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ORTHOGONALITY OF FACTORIAL EFFECTS

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A necessary and sufficient condition is given for a specified factorial effect to be orthogonal to every other factorial effect, after adjustment is made for blocks. The results are extended to the case of regular disconnected designs. The structure of a generalized inverse of the intrablock matrix is investigated when certain pairs of factorial spaces are orthogonal. A useful class of designs exhibiting partial orthogonal factorial structure is identified and examples are given.

1. Introduction. When a factorial experiment is arranged as an incomplete block design, some degree of nonorthogonality is necessarily introduced into the analysis. For ease of interpretation, therefore, it is frequently desirable to use a design which admits an orthogonal analysis of the main effects and interactions (after adjusting for block effects). Such designs are said to have "orthogonal factorial structure." A set of sufficient conditions for a design to have orthogonal factorial structure was given by Cotter, John, and Smith (1973), but there exist many designs which are orthogonal yet do not satisfy these conditions. Mukerjee (1979) gave a set of necessary and sufficient conditions for orthogonal factorial structure which are useful for constructing classes of such designs [see Mukerjee (1981)]. As pointed out by John and Smith (1972) and Mukerjee (1979), several of the well-known classes of designs (such as group divisible, generalized cyclic) exhibit orthogonal factorial structure for particular sets of factor levels. It is frequently the case, however, that in factorial experimentation high-order interactions are assumed to be negligible and, in such cases, a design with complete orthogonal factorial structure is unnecessary. All that is required is a design which admits an orthogonal partition of contrasts belonging to the low-order factorial effects. Mukerjee (1980) considered this problem and adapted his 1979 conditions to give a set of necessary and sufficient conditions for the orthogonality of all interaction effects of order less than or equal to a fixed number, t.

Unfortunately, the conditions of Mukerjee (1979, 1980) give no information on the orthogonality of any specified pair of factorial spaces. Thus if the conditions are violated, it is not known which of the factorial spaces are orthogonal and which are nonorthogonal (see Example 1). The purpose of this paper is to show that part of this information is, in fact, available in the course of checking Mukerjee's conditions (see Section 3), and to show that useful designs exhibiting partial orthogonal factorial structure are readily available (see Section 4). The results are extended to regular disconnected designs in Section 5. In addition, in Section 3 the structure of a generalized inverse of the intrablock matrix is investigated when certain pairs of factorial spaces are orthogonal.

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2. Notation and preliminaries. Consider a factorial experiment with p factors $F_1, F_2, \ldots, F_p$ where the jth factor has m_j levels and $v = \prod_{j=1}^{p} m_j$. The v treatment combinations are written as p-tuples, $a = (a_1, a_2, \ldots, a_p)$ where $0 \le a_j \le m_j - 1$, $j = 1, \ldots, p$. We use the convention of writing the treatment combinations in lexicographical order (i.e., ascending numerical order when viewed as p-digit numbers). We represent a generalized interaction by α^x, where $x = (x_1, x_2, \ldots, x_p)$ and $x_j = 1$ if factor F_j is present in the interaction, and $x_j = 0$ otherwise. For brevity, we use the term "interaction" to mean "main effect or interaction." Let Φ^* be the lexicographically ordered set of all binary vectors $x = (x_1, \ldots, x_p)$ and denote the ith element of Φ^* by ϕ_i, $i = 1, \ldots, 2^p$. Then α^{ϕ_1} denotes the general mean and is nonestimable in any incomplete block design. Let $\Phi = \{\phi_2, \phi_3, \ldots, \phi_n\}$ where $n = 2^p$. For $x \neq y \in \Phi$, α^x and α^y represent different generalized interactions and hence define different factorial spaces. The factorial space corresponding to α^x, can be represented by a vector space V_x, of dimension $\prod_{j=1}^{p}(m_j - 1)^{x_j}$. A set of basis vectors for V_x is given by a set of orthogonal contrasts in the treatment parameters corresponding to the interaction α^x. We shall be interested in the independence of the estimators of contrasts in the treatment parameters, having adjusted for block effects, where the contrasts belong to different vector spaces V_x and V_y, $x \neq y \in \Phi$.

Let the factorial experiment be arranged in b blocks where the jth block is of size $k_j \le v$, and the ith treatment combination is observed a total of r_i times in the design ($i = 1, \ldots, v$; $j = 1, \ldots, b$).

The usual intrablock model will be assumed, namely

$$y_{ij} = \mu + \tau_i + \beta_j + e_{ij} \qquad (i = 1, \ldots, v;\ j = 1, \ldots, b),$$

where y_{ij} is the yield of the plot in the jth block which received the ith treatment combination, τ_i is the effect of the ith treatment combination, β_j is the effect of the jth block, μ is a constant, and e_{ij} are independent normal random variables with zero means and homogeneous variances σ^2.

The reduced normal equations for estimating the treatment effects having adjusted for blocks are

$$A\hat{\tau} = Q,$$

where

$$Q = T - Nk^{-\delta}B, \qquad A = r^{\delta} - Nk^{-\delta}N', \tag{2.1}$$

and where T and B are vectors of treatment totals and block totals respectively, N is the incidence matrix for the design, r^{δ} and k^{δ} are diagonal matrices of treatment replications and block sizes respectively, $\hat{\tau}$ is the intrablock estimator of $\tau = (\tau_1, \tau_2, \ldots, \tau_v)'$, and $'$ denotes transpose.

A solution to the normal equations is given by

$$\hat{\tau} = \Omega Q,$$

where Ω is any generalized inverse of A, that is $A\Omega A = A$.

Apart from Section 5 where disconnected designs are considered, it will be assumed that $\text{rank}(A) = v - 1$, so that all contrasts in the treatment parameters are estimable. Let C^x be a $q \times v$ matrix where $q \ge \prod(m_j - 1)^{x_j}$, such that the rows of C^x form a set of contrasts spanning the vector space V_x, $x \in \Phi$. Symbolically we may write $\alpha^x = C^x\tau$. It was shown by Kurkjian and Zelen (1962) that one such set of contrasts is given by $C^x = v^{-1}M^x$, where

$$M^x = M_1^{x_1} \otimes M_2^{x_2} \otimes \cdots \otimes M_p^{x_p} \tag{2.2}$$

and $\otimes$ denotes the Kronecker product of matrices, and

$$M_j^{x_j} = \begin{cases} \left(m_j I_{m_j} - J_{m_j}\right) & \text{if } x_j = 1, \\ m_j^{1/2} e'_{m_j} & \text{if } x_j = 0, \end{cases}$$

where e is a column vector of m_j elements each equal to $m_j^{-1/2}$, $J_{m_j} = e_{m_j} e'_{m_j}$, and I_{m_j} is an $m_j \times m_j$ identity matrix.

Any other set of contrasts spanning V_x may be written as

$$C^x = C_1^{x_1} \otimes C_2^{x_2} \otimes \cdots \otimes C_p^{x_p}, \tag{2.3}$$

where $C_j^{x_j} = R_j^{x_j} M_j^{x_j}$ with $R_j^{x_j} = m_j$ for $x_j = 0$, and for $x_j = 1$, $R_j^{x_j}$ is any $s_j \times m_j$ matrix of rank $(m_j - 1)$ where $s_j \geq m_j - 1$, $j = 1, \ldots, p$. Thus C^x may be expressed as $C^x = R^x M^x$ where $R^x = R_1^{x_1} \otimes R_2^{x_2} \otimes \cdots \otimes R_p^{x_p}$.

Without loss of generality, we select C^x so that $C^x C^{x'} = I$. Hence the rows of C^x form an orthonormal basis for V_x, and $s_j = (m_j - 1)$ when $x_j = 1$ for $j = 1, \ldots, p$. Note that for $x \neq y \in \Phi^*$, $C^x C^{y'} = 0$,

The covariance between the minimum variance unbiased estimators of the parametric functions $C^x\tau$ and $C^y\tau$, after adjusting for block effects, is given by $\operatorname{Cov}(C^x\hat{\tau}, C^y\hat{\tau}) = C^x \Omega C^{y'} \sigma^2$, for any generalized inverse, Ω, of A. A design has orthogonal factorial structure if and only if $C^x \Omega C^{y'} = 0$ for all $x \neq y \in \Phi$ [see, for example, Cotter, John, and Smith (1973)].

3. Orthogonality of factorial spaces. In Theorem 1 a necessary and sufficient condition is given for a specified interaction, α^x, to be orthogonal to all other interactions, α^y. Thus it follows that, in the course of checking the conditions given by Mukerjee [(1979), Theorem 3.3 and (1980), Theorem 2.2] for orthogonal factorial structure, information is in fact obtainable on the pairwise orthogonality of factorial spaces.

Let C^x, V_x, A, and $\Phi = [\phi_2, \ldots, \phi_n]$, $n = 2^p$, be defined as in Section 2. The proof of Theorem 1 requires the following lemma.

LEMMA 1. *Let W and Q be two real symmetric $v \times v$ matrices, and let Q^+ be the Moore–Penrose inverse of Q. If $QW = WQ$, then* (i) $QQ^+W = WQQ^+$ *and* (ii) $Q^+W = WQ^+$.

PROOF. Using the properties of the Moore–Penrose inverse, and the fact that $QW = WQ$,

$$\text{(i)}\quad \begin{aligned} QQ^+W &= Q^+QW = Q^+WQ = Q^+WQQ^+Q = Q^+QWQQ^+ \\ &= QQ^+QWQ^+ = QWQ^+ = WQQ^+ \end{aligned}$$

$$\text{(ii)}\quad \begin{aligned} WQ^+ &= WQ^+QQ^+ = Q^+QWQ^+, \quad \text{using (i)} \\ &= Q^+WQQ^+ = Q^+QQ^+W, \quad \text{using (i)} \\ &= Q^+W. \end{aligned}$$

□

THEOREM 1. *A necessary and sufficient condition for α^x, $x = \phi_i \in \Phi$, to be orthogonal to α^y, for all $y = \phi_j \in \Phi$, $j \neq i$, is that $C^{x\prime}C^x$ commutes with A.*

PROOF. (i) Necessity. Assume that α^x is orthogonal to all α^y, $x = \phi_i$, $y = \phi_j$, $\phi_i \neq \phi_j \in \Phi$, and let $H^x = [C^{\phi_2\prime}, \dots, C^{\phi_{i-1}\prime}, C^{\phi_{i+1}\prime}, \dots, C^{\phi_n\prime}]'$. Then, following the arguments in the proof of Theorem 3.1 of Mukerjee (1979),

$$A = C^{x\prime}C^x A C^{x\prime}C^x + H^{x\prime}H^x A H^{x\prime}H^x. \tag{3.1}$$

Since $C^x C^{x\prime} = I$ and $C^x H^{x\prime} = 0$, it follows from (3.1) that $C^{x\prime}C^x$ and A commute.

(ii) Sufficiency. Assume that $C^{x\prime}C^x$ and A commute, then from Lemma 1, $C^{x\prime}C^x$ and A^+ commute, hence

$$C^x A^+ H^{x\prime} = C^x C^{x\prime} C^x A^+ H^{x\prime} = C^x A^+ C^{x\prime} C^x H^{x\prime} = 0.$$

Therefore, setting $\Omega = A^+$, the orthogonality of α^x and all α^y, $y \neq x \in \Phi$, follows. □

Note that for all contrast matrices C^x as defined in (2.3), $C^{x\prime}C^x = bM^{x\prime}M^x$ for some constant b, where M^x is defined in (2.2) [see Dean (1978), Lemma 1]. Therefore $C^{x\prime}C^x$ commutes with A if and only if $M^{x\prime}M^x$ commutes with A. Hence if the condition of Theorem 1 is satisfied *for all* $x \in \Phi$ of order $\leq t$, then Theorem 1 implies Theorem 2.2 of Mukerjee (1980), and if $t = p$, then Theorem 3.3 of Mukerjee (1979) follows.

Note also that, since A is symmetric, the condition of Theroem 1 is equivalent to the condition of symmetry of $C^{x\prime}C^x A$, and hence that of $M^{x\prime}M^x A$.

EXAMPLE 1. Consider a design for a $4 \times 2 \times 2$ experiment in 16 blocks of size 6 obtained by adding in turn the treatment combinations (000, 100, 200, 300) to the following four blocks

000	001	100	101	210	311
001	010	101	110	211	300
010	011	110	111	200	301
011	000	111	100	201	310

where addition of two treatment combinations $a_1a_2a_3$ and $b_1b_2b_3$ is defined by

$$c_1c_2c_3 = a_1a_2a_3 + b_1b_2b_3, \quad \text{where } c_i = a_i + b_i \bmod m_i,\ i = 1, 2, 3.$$

The concurrence matrix NN' is block circulant of the form $\{Q_1, Q_2, Q_3, Q_4\}$ where each Q_j is a 4×4 circulant matrix; $Q_1 = \{6, 2, 0, 2\}$; $Q_2 = \{2, 4, 2, 1\}$; $Q_3 = \{0, 2, 4, 2\}$; $Q_4 = \{2, 1, 2, 4\}$. It may be verified that $M^{x\prime}M^x NN'$ (and hence $M^{x\prime}M^x A$, since the design is proper and equireplicate), is symmetric for all x except for $x = (110)$ and $x = (111)$. Hence all pairs of interactions are orthogonal with the possible exception of $(\alpha^{110}, \alpha^{111})$.

The condition of Mukerjee [(1980), Theorem 2.2] does not hold for this example, and therefore could not be used to deduce the orthogonality of all pairs of effects of order less than or equal to 2.

COROLLARY 1. *If α^x is orthogonal to all α^y, $y \neq x \in \Phi$, then the intrablock matrix, A, can be expressed as $A = A_1 + A_2$ where the rows and columns of A_1 belong to V_x, and the rows and columns of A_2 are orthgonal to V_x.*

PROOF. Follows directly from (3.1). □

Corollary 1 relates the orthogonal factorial properties of the design directly to the structure of the intrablock matrix. Theorem 2 shows that any generalized inverse of the intrablock matrix exhibits a similar structure when two factorial spaces are orthogonal even if the conditions of Theorem 1 are not satisfied.

LEMMA 2. *Let P, X, and Q be real nonzero matrices such that the product PXQ exists, then $PXQ = 0$ if and only if $X = X_1 + X_2$ where the columns of X_1 are orthogonal to the rows of P, and the rows of X_2 are orthogonal to the columns of Q.*

PROOF. Follows from Rao and Mitra (1971), Theorem 2.3.2 and the proof of Theorem 2.4.1b. □

THEOREM 2. $\mathrm{Cov}(C^x\hat{\tau}, C^y\hat{\tau}) = 0$ *for a specified pair of interactions α^x and α^y, $x, y \in \Phi$, $x \neq y$, if and only if any generalized inverse, Ω, of the intrablock matrix can be expressed as $\Omega = \Omega_1^x + \Omega_2^y$, where the columns of Ω_1^x are orthogonal to V_x and the rows of Ω_2^y are orthogonal to V_y.*

PROOF. Follows directly from Lemma 2. □

COROLLARY 2. *If α^y represents a main effect (of the first factor without loss of generality), then* $\mathrm{Cov}(C^x\hat{\tau}, C^y\hat{\tau}) = 0$ *for any specified α^x, $x \neq y \in \Phi$ if and only if $C^x\Omega P^{y'}$ has constant rows, where $P^{y'} = I_{m_1} \otimes e_s$ and $s = v/m_1$.*

PROOF. (i) Sufficiency. $C^y = K^y(I_{m_1} \otimes e_s') = K^yP^y$, for some $(m_1 - 1) \times m_1$ orthonormal contrast matrix K^y. Hence if $C^x\Omega P^{y'}$ has constant rows then $C^{x'}\Omega C^{y'} = (C^x\Omega P^{y'})K^{y'} = 0$.

(ii) Necessity. If $C^x\Omega C^{y'} = 0$ then, from Theorem 2, $C^x\Omega = C^x\Omega_2^y$, where the rows of $C^x\Omega_2^y$ are orthogonal to V_y. Hence $C^x\Omega_2^y = e_{m_1}' \otimes B$, where B is some $q \times s$ matrix and $q = \prod(m_j - 1)^{x_j}$. Hence $C^x\Omega P^{y'}$ is of the form $e_{m_1}' \otimes Be_s = e_{m_1}' \otimes b_q$ (where b_q is a vector of length q) as required. □

COROLLARY 3. (a) *If α^y represents a first-order interaction (between the first two factors without loss of generality), then* $\mathrm{Cov}(C^x\hat{\tau}, C^y\hat{\tau}) = 0$ *for any specified α^x, $x \neq y \in \Phi$, if and only if $C^x\Omega P^{y'}$ can be expressed as $[D_1, D_2, \ldots, D_{m_1}]$, where D_ℓ is of dimension $q \times m_2$, $q = \prod(m_j - 1)^{x_j}$, and $D_\ell - D_1$ has constant rows, $\ell = 1, \ldots, m_1$, and where $P^{y'} = I_{m_1} \otimes I_{m_2} \otimes e_t$, $t = v/m_1m_2$.*

(b) *If* (a) *holds and $D_1 = D_2 = \cdots = D_{m_1}$, then α^x is also orthogonal to the main effect of the first factor.*

(c) *If* (a) *holds and D_ℓ has constant rows, $\ell = 1, \ldots, m_1$, then α^x is also orthogonal to the main effect of the second factor.*

PROOF. (a) (i) Sufficiency. $C^y = K^y P^{y'}$ where $K^y = K_1 \otimes K_2$ and K_i is an $(m_i - 1) \times m_i$ orthonormal contrast matrix, $i = 1, 2$. Also, by assumption,

$$
\begin{aligned}
C^x \Omega P^{y'} &= \left(e'_{m_1} \otimes D_1\right) + \left[0, D_2 - D_1, \ldots, D_{m_1} - D_1\right] \\
&= \left(e'_{m_1} \otimes D_1\right) + \left(Q \otimes e'_{m_2}\right).
\end{aligned}
\tag{3.2}
$$

Hence $C^x \Omega C^{y'} = (C^x \Omega P^{y'})(K_1 \otimes K_2) = 0$.

(ii) Necessity. If $C^x \Omega C^{y'} = 0$ then, from Theorem 2, $C^x \Omega = C^x \Omega_2^y$ where the rows of $C^x \Omega_2^y$ are orthogonal to V_y. Hence $C^x \Omega_2^y = (e'_{m_1} \otimes B_1 \otimes B_3) + (B_2 \otimes e'_{m_2} \otimes B_4)$ for some matrices B_1, B_2, B_3, B_4 of dimensions $q \times m_2$, $q \times m_1$, $q \times t$, $q \times t$ respectively, $q = \prod (m_j - 1)^{x_j}$, $t = v / m_1 m_2$. Hence $C^x \Omega P^{y'} = (e'_{m_1} \otimes B_1 \otimes h_1) + (B_2 \otimes e'_{m_2} \otimes h_2)$ where h_1 and h_2 are vectors of length q. Let $B_2 = [b_1, b_2 \ldots b_{m_1}]$ and let $D_\ell = (B_1 \otimes h_1) + (b_\ell \otimes e'_{m_2} \otimes h_2)$, $\ell = 1, \ldots, m_1$, and the result follows.

(b) Assume that (a) holds, and $D_\ell = D_1$ for all $\ell = 2, \ldots, m_1$ then from (3.2), $C^x \Omega P^{y'} = e'_{m_1} \otimes D_1$. Let α^z represent the main effect of the first factor, then $P^{z'} = P^{y'}(I_{m_1} \otimes e_{m_2})$. Hence $C^x \Omega P^{z'} = e'_{m_1} \otimes D_1 e_{m_2}$ which has constant rows. Hence from Corollary 2, α^x and α^z are orthogonal.

(c) Assume that (a) holds and $D_\ell = d_\ell e'_{m_2}$ for some constant d_ℓ, $\ell = 1, \ldots, m_1$. Hence from (3.2)

$$
C^x \Omega P^{y'} = d_1\left(e'_{m_1} \otimes e'_{m_2}\right) + \left[0, d_2 - d_1, \ldots, d_{m_1} - d_1\right] \otimes e'_{m_2}.
$$

Let α^z represent the main effect of the second factor then $P^{z'} = P^{y'}(e_{m_1} \otimes I_{m_2})$. Hence $C^x \Omega P^{y'} = [d_1 + \gamma] e'_{m_2}$, where $\gamma = \Sigma(d_\ell - d_1)$, which has constant rows. Hence from Corollary 2, replacing the first main effect by the second, α^x and α^z are orthogonal. □

EXAMPLE 2. Consider the design of Example 1. We have already shown, using Theorem 1, that all pairs of main effects are orthogonal for a $4 \times 2 \times 2$ experiment. However, this is a convenient example to illustrate the use of Corollaries 2 and 3. To check the orthogonality of α^{010} and α^{110} using Corollary 2, we calculate $C^x \Omega P^{y'}$ where $x = 010$ and $y = 100$. Without loss of generality, choosing $C^x = R^x M^x$ gives

$$
C^x \Omega P^{y'} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

as required.

To check the orthogonality of α^{010} and α^{110} using Corollary 3,

$$
\begin{aligned}
C^x \Omega P^{y'} &= R^x\left(e'_4 \otimes \left(m_2 I_2 - J_2\right) \otimes e'_2\right) \Omega\left(I_4 \otimes I_2 \otimes e_2\right) \\
&= R^x \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{bmatrix} \\
&= \left[D_1, D_2, D_3, D_4\right],
\end{aligned}
$$

where D_ℓ is 2×2 and $D_\ell - D_1$ has constant rows, $\ell = 1, \ldots, 4$ as required. Since $D_1 = D_2 = D_3 = D_4$, Corollary 3(ii) verifies the orthogonality of α^{010} and α^{100}.

Classes of designs exist in which certain pairs of factorial spaces are orthogonal. One such example is given in Section 4. More generally the characterization of designs with certain orthogonality properties can be based on Theorem 2 or Corollaries 2 and 3. However, this is a problem for further research and will not be pursued in this paper.

4. Designs with partial orthogonal factorial structure. For a given design, if there exists an $x \in \Phi$ for which $C^{x\prime}C^x$ and A do not commute, then from Theorem 1, there exists at least one $y \neq x \in \Phi$ such that α^x and α^y are nonorthogonal. Such interactions, α^y, can be identified by calculating the covariances $C^x\Omega C^{y\prime}$ for all $y \neq x$.

Designs which exhibit orthogonality between many, but not all, pairs of factorial spaces will be said to have *partial orthogonal factorial structure*. In this section a useful class of designs possessing such a structure is identified and an example is given.

DEFINITION. Let $D(m_1, m_2, \ldots, m_p)$ denote the class of designs such that if $d \in D(m_1, m_2, \ldots, m_p)$ then the Moore–Penrose inverse, A_d^+, of the intrablock matrix of d can be written in the form

$$A_d^+ = \sum_{j=1}^{w} \xi_j \left(R_{j1}^{m_1} \otimes \cdots \otimes R_{jr}^{m_r} \otimes Q_j \otimes R_{js}^{m_s} \otimes \cdots \otimes R_{jp}^{m_p} \right), \tag{4.1}$$

where w is some integer, $\xi_1, \ldots, \xi_w$ some constants, $R_{ji}^{m_i}$ is an $m_i \times m_i$ permutation matrix, $r < s - 1$, and Q_j is some square matrix of dimension $m_{r+1}m_{r+2} \cdots m_{s-1}$.

THEOREM 3. *If $d \in D(m_1, m_2, \ldots, m_p)$ then the design d is guaranteed to have partial orthogonal factorial structure for a p-factor experiment whose ith factor has m_i levels, $i = 1, \ldots, p$.*

PROOF. If $d \in D(m_1, m_2, \ldots, m_p)$ then A_d^+ is given by (4.1). Let $g = \{r + 1, r + 2, \ldots, s - 1\}$. Consider the generalized interactions α^x and α^y, where $x = (x_1, \ldots, x_p)$ and $y = (y_1, \ldots, y_p)$ such that there exists at least one $i \notin g$ for which $x_i = 1$ and $y_i = 0$. Then with $\Omega = A_d^+$, using (4.1) and (2.3), $C_i^{x_i} R_{ji}^{m_i} C_i^{y_i\prime} = C_i^{x_i} e_{m_i} = 0$. Hence, α^x and α^y are orthogonal. If $x_i = y_i$ for all $i \leq r$ and $i \geq s$, then the orthogonality of α^x and α^y depends upon the structure of Q_j, $j = 1, \ldots, w$. Hence the design is guaranteed to have partial orthogonal factorial structure. □

Examples of designs in the class $D(m_1, m_2, \ldots, m_p)$ can be derived from designs in the class $D_0(n_1, n_2, \ldots, n_q)$ which have orthgonal factorial structure for a q-factor experiment where the ith factor has n_i levels, $i = 1, \ldots, q$ and where $\prod n_i = \prod m_i$, by factorizing and/or combining factor levels.

EXAMPLE 3. The generalised cyclic design, d_0, in 16 blocks of size 6 with generating block (00, 01, 10, 11, 22, 33) has orthogonal factorial structure for an experiment with two factors each at four levels. Hence $d_0 \in D_0(4,4)$. It can be verified that the Moore–Penrose inverse of the intrablock matrix is of the form

$$A_{d_0}^{+} = \sum_{\ell=1}^{u} \xi_\ell^*\left(R_{\ell 1}^4 \otimes R_{\ell 2}^4\right),$$

where $R_{\ell i}^4$ is a 4×4 circulant permutation matrix, $i = 1,2$ [see John and Smith (1972)].

Consider the design $d \in D(4,2,2)$, formed from d_0 by mapping the levels of the second factor to the levels of two new factors each at two levels, respectively. If the mapping gives a lexicographical ordering of the treatment combinations then

$$\begin{aligned} A_d^{+} = A_{d_0}^{+} &= \sum_{\ell=1}^{u} \xi_\ell^*\left(R_{\ell 1}^4 \otimes R_{\ell 2}^4\right) \\ &= \sum_{j=1}^{w} \xi_j\left(R_{j1}^4 \otimes R_{j2}^2 \otimes Q_{j3}^2\right) \end{aligned}$$

since any 4×4 circulant permutation matrix R_ℓ^4 can be expressed as $\Sigma(R_{\ell i}^2 \otimes Q_{\ell i}^2)$ where $R_{\ell i}^2$ is a 2×2 circulant permutation matrix, and $Q_{\ell i}^2$ is a 2×2 matrix of 0's and 1's. The design d is the design considered in Example 1. It follows from Theorem 3 that for this design all the pairs of interactions $(\alpha^{x_1x_2x_3}, \alpha^{y_1y_2y_3})$ are orthogonal for $x_1x_2 \neq y_1y_2$. The orthogonality of the remaining pairs of interactions $(\alpha^{100}, \alpha^{101})$; $(\alpha^{010}, \alpha^{011})$; $(\alpha^{110}, \alpha^{111})$ may be checked using Theorem 1 or by calculating the covariances directly. In Example 1, Theorem 1 was applied to show orthogonality of the first two pairs. It is possible for this design to deduce that $(\alpha^{110}, \alpha^{111})$ cannot be orthogonal, since otherwise in Example 1, $M^xM^{x\prime}NN'$ would have been symmetric for $x = 110$ and $x = 111$. If α^{111} can be assumed to be negligible the orthogonality of this pair of interactions is of little importance.

Note that designs in the classes $D(2,2,4)$, $D(2,4,2)$, $D(2,2,2,2)$, $D(2,8)$, and $D(8,2)$ may be obtained by similar methods and Theorems 1 and 3 applied in each case.

5. Disconnected designs. Let A, Φ, C^x, and V_x be defined as in Section 2, and let V be the vector space spanned by the rows of A. If $\operatorname{rank}(A) < v - 1$ then the design is disconnected. Let $V_x^* = V_x \cap V$, then V_x^* is the vector space of all estimable contrasts corresponding to α^x.

DEFINITION (Mukerjee, 1979). A disconnected incomplete block design is *regular* if $\oplus V_x^* = V$, where $\oplus$ represents direct sum over all $x \in \Phi$.

In irregular designs estimable contrasts belonging to factorial effects do not span the space of all estimable treatment contrasts. For regular designs results corresponding to Theorems 1, 2 and Corollaries 1, 2, and 3 hold.

Let B^x be a matrix whose rows form an orthonormal basis for V_x^*.

THEOREM 4. *If A is the intrablock matrix of a regular disconnected incomplete block design, then the estimable contrasts corresponding to α^x, $x = \phi_\iota \in \Phi$,* *are orthogonal to all other estimable contrasts if and only if* $B^{x\prime}B^x$ *commutes with* A.

PROOF. Follows exactly the lines of the proof of Theorem 1, using Mukerjee (1979), Theorem 4.1. □

Theorem 2 can be extended to regular disconnected designs as follows. V_x^* and V_y^* are assumed to be nonnull, otherwise the theorem is trivially true.

THEOREM 5. $\mathrm{Cov}(B^x\hat{\tau}, B^y\hat{\tau}) = 0$ *for a specified pair of interactions* α^x *and* α^y, $x, y \in \Phi$, $x \neq y$, *if and only if* Ω *can be expressed as* $\Omega_1^x + \Omega_2^y$, *where the columns of* Ω_1^x *are orthogonal to* V_x^* *and the rows of* Ω_2^y *are orthogonal to* V_y^* (*where* V_x^* *and* V_y^* *are nonnull*).

PROOF. Follows directly from Lemma 2. □

Corollaries 1, 2, and 3 can be extended in the obvious way by replacing C^x with B^x, V_x with V_x^*, and redefining H^x.

EXAMPLE 4. The generalized cyclic design, d_0, in 18 blocks of size 8, with generating block (00 11 15 20 33 42 44 53) has orthogonal factorial structure for an experiment with two factors each at 6 levels. Hence $d_0 \in D_0(6,6)$. The design d_0 is disconnected, the confounded contrast belonging to α^{11}. Consider the design $d \in D(2,3,6)$ formed by a lexicographical mapping of the levels of the first factor to the levels of two new factors.

$$A_d^+ = A_{d_0}^+ = \sum_{\ell=1}^{u} \xi_\ell^*\left(R_{\ell 1}^6 \otimes R_{\ell 2}^6\right)$$
$$= \sum_{j=1}^{w} \xi_j\left(R_{j1}^2 \otimes Q_{j2}^3 \otimes R_{j3}^6\right).$$

From Theorem 3 all pairs of interactions are orthogonal with the possible exception of $(\alpha^{001}, \alpha^{011})$, $(\alpha^{100}, \alpha^{110})$, and $(\alpha^{101}, \alpha^{111})$. Checking Theorem 4 shows that $B^xB^{x\prime}A$ is symmetric for $x = 001$ and 100, but not for $x = 101$ nor 111. Hence the first two pairs of interactions are orthogonal but not the last pair. Alternatively Corollaries 2 and 3 can be checked. The confounded contrast lies in $V_{101} \oplus V_{111}$.

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