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ESTIMATION OF A UNIMODAL DISTRIBUTION FUNCTION

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This paper deals with the problem of efficiently estimating (asymptotically minimax) a distribution function when essentially nothing is known about it except that it is unimodal.

The sample distribution function F_n is shown to be asymptotically minimax among the family $\mathscr{E}$ of all unimodal distribution functions. Since F_n does not belong to this family, estimators belonging to this family are constructed and are shown to be asymptotically minimax relative to the collection of subfamilies of $\mathscr{E}$.

1. Introduction. In their pioneering paper, Dvoretzky, Kiefer, and Wolfowitz (1956) proved that the sample distribution function F_n is asymptotically minimax (a.m.) in the collection of all continuous distribution functions (d.f.'s). After 20 years, Kiefer and Wolfowitz (1976), motivated by reliability theory (see Barlow et al. (1972)), reopened the problem and proved that the sample d.f. is still a.m. either in the class of all concave d.f.'s or in the class of all convex d.f.'s. Furthermore, in the same paper, by using Marshall's lemma (1970) they immediately got that C_n (the least concave majorant or the greatest convex minorant of F_n), which is concave (convex) and hence suitable to be used as an estimator, is also a.m. for estimating F. In the same paper, Kiefer and Wolfowitz noted some interesting open problems which are related to reliability theory. Two of them are estimating increasing (decreasing) failure rate distributions and estimating unimodal distributions. The first problem was later considered by Millar (1979); he showed that the sample d.f. is still a.m. among the class of all increasing (decreasing) failure rate distribution functions. Wang (1982) showed that under some additional assumptions it is possible to find an estimator C_n which is a.m. such that C_n itself is in the class of increasing failure rate distributions. The present paper considers the second problem; i.e., estimating a unimodal distribution function. In the next section, the author gives the definition of a unimodal distribution function and proves that the sample d.f. F_n is still a.m. among the family $\mathscr{E}$ of all unimodal distribution functions (Theorem 2.1). Since F_n does not belong to this family $\mathscr{E}$, estimators $(\hat{F}_n)$ belonging to this family are constructed and are shown to be $\sqrt{n}$-close (in supremum norm) to the sample d.f. uniformly among the subfamily $\mathscr{E}^*(\delta_0, M, k)$ of $\mathscr{E}$ (see (2.4)). A slightly weaker concept "a.m. relative to a family" is defined (see (2.5)), and the estimator $\hat{F}_n$ (as well as F_n) is proved to be a.m. relative to the family $\{\mathscr{E}^*(\delta_0, M, k)\}$ (Theorem 2.2). Section 2 contains our main results. All the proofs are given in Section 3.

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2. Main results. A function f is unimodal at θ if and only if f is nondecreasing at x for $x \le \theta$ and f is nonincreasing at x for $x \ge \theta$. We consider the collection $\mathscr{E}$ as follows:

$$\mathscr{E} = \{F(x);\ F(x) \text{ is an absolutely continuous d.f. with a unimodal density function } f(x)\}.$$

Let B denote the collection of all cumulative distribution functions on the real line. In this paper, we consider the loss function for a sample size n as L_n: $B \times B \to R^+ = [0, \infty)$ with $L_n(F, G) = l(n^{1/2}(F - G))$, where l is subconvex with the properties that $El(n^{1/2}(F_n - F))$ converges to $El(W^0(F))$, and $W^0(F)$ is the Brownian bridge process composed with F. These assumptions are essentially the same as the ones used by Millar (1979), and also cover the classical loss functions such as Kolmogorov distance and von Mises distance used by Kiefer and Wolfowitz (1956, 1976).

An estimator ϕ_n of F is a.m. in $\mathscr{E}$ if

$$\lim_{n\to\infty} \frac{\sup_{F\in\mathscr{E}} E_F l\big(n^{1/2}(\phi_n - F)\big)}{\inf_b \sup_{F\in\mathscr{E}} E_F\big\{\int l\big(n^{1/2}(y - F)\big) b\big(x_{(n)}, dy\big)\big\}} = 1, \tag{2.1}$$

where $x_{(n)}$ denotes $(x_1, x_2, \dots, x_n)$ and b runs over all randomized procedures.

One can use Millar's (1979) sufficient conditions to prove the following theorem.

THEOREM 2.1. *Let L_n be described as above. Then the sample d.f. F_n is a.m. (in the sense of (2.1)) among the collection $\mathscr{E}$.*

The proof of this theorem is deferred and will be given in the next section. Since the sample d.f. may not belong to $\mathscr{E}$, it is not a proper estimator to use in some situations. Therefore, we are going to construct some estimators $\hat{F}_n$ (modified by F_n) which belong to $\mathscr{E}$ and are close to F_n. The constructions involve the estimation of the mode. The problems of estimating a mode have been studied by Chernoff (1964), Grenander (1965), and Venter (1967). The following proposition is proved in Venter (1967). The rates of convergence have been shown to be the best possible (see Hasminskii (1974)).

PROPOSITION 1 (Venter, 1967). *Suppose $f(x)$ has a unique mode at θ. Let $\delta > 0$ and write*

$$\alpha_1(\delta) = \min\{f(x);\ \theta - \delta \le x \le \theta + \delta\},$$
$$\alpha_2(\delta) = \max\{f(x);\ x \le \theta - 2\delta, \theta + 2\delta \le x\},$$
$$\alpha(\delta) = \alpha_1(\delta)/\alpha_2(\delta).$$

Suppose the following condition holds:

$$\text{For all } \delta \text{ small enough} \quad \alpha(\delta) \ge 1 + \rho\delta^k, \quad \text{where } \rho \text{ and } k \text{ are positive constants.} \tag{2.2}$$

Then one can find proper estimators $\hat{\theta}_n$, such that $\hat{\theta}_n = \theta + o(\delta_n)$ w.p. 1, where

$$\delta_n = n^{-1/(1+2k)}(\log n)^{1/k} \quad \text{if } k \geq \tfrac{1}{2} \tag{2.3}$$
$$= n^{-1/2}(\log n)^{1/k} \quad \text{if } k < \tfrac{1}{2}.$$

Note that the speed of convergence of $\hat{\theta}_n$ to θ depends on the knowledge of smoothness of f near θ. Consider the following subcollections of $\mathscr{E}$:

Let δ_0 be a small positive number, and let K, M be two positive constants. Define

$$\mathscr{E}^*(\delta_0, M, K) = \{F;\ F \in \mathscr{E} \text{ and there exists a } \rho^* \leq M \text{ such that} \tag{2.4}$$
$$1 + \rho^*\delta^k \geq \alpha(\delta) \geq 1 + \rho\delta^k \text{ for all } \delta \leq \delta_0\}.$$

It follows from the results in Venter (1967) that among the subcollection $\mathscr{E}^*(\delta_0, M, K)$, $\hat{\theta}_n$ has the property that the speed of convergence of $\hat{\theta}_n$ to θ is given by (2.3) uniformly in $\mathscr{E}^*(\delta_0, M, K)$.

Consider the estimator $\hat{F}_n(x)$ of $F(x)$ as follows:

Let $\hat{\hat{F}}_n$ be constructed as the least concave majorant (LCM) of $F_n(x)$ on $x \geq \hat{\theta}_n$ and the greatest convex minorant (GCM) on $x \leq \hat{\theta}_n$. It is easy to construct a modified version, say $\hat{F}_n$, of $\hat{\hat{F}}_n$ such that $\|\hat{F}_n - \hat{\hat{F}}_n\| \leq 1/n$ w.p. 1, and $\hat{F}_n$ is in $\mathscr{E}$ and has $\hat{\theta}_n$ as its unique mode.

The following theorem tells us that the difference $n^{1/2}\|\hat{F} - F\|_\infty$ is essentially no bigger than $n^{1/2}\|F_n - F\|_\infty$ in each subcollection $\mathscr{E}^*(\delta_0, M, K)$, and hence yields a slightly weaker a.m. result as follows:

An estimator ϕ_n is a.m. relative to the family $\{\mathscr{E}^*(\delta_0, M, K);\ \delta_0, M, K > 0\}$ if

$$\sup_{(\delta_0, M, K)} \lim_{n\to\infty} \frac{\sup_{F\in\mathscr{E}^*(\delta_0, M, K)} E_F l(n^{1/2}(\delta_n - F))}{\inf_b \sup_{F\in\mathscr{E}^*(\delta_0, M, K)} E_F\{\int l[n^{1/2}(y - F)]\, b(x_{(n)}, dy)\}} = 1. \tag{2.5}$$

THEOREM 2.2. *For every $\mathscr{E}^*(\delta_0, M, K)$ described as above,*

$$\sqrt{n}\,\|\hat{F}_n - F\|_\infty \leq \sqrt{n}\,\|F_n - F\|_\infty + o_p(1) \tag{2.6}$$

uniformly in $F \in \mathscr{E}^*(\delta_0, M, K)$. Furthermore, $\hat{F}_n$ is a.m. relative to the family $\{\mathscr{E}^*(\delta_0, M, K)\}$.

REMARK 1. The first part of Theorem 2.2 does not imply that $\hat{F}_n$ is a.m. among $\mathscr{E}^*(\delta_0, M, K)$ since the sample d.f. F_n may not be a.m. among $\mathscr{E}^*(\delta_0, M, K)$.

REMARK 2. From the proof (given the the next section) of the second part of Theorem 2.2, one can show that (2.5) holds with fixed $K = 2$.

REMARK 3. Note that $\mathscr{E}^*(\delta_1, M, K) \subset \mathscr{E}^*(\delta_2, M, K)$ for $\delta_2 < \delta_1$, and for every fixed M and K. Let $\mathscr{E}^*(M, K) = \bigcup_{\delta_0 > 0}\mathscr{E}^*(\delta_0, M, K)$. It can be shown (see the proof of Theorem 2.2) that F_n is a.m. among $\mathscr{E}^*(M,2)$, for some $M > 0$, but it is not clear at this moment whether F_n is a.m. among $\mathscr{E}^*(M, K)$ for $K \neq 2$.

Before closing this section, we give an example.
Suppose f satisfies

(2.7) $$f(x) = \gamma_0 - \gamma(x-\theta)^2 + o(|x-\theta|^2) \quad \text{as } x \to \theta \text{ for } \gamma_0, \gamma > 0.$$

There exists a $\delta_0 \le \gamma_0^{1/2}/10\gamma^{1/2}$ such that the term

$$|o(|x-\theta|^2)| \le (X-\theta)^2 \min(\gamma/10, \gamma_0/10) \quad \text{if } |x-\theta| \le \delta_0.$$

Therefore, if $|\Delta| \le \delta_0$, one can write

$$\frac{f(\theta+\Delta)}{f(\theta+2\Delta)} = \frac{\gamma_0 - (\gamma - o(\Delta^2)/\Delta^2)\Delta^2}{\gamma_0 - 4(\gamma - o(\Delta^2)/\Delta^2)\Delta^2} = 1 + \frac{3(\gamma - o(\Delta^2)/\Delta^2)\Delta^2}{\gamma_0 - 4(\gamma - o(\Delta^2)/\Delta^2)\Delta^2}$$

$$\ge 1 + \frac{3(\gamma - o(\Delta^2)/\Delta^2)\Delta^2}{\gamma_0} \ge 1 + \frac{3(\frac{9}{10}\gamma)\Delta^2}{\gamma_0} = 1 + \left(\frac{27}{10}\right)\frac{\gamma}{\gamma_0}\Delta^2.$$

On the other hand,

$$\frac{f(\theta+\Delta)}{f(\theta+2\Delta)} \le 1 + \frac{\frac{33}{10}\gamma\Delta^2}{\gamma_0 - \frac{36}{1000}\gamma_0} = 1 + \frac{3300}{964}\frac{\gamma}{\gamma_0}\Delta^2.$$

Therefore, the corresponding d.f. $F(x) \in \mathscr{E}^*(\gamma_0^{1/2}/10\gamma^{1/2}, M, 2)$ for any $M \ge 3300/964\ (\gamma/\gamma_0)$.

3. Proofs.

Proof of Theorem 2.1. Take $F_0(x) = \Phi(x)$ to be the standard normal d.f. with density $\phi(x) = (1/\sqrt{2\pi})e^{-x^2/2}$. It is clear that $\Phi \in \mathscr{E}$. It suffices to show that $\mathscr{E}$ is radially dense at Φ as Millar (1979) pointed out.

Consider the densities of the form

$$\phi(x; n^{-1/2}h(x)) = \phi(x)(1 + n^{-1/2}h(x)).$$

$\phi(x; n^{-1/2}h(x))$ is a density if $\int_{-\infty}^{\infty}\phi(x)h(x)\,dx = 0$ and $\sup_x|n^{-1/2}h(x)| \le \frac{1}{2}$. To assure $\phi(x; n^{-1/2}h)$ in $\mathscr{E}$, consider

$$H_n = \Big\{h; \int_{-\infty}^{\infty} h(x)\phi(x)\,dx = 0, \int_{-\infty}^{\infty} h^2(x)\phi(x)\,dx < \infty, \sup_x |n^{-1/2}h(x)| \le \tfrac{1}{3},$$
$$h(x) = 0 \text{ if } x \in [-\varepsilon_n, \varepsilon_n], \text{ and } |h'(x)| \le \tfrac{1}{3}n^{1/2}\varepsilon_n \text{ if } x \notin [-\varepsilon_n, \varepsilon_n]\Big\},$$

where $\{\varepsilon_n\}$ is a positive sequence tending to zero with $n^{1/2}\varepsilon_n \to \infty$ as $n \to \infty$. Clearly, $\bigcup_{n=1}^{\infty} H_n$ is dense in $H(\Phi)$ where $H(\Phi)$ is defined in Millar (1979) as

$$H(\Phi) = \Big\{h; \int_{-\infty}^{\infty} h(x)\phi(x)\,dx = 0 \text{ and } \int_{-\infty}^{\infty} h^2(x)\phi(x)\,dx < \infty\Big\}.$$

Direct calculation of $\phi'(x; n^{-1/2}h)$ shows that $\phi(x; n^{-1/2}h)$ is unimodal, and hence in $\mathscr{E}$ when $h \in H_n$. This shows that Φ is a radial cluster point, and the theorem thus follows. □

We need some lemmas to prove Theorem 2.2. For any $F \in \mathscr{E}^*(\delta_0, M, K)$, let f denote the density of F. Let $\mathbf{f}(\theta \pm \delta_n) = \inf\{f(\theta + x);\ |x| \le \delta_n\}$ for $\delta_n \le \delta_0$.

LEMMA 1. *Assume $F \in \mathscr{E}^*(\delta_0, M, K)$. Then*

$$\Delta_n = \int_{\theta - \delta_n}^{\theta + \delta_n} f(x)\,dx - 2\delta_n \mathbf{f}(\theta \pm \delta_n) = o(n^{-1/2}) \tag{3.1}$$

uniformly in $\mathscr{E}^*(\delta_0, M, K)$, where δ_n is defined as in (2.3).

PROOF. First note that $2\delta_n \mathbf{f}(\theta \pm \delta_n) \le 1$; therefore, $\mathbf{f}(\theta \pm \delta_n) \le 1/2\delta_n$. (This is true for all F in $\mathscr{E}^*(\delta_0, M, K)$.) By the definition of $\mathscr{E}^*(\delta_0, M, K)$, one can write

$$\mathbf{f}(\theta \pm \delta_n/2^p) \le \mathbf{f}(\theta \pm \delta_n) \prod_{j=1}^{p} \left[1 + M(\delta_n/2^j)^k\right]. \tag{3.2}$$

Taking the log, we obtain

$$\begin{aligned} \log \prod_{j=1}^{p} \left[1 + M(\delta_n/2^j)^k\right] &\le M\delta_n^k \sum_{j=1}^{p} \left(\frac{1}{2}\right)^{jk} + \frac{M^2\delta_n^{2k}}{2} \sum_{j=1}^{p} \left(\frac{1}{2}\right)^{2jk} \\ &= \varepsilon_{n,p} \quad \text{(say)}, \end{aligned} \tag{3.3}$$

since $\log(1 + x) \le x + x^2/2$ if $x > 0$. Therefore,

$$\prod_{j=1}^{p} \left[1 + M(\delta_n/2^j)^k\right] \le e^{\varepsilon_{n,p}} \le 1 + \varepsilon_{n,p}$$

($\varepsilon_{n,p} < 1$ if δ_n is small enough). We obtain

$$\mathbf{f}(\theta \pm \delta_n/2^p) \le \mathbf{f}(\theta \pm \delta_n)\left(1 + L_{n,p}\delta_n^k\right), \tag{3.4}$$

where

$$L_{n,p} = M \sum_{j=1}^{p} \left(\frac{1}{2}\right)^{jk} + \frac{M^2\delta_n^k}{2} \sum_{j=1}^{p} \left(\frac{1}{2}\right)^{2jk} \to L_n < \infty \quad \text{as } p \to \infty.$$

This together with the fact that $\mathbf{f}(\theta \pm \delta_n) \le 1/2\delta_n$ implies $f(\theta) \le L_n/2\delta_n$ for $\delta_n \le \delta_0$. This shows that the densities of $\mathscr{E}^*(\delta_0, M, K)$ are uniformly bounded.

From (3.4), we have

$$\begin{aligned} \mathbf{f}(\theta \pm \delta_n/2^p) - \mathbf{f}(\theta \pm \delta_n) &\le \mathbf{f}(\theta \pm \delta_n) L_{n,p}\delta_n^k \\ &\le f(\theta) L_n \delta_n^k = O(1)\delta_n^k \end{aligned} \tag{3.5}$$

uniformly in $\mathscr{E}^*(\delta_0, M, K)$.

From Proposition 1 and (3.5),

$$\begin{aligned} \Delta_n &\le O(1)\delta_n^k o(\delta_n) \le o\left(\delta_n^{k+1}\right) \\ &= \begin{cases} n^{-(k+1)/(1+2k)}(\log n)^{1/k} & \text{if } k \ge \frac{1}{2} \\ n^{-(k+1)/2}(\log n)^{k+1/k} & \text{if } k < \frac{1}{2} \end{cases} \\ &= o(n^{-1/2}). \end{aligned}$$

□

LEMMA 2. *Under the assumptions of Lemma* 1, *let* f_n *be any unimodal density function which is identical with* $f(x)$ *outside* $I_n = (\theta - \delta_n, \theta + \delta_n)$, *and let* F_n^* *denote the distribution function of* f_n. *Then*

$$\sup_x |F_n^*(x) - F(x)| = o(n^{-1/2}) \tag{3.6}$$

uniformly in $\mathscr{E}^*(\delta_0, M, K)$.

PROOF. It suffices to show that

$$\sup_{x \in I_n} |F_n^*(x) - F(x)| = o(n^{-1/2})$$

uniformly in $\mathscr{E}^*(\delta_0, M, K)$. F_n^* unimodal implies that $f_n(x) \ge \mathbf{f}(\theta \pm \delta_n)$ if $x \in I_n$. Since F_n^* is a d.f.,

$$\int_{\theta-\delta_n}^{\theta+\delta_n} f_n(t)\,dt - \mathbf{f}(\theta \pm \delta_n)2\delta_n = \int_{\theta-\delta_n}^{\theta+\delta_n} f(t)\,dt - 2\delta_n \mathbf{f}(\theta \pm \delta_n) \le \Delta_n;$$

Δ_n is defined as in (3.1). Therefore, for $x \in I_n$,

$$\begin{aligned} |F_n^*(x) - F(x)| &= \left| \int_{\theta-\delta_n}^{x} f_n(t)\,dt - \int_{\theta-\delta_n}^{x} f(t)\,dt \right| \\ &\le \left| \int_{\theta-\delta_n}^{x} f_n(t)\,dt - (x - \theta + \delta_n)\mathbf{f}(\theta \pm \delta_n) \right| \\ &\quad + \left| \int_{\theta-\delta_n}^{x} f(t) - (x - \theta + \delta_n)\mathbf{f}(\theta \pm \delta_n) \right| \\ &\le 2\Delta_n. \end{aligned}$$

The lemma thus follows from Lemma 1. □

LEMMA 3. *Suppose* $\hat{\theta}_n \in I_n$. *Let* f_n, F_n^* *be as in Lemma* 2 *with the mode of* f_n *at* $\hat{\theta}_n$. *Then*

$$\sup_x |\hat{F}_n(x) - F_n^*(x)| \le \sup_x |F_n(x) - F_n^*(x)|. \tag{3.7}$$

PROOF. Recall that $\hat{F}_n$ is constructed in Section 2. Since $\hat{F}_n$, F_n^* are both convex if $x \le \hat{\theta}_n$ and both concave if $x \ge \hat{\theta}_n$, the lemma follows directly from Marshall's lemma (1970). □

PROOF OF THEOREM 2.2. From (3.6) and (3.7),

$$\begin{aligned} \sup_x |\hat{F}_n(x) - F(x)| &\le \sup_x |\hat{F}_n(x) - F_n^*(x)| + \sup_x |F_n^*(x) - F(x)| \\ &\le \sup_x |F_n(x) - F_n^*(x)| + o_p(n^{-1/2}) \\ &\le \sup_x |F_n(x) - F(x)| + o_p(n^{-1/2}) \end{aligned}$$

uniformly in $\mathscr{E}^*(\delta_0, M, K)$.

The first part of the theorem follows immediately from the above fact.

To show $\hat{F}_n$ is a.m. relative to the family $\{\mathscr{E}^*(\delta_0, M, K)\}$, it suffices to show that F_n is a.m. relative to the family $\{\mathscr{E}^*(\delta_0, M, K)\}$. If we can show that F_n is a.m. (in the sense of (2.1)) among the collection $\mathscr{E}^*(M, 2) = \bigcup_{\delta_0 > 0} \mathscr{E}^*(\delta_0, M, 2)$ for some $M > 0$, then this, together with the fact that $\lim_{\delta_0 \searrow 0} \mathscr{E}^*(\delta_0, M, 2) = \mathscr{E}^*(M, 2)$, will imply

$$(3.8)\quad \sup_{\delta_0} \lim_{n\to\infty} \frac{\sup_{F \in \mathscr{E}^*(\delta_0, M, 2)} E_F l\left(n^{1/2}(\phi - F)\right)}{\inf_b \sup_{F \in \mathscr{E}^*(\delta_0, M, 2)} E_F\left\{\int l\left[n^{1/2}(y - F)\right] b\left(x_{(n)}, dy\right)\right\}} = 1.$$

So, it suffices to show F_n is a.m. among the collection $\mathscr{E}^*(M, 2)$.

To see this, we claim that Φ, the standard normal d.f. is again a radial cluster point in the family $\mathscr{E}^*(M, 2)$. Since $\phi(x) = (1/\sqrt{2\pi})e^{-x^2/2}$ satisfies (2.7) with $\gamma_0 = \gamma = 1/\sqrt{2\pi}$, we have $\Phi \in \mathscr{E}^*(\delta_0, M, 2) \subset \mathscr{E}^*(M, 2)$ for some proper δ_0 and M. For any $h \in H_n$ (defined in the beginning of this section), it is easy to check that $\phi(x; n^{-1/2}h) \in \mathscr{E}^*(\delta^*, M, 2)$ for some $\delta^* > 0$. Since $\bigcup_{n=1}^{\infty} H_n$ is dense in $H(\Phi)$, this shows that Φ is a radial cluster point in $\mathscr{E}^*(M, 2)$, and the theorem thus follows. □

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