throughout the paper we assume that the trajectories of the considered process are right-continuous and that the following conditions are fulfilled:

(i) $T(0) = 0,\ X_i(0) = 0,\ i = 1, 2, \dots, n;$

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(ii) for some $\theta^0 \in \Theta$ and every $\theta \in \Theta$ we have

$$\frac{dP_{\theta,t}}{dP_{\theta^0,t}} = \exp\left[\sum_{i=1}^{n} \theta_i X_i(t) + \varphi(\theta)T(t)\right],$$

where $P_{\theta,t}$ is the restriction of P_θ, to the σ algebra $\mathscr{F}_t$, $\mathscr{F}_t := \sigma\{X_1(s),\dots, X_n(s), T(s)\colon s \le t\}$, and $\varphi(\cdot)$ is a continuously differentiable function;

(iii) $T(t)$, $t \ge 0$, are nonnegative random variables ($T(t)$ may be nonrandom as well), such that $T(t)$ is strictly increasing and continuous as a function of t and $T(t)\uparrow +\infty$ as $t \uparrow +\infty$, P_θ-a.s.

This class of processes is quite large. It contains not only sufficient statistics of the processes mentioned above, but also sufficient statistics of many other point, branching, diffusion, etc., processes. As far as the form of the likelihood functions for diffusion or point processes is concerned, one can refer to Liptser and Shiryayev (1977, 1978) and Basawa and Prakasa Rao (1980).

The aim of the present paper is to give a solution to the problem of efficient sequential estimation for the processes satisfying conditions (i)–(iii). The basic result is Theorem 1, which asserts that after a suitable random (nonrandom in the case when $T(t)$ is nonrandom) time transformation, the processes considered become processes with stationary independent increments. Theorem 1 is interesting in its own right. In particular, from it we obtain well known important results; namely after a suitable time transformation (1) the sufficient statistic in the linear birth–death process becomes a compound Poisson process [see Keiding (1975)]; (2) some diffusion processes become Wiener processes [see Liptser and Shiryayev (1978), Section 17.3]; (3) some point processes become Poisson processes [see Liptser and Shiryayev (1978), Section 18.5]. Theorem 1 together with the result of Stefanov (1985) yields the desired result (Theorem 2). Theorem 1 can also be applied for other purposes: to prove that sequential maximum likelihood estimators are asymptotically normally distributed [as in Keiding (1975)] in exponential models; to prove limit-type theorems for Markov branching processes, etc.

2. Main results. Let

$$t_s := \inf\{t\colon T(t) \ge s\}, \qquad s \ge 0,$$

$$\tilde{X}_i(s) := X_i(t_s), \qquad i = 1,2,\dots, n.$$

THEOREM 1. *The random process $\{\tilde{X}(s)\}_{s\ge 0}$ has stationary independent increments.*

LEMMA. *Let $\{Y(t)\}_{t\ge 0}$ be a strong Markov process. Denote by $P^x_{\theta,[\tau_1,\tau_2]}$ the restriction of P_θ to $\sigma\{Y((s\wedge\tau_2)\vee\tau_1)\colon s\ge 0\}$ under the condition $Y(\tau_1) = x$, where τ_1, τ_2 are finite Markov stopping times. Assume that for every finite* *stopping time τ we have for some θ^0*

$$P^x_{\theta,[0,\tau]} \ll P^x_{\theta^0,[0,\tau]}.$$

Then for every finite Markov stopping time $\tau_1, \tau_2, \tau_1 < \tau_2$,

$$\frac{dP^x_{\theta,[0,\tau_2]}}{dP^x_{\theta^0,[0,\tau_2]}} = \frac{dP^x_{\theta,[0,\tau_1]}}{dP^x_{\theta^0,[0,\tau_1]}} \frac{dP^{Y(\tau_1)}_{\theta,[\tau_1,\tau_2]}}{dP^{Y(\tau_1)}_{\theta^0,[\tau_1,\tau_2]}}. \tag{1}$$

PROOF. The proof coincides with that of Gihman and Skorohod [(1980), Chapter 7, Section 6] for nonrandom τ_1, τ_2, taking into account the strong Markov property and the fact that $\mathscr{F}_\tau^Y = \sigma\{Y(t \wedge \tau): t \geq 0\}$ in the case when Ω is sufficiently "rich." □

PROOF OF THEOREM 1. According to Döhler (1981), from (ii) we obtain that for every stopping time τ, which is finite,

$$\frac{dP_{\theta,\tau}}{dP_{\theta^0,\tau}} = \exp\left[\sum_{i=1}^{n} \theta_i X_i(\tau) + \varphi(\theta) T(\tau)\right]. \tag{2}$$

Of course, in view of condition (iii), we have that t_s, $s > 0$, are finite stopping times and $T(t_s) = s$, P_θ-a.s. Thus, in view of (1) and (2) we obtain

$$\frac{dP^{X(t_s)}_{\theta,[t_s,t_m]}}{dP^{X(t_s)}_{\theta^0,[t_s,t_m]}} = \exp\left[\sum_{i=1}^{n} \theta_i (X_i(t_m) - X_i(t_s)) + \varphi(\theta)(m-s)\right], \qquad m > s. \tag{3}$$

From (3), taking into account that t_s, $s > 0$, are crossing times, we obtain for the conditional moment generating function of $X(t_m) - X(t_s)$,

$$E_\theta\left(\exp\left[\sum_{i=1}^{n} v_i (X_i(t_m) - X_i(t_s))\right] \middle| \mathscr{F}_{t_s}\right) = \exp[(\varphi(\theta) - \varphi(\theta + v))(m-s)], \tag{4}$$

where $v := (v_1, \dots, v_n)$ is such that $\theta + v \in \Theta$. Such a v always exists, since Θ is an open set. In view of the right-hand side of (4), the increments of $\{\tilde{X}(s)\}_{s \geq 0}$ are stationary and independent. □

REMARK 1. Another idea for proving Theorem 1 is hiding in the proof of a similar result obtained by Le Cam [Proposition 1 in Le Cam (1979)].

The basic definitions regarding efficient sequential estimation can be found in Stefanov (1985) and Basawa and Prakasa Rao (1980). Since, for the most part, the preliminaries in Stefanov (1985) hold true in the case of processes satisfying conditions (i)–(iii), we shall not repeat them. Using well-known arguments [see DeGroot (1959)] it follows that if the stopping time τ is efficient, then

$$\sum_{i=1}^{n} a_i X_i(\tau) + a_{n+1} T(\tau) = a_{n+2},$$

P_θ-a.s. for each θ from some n-dimensional interval I, $I \subset \Theta$, where

$a_1, a_2, \ldots, a_{n+1}$ are real numbers, not all zero, and $a_{n+2} > 0$. In other words, the measure generated by the sufficient statistic in an efficient stopping time is supported by some hyperplane A:

$$A := \left\{(x_1, \ldots, x_{n+1}) \in R^{n+1}: \sum_{i=1}^{n+1} a_i x_i = a_{n+2}\right\}.$$

We shall consider only stopping times, which are moments of first attainment of such hyperplanes. Let

$$\tau_A := \inf\left\{t: \sum_{i=1}^{n} a_i X_i(t) + a_{n+1} T(t) = a_{n+2}\right\},$$

$$\eta_s := \inf\left\{t: \sum_{i=1}^{n} a_i X_i(t) + a_{n+1} T(t) \geq s\right\}, \qquad s > 0,$$

$$\Theta_{a_1 \cdots a_{n+1}} := \left\{\theta \in \Theta: -\sum_{i=1}^{n} a_i \, \partial\varphi/\partial\theta_i + a_{n+1} > 0\right\}.$$

DEFINITION. We say that the hyperplane A can "be passed" by the process $\{X(t), T(t)\}_{t \geq 0}$ if, for each $\theta \in \Theta_{a_1 \cdots a_{n+1}}$,

$$P_\theta\left\{\sum_{i=1}^{n} a_i X_i(\eta_{a_{n+2}}) + a_{n+1} T(\eta_{a_{n+2}}) > a_{n+2}\right\} > 0.$$

Let

$$\hat{\tau}_A := \inf\left\{s: \sum_{i=1}^{n} a_i \tilde{X}_i(s) + a_{n+1} s = a_{n+2}\right\}.$$

Taking into account condition (iii), we see that the hyperplane A can "be passed" by $\{X(t), T(t)\}_{t \geq 0}$ if and only if A can "be passed" by $\{\tilde{X}(s), s\}_{s \geq 0}$. Also

$$\tilde{X}_i(\hat{\tau}_A) = X_i(\tau_A), \qquad \hat{\tau}_A = T(\tau_A). \tag{5}$$

Thus, since $\{\tilde{X}(s)\}_{s \geq 0}$ has stationary independent increments, in view of Stefanov (1985) and (5), we have

$$E_\theta X_i^2(\tau_A) < +\infty, \qquad E_\theta T^2(\tau_A) < +\infty, \qquad i = 1, 2, \ldots, n, \qquad \theta \in \Theta_{a_1 \cdots a_{n+1}},$$

in the case when the hyperplane A cannot "be passed." This implies that Cramér–Rao regularity conditions are fulfilled. The rest of the proof of the following theorem works in the same way as that of Theorem 1 in Stefanov (1985).

THEOREM 2. *A stopping time is efficient if and only if it represents a moment of first attainment of a hyperplane which cannot "be passed." Every* τ_A, *where* A *is a hyperplane that cannot "be passed," is efficient for* $\Theta_{a_1 \cdots a_{n+1}}$ *and*

$\Theta_{a_1 \cdots a_{n+1}}$ *is the largest subset of* Θ *for which* τ_A *is efficient. The real function* $g(X(\tau_A), T(\tau_A))$ *(A cannot "be passed") is an efficient estimator if and only if* $g(X(\tau_A), T(\tau_A))$ *is a linear function of* $X_1(\tau_A), \ldots, X_n(\tau_A)$, $T(\tau_A)$. *All efficiently estimable functions are of the form*

$$E_\theta\left(\sum_{i=1}^{n} c_i X_i(\tau_A) + c_{n+1}T(\tau_A) + c_{n+2}\right)$$

$$= \frac{a_{n+2}(\sum_{i=1}^{n} - c_i\, \partial\varphi/\partial\theta_i + c_{n+1})}{a_{n+1} - \sum_{i=1}^{n} a_i\, \partial\varphi/\partial\theta_i} + c_{n+2},$$

where $c_1, c_2, \ldots, c_{n+2}$ *are arbitrary real numbers and* $a_1, a_2, \ldots, a_{n+2}$ *are those that determine A.*

NOTE. Theorem 1 also is true without the assumption of the Markov property for the process considered. Actually, if instead of the lemma we apply the equality $E(Z_{t_m}/Z_{t_s}|\mathscr{F}_{t_s}) = 1$, $m > s$, to the martingale $(Z_{t_m}, \mathscr{F}_{t_m})_{m \geq 0}$, $Z_{t_m} := dP_{\theta, t_m}/dP_{\theta^0, t_m}$, we can obtain (4).

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