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FINITE STOPPING IN SEQUENTIAL SAMPLING WITHOUT RECALL FROM A DIRICHLET PROCESS

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This paper shows that for sequential sampling without recall from a Dirichlet process, there exists a finite bound beyond which one will cease sampling with probability 1. The result is valid for Dirichlet processes defined on closed bounded intervals of the real line.

1. Introduction. This paper shows that for sequential sampling without recall from a Dirichlet process on a bounded interval of the real line, there exists a finite constant that is an almost sure upper bound to the sample size of an optimal procedure. As a referee has stated, "[This] shows that not even God can fool a Bayesian forever." This section introduces the problem and notation. Section 2 contains the result. Christensen (1983) contains additional results on sampling without recall from Dirichlet processes.

A shopper must buy an item. He can elicit price quotations sequentially, but must pay for each quotation. A price obtained is valid only at the time of the quotation. The shopper can buy the item at the current price but is not allowed to return to previous prices to buy the item. The shopper seeks to minimize his total cost, i.e., the price paid plus the cost of the price quotations obtained.

Suppose a sequential random sample $X_1, X_2, \dots$ is available from a possibly random distribution, say F. Without loss of generality, the first variable X_1 is observed at no cost. The observer (shopper) can choose to stop sampling and accept the observation X_1, or pay a cost, say C, and take another observation. For any $j \geq 1$, after $X_1, X_2, \dots, X_j$ have been obtained the observer can either accept X_j and make no further observations, or pay C and observe X_{j+1}. If X_j is accepted, the observer's total cost is X_j plus the sampling costs up to that point, $(j-1)C$. The observer's goal is to find a way to minimize his total payments. Clearly, the observer's problem is to find the best method of determining when to stop taking observations. If N is any stopping rule, the observer seeks a stopping rule N' so that

$$E(X_{N'} + (N'-1)C) = \min_N E(X_N + (N-1)C).$$

This problem is known as sequential sampling without recall.

The distribution F has not yet been discussed. Most of the literature on this problem deals with the special case where $X_1, X_2, \dots$ are independent and identically distributed (i.i.d.) from the fixed distribution F. DeGroot (1968, 1970) considered the problem where F is a normal distribution with unknown mean W and variance 1. Prior beliefs about F are incorporated by putting a normal

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distribution with known mean μ and known variance σ_0^2 on the parameter W. Rothschild (1974) restricted F to the class of multinomial distributions with a fixed number of categories, say n. The probabilities for each category were unknown but prior beliefs were modeled with an n-dimensional Dirichlet distribution. Rothschild suggested generalizing his work by taking the random probability measure F to have a Dirichlet process.

In related work, Ferguson (1974) briefly considers the problem of sampling with recall from a Dirichlet process. Clayton (1985) has found a similar bound on sampling in a sequential testing problem using Dirichlet processes.

The notation follows Ferguson (1973) and Rothschild (1974). For a Dirichlet process defined on a subset of the real line, say $\mathscr{X}$, with parametric measure $\alpha(\cdot)$, the notation $F \sim \mathscr{D}(\alpha)$ is used to indicate that F is a random observation from the Dirichlet process. The parameter α is assumed to be a finite measure. The weight of the measure α is denoted $W = \alpha(\mathscr{X})$. When $F \sim \mathscr{D}(\alpha)$, F is used to denote both the Dirichlet process and a realization of the process.

The marginal distribution of an observation, p, from F is $\alpha(\cdot)/W$. This marginal distribution is also denoted as $E(F)$ and EF, because the marginal distribution of p, can be used to define the expected value of F when considering F as a distribution function.

When p has been observed from a Dirichlet process F, the posterior distribution of F given p is denoted $F|p$. $F|p \sim \mathscr{D}(\alpha(\cdot) + \delta_p(\cdot))$, where $\delta_p(\cdot)$ is a measure that gives point mass 1 to p. If p' is an observation from $F|p$, the marginal distribution of p' can be written as

$$E(F|p) = \left[\alpha(\cdot) + \delta_p(\cdot)\right]/[W + 1].$$

Some additional notation: $\tilde{p}_j = (p_1, \ldots, p_j)$, $\tilde{p}$ without a subscript is used for $\tilde{p}_n$, $E_{\tilde{p}}(\cdot)$ denotes expectation with respect to the distribution of $\tilde{p}$, $E_F(\cdot)$ denotes expectation with respect to the joint distribution of a sample $p_1, p_2, \ldots$ from F, and $P_F(\cdot)$ denotes the probability measure of sets depending on $p_1, p_2, \ldots$.

The observer's problem is deciding when to stop sampling. Let τ be a stopping time for sequences $p_1, p_2, \ldots,$ of observations on F. The terms stopping time, stopping rule, strategy, and procedure will be used interchangably. Following DeGroot (1970), without loss of generality attention can be restricted to strategies in $\Delta = \{\tau | P_F(\tau < \infty) = 1\}$. For the expected payment under the strategy τ, the notation

$$V(F, \tau) = E_F\left(p_\tau + (\tau - 1)C\right)$$

is used. The minimal expected payment is also known as the value of the search problem. It is denoted

$$V(F) = \inf_{\tau \in \Delta} V(F, \tau).$$

A strategy τ_0 in Δ is optimal if

$$V(F, \tau_0) = V(F).$$

The following theorem shows that an optimal strategy exists under a regularity condition.

THEOREM. *Let $p_1, p_2, p_3, \ldots$ be a sequence of identically distributed random variables whose common distribution function is A. If $\operatorname{Var}_A(p_1) < \infty$ then there exists $\tau_0 \in \Delta$ such that $V(F, \tau_0) = V(F)$.*

PROOF. See DeGroot (1970, page 352). □

In particular, if the set $\mathscr{X}$, on which A is defined, is a bounded set, then an optimal stopping rule will exist.

Henceforth, any Dirichlet process referred to will be assumed to have a parameter α that gives a finite variance for the distribution of prices, ensuring the existence of an optimal rule.

Unfortunately, the class of optimal rules is a very broad one. Let τ be a stopping rule, we say that τ' is a version of τ if $\tau = \tau'$ almost surely (a.s.).

It is convenient when describing optimal procedures to think of p as the current observation and F as the process updated by all observations except the current one. The function $V(F|p)$ is the function $V(\cdot)$ evaluated at the process $F|p$. Any optimal procedure dictates stopping, a.s., if

$$p < V(F|p) + C \tag{1.1}$$

and dictates continued sampling, a.s., if

$$p > V(F|p) + C. \tag{1.2}$$

With this convention, we say that it is uniquely optimal to accept p if (1.1) holds and it is uniquely optimal to reject p if (1.2) holds. We say that it is optimal to accept p if

$$p \leq V(F|p) + C$$

and it is optimal to reject p if

$$p \geq V(F|p) + C.$$

The discussion above implies that V satisfies the functional equation $V(F) = \int \min[p, V(F|p) + C]\, dEF$.

A truncated version of the search problem is one in which the number of price quotations the observer is allowed to elicit is bounded. If the values of truncated problems converge to the value of the untruncated problem as the number of price quotations increases, the results proven for the truncated problem can be extended to the untruncated problem.

We need notation for the value of a truncated problem. The value of a problem truncated at the initial step is

$$V_1(F) = \int p\, dEF(p).$$

The value of a problem truncated so that an observer is allowed no more than $T \geq 2$ price quotations is

$$V_T(F) = \int \min[p, V_{T-1}(F|p) + C]\, dEF(p).$$

Yahav (1966) has shown that if $\int p^2\, dEF(p) < \infty$ then $V_T \to V(F)$.

In Section 2, an example and a corollary refer to the reservation price property. This property is simply that if it is optimal at any stage to accept a price p_0, then it is also optimal at that stage to accept any price $p < p_0$. It is by no means clear that this property holds for sequential sampling without recall from a Dirichlet process. Rothschild (1974) proved that this property holds for the Dirichlet multinomial sampling problem. Unfortunately, his proof does not seem to extend to the Dirichlet process problem. Christensen (1983) contains a proof valid for Dirichlet processes.

2. Finiteness of search. The result is that, when $\mathscr{X} = [-M, M]$, an optimal strategy is a truncated strategy, i.e., there exists some number, say N, such that an optimal procedure will not continue sampling past the Nth stage. The crucial result needed is a bound on the cost of continued sampling. The cost of continued sampling is the cost of another observation, C, plus the expected cost for proceeding with an optimal strategy after determining to take another observation. Lemma 2.1 considers the interval $[-M, -M + kC/2)$ for an arbitrary positive integer k. If one samples for a sufficiently long time and never gets an observation less than $-M + kC/2$, one becomes convinced that such observations are unlikely to occur. One essentially concludes that the best observation one can get is $-M + kC/2$. The cost of taking another observation and then following an optimal rule is at least $(-M + kC/2) + C$. In fact, one cannot do quite that well because one is not absolutely sure that observations less than $-M + kC/2$ cannot occur. In the following theorem we obtain a lower bound of $-M + (k + 1)C/2$ for the cost of continued sampling.

LEMMA 2.1. *Suppose that $F \sim \mathscr{D}(\alpha)$, and $\tilde{p}_n = (p_1, \dots, p_n)$ is observed. For a positive integer k assume that $p_i \geq -M + kC/2$ for $i = 1, 2, \dots, n$; then there exists an n_k finite such that if $n \geq n_k$ then the cost of continued sampling at the nth stage is at least $-M + (k + 1)C/2$. The n_k's can be chosen to form a nondecreasing sequence for $k = 1, 2, \dots$ with $n_1 = 1$.*

PROOF. The proof is by induction on k.

(a) The initial step: Let $n_1 = 1$. At any stage (the nth stage is where $\tilde{p}_n$ has been observed) the cost of continued sampling is at least $-M + C$, the cost of an observation plus the best price that can be obtained. Since $-M + C/2 < -M + C$, the theorem holds for $k = 1$.

(b) The inductive step: Assume that the theorem is true for k and show that it is true for $k + 1$. The hypothesis of the theorem for $k + 1$ is that $p_i \geq -M + (k + 1)C/2$, $i = 1, 2, \dots, n$. Clearly, it is also true that $p_i \geq -M + kC/2$, $i = 1, 2, \dots, n$. Using the induction hypothesis, there exists n_k such that if $n \geq n_k$, the cost of continued sampling is at least $-M + (k + 1)C/2$.

A better bound is needed. Let $\beta_{n, k+1}$ be the probability that the $(n + 1)$st observation is in the interval $[-M, -M + (k + 1)C/2)$ given that none of the first n observations were in the interval. Since sampling is from a Dirichlet process,

$$\beta_{n,k+1} = \alpha\left(\left[-M, -M + \frac{k+1}{2}C\right)\right) \Big/ (W+n).$$

When sampling is continued, the cost of an observation, C, is always incurred. Having obtained the observation p_{n+1}, two cases are considered. First, if p_{n+1} is less than $-M + (k+1)C/2$, the least additional cost possible (regardless of whether p_{n+1} is accepted or not) is $-M$. The second case is when p_{n+1} is at least $-M + (k+1)/2$. If sampling stops, the cost is at least $-M + (k+1)C/2$. If sampling continues, the induction hypothesis applies because $p_i \geq -M + (k+1)C/2 \geq -M + kC/2$ for $i = 1, 2, \ldots, n, n+1$ and $n+1$ is greater than n_k. Since the induction hypothesis applies, the cost of continued sampling is at least $-M + (k+1)C/2$. Thus, in this case, regardless of whether or not we accept p_{n+1}, the cost is at least $-M + (k+1)C/2$.

Since the probability that p_{n+1} is less than $-M + (k+1)C/2$, is $\beta_{n,k+1}$, the cost of taking the $(n+1)$st observation is at least

$$C + \left[-M\beta_{n,k+1} + \left(-M + \frac{k+1}{2}C\right)(1 - \beta_{n,k+1})\right]. \tag{2.1}$$

Rewriting (2.1) gives the lower bound for the cost of continued sampling as

$$-M + \frac{k+1}{2}C + \left(1 - \beta_{n,k+1}\frac{k+1}{2}\right)C.$$

Since $\beta_{n,k+1} \to 0$ as $n \to \infty$ we can pick $n_{k+1} \geq n_k$ such that if $n \geq n_{k+1}$ then $(1 - \beta_{n,k+1}(k+1)/2) \geq \frac{1}{2}$. Thus for $n \geq n_{k+1}$,

$$\begin{aligned} -M + \frac{k+2}{2}C &= -M + \frac{k+1}{2}C + \frac{C}{2} \\ &\leq -M + \frac{k+1}{2}C + \left(1 - \beta_{n,k+1}\frac{k+1}{2}\right)C \end{aligned}$$

and so the lower bound, $-M + (k+2)C/2$, is established, proving the theorem. □

Lemma 2.1 is not as strong a result as it may at first appear. There are many sequences of observations for which it does not apply. The following example produces one sequence of observations about which the lemma has nothing to say.

EXAMPLE. Suppose $\alpha([-M, M]) = 100$ and $\alpha([-M, -M + C/2)) = 75$. n_k can be defined so that if $n \geq n_k$ then $[1 - \beta_{n,k}(k/2)] \geq \frac{1}{2}$ or equivalently $\beta_{n,k} \leq 1/k$. Recall $\beta_{n,k} = \alpha([-M, -M + Ck/2))/(W+n)$, so in particular, $\beta_{n,2} \geq 75/(100+n)$ and $n_2 \geq 50$.

From Lemma 2.1 if $k = 1$, $n \geq n_1 = 1$, and $p_i \geq -M + C/2$ for $i = 1, \ldots, n$ then any value of p_n between $-M + C/2$ and $-M + C$ will be accepted. By the reservation price property any value of $p_n \leq -M + C$ will be accepted. Similarly, if $k = 2$, $n > n_2 \geq 50$, and $p_i \geq -M + C$ for $i = 1, \ldots, n-1$ then any $p_n \leq -M + 3C/2$ will be accepted.

Choose $\varepsilon_1, \varepsilon_2$ with, $0 < \varepsilon_i < C/2$. Let $p_1 = -M + C + \varepsilon_1$, and take $p_i > -M + 3C/2$ for $i = 2, \dots, n$ where $n > n_3$. Lemma 2.1 has nothing to say about such a sequence of observations. The lemma applies with $k = 1$ but all the p_i's are too large to say anything about. For $k = 2$, the lemma has nothing to say about $p_1, \dots, p_{n_2-1}$ $(n_2 - 1 > 49)$ and for $i \geq 50$ the p_i's are too large to say anything about. For $k = 3, 4, 5, \dots$ the lemma does not apply because $p_1 < -M + kC/2$.

Since at the nth stage the cost of continued sampling is $V(F|\tilde{p}_n) + C$, Lemma 2.1 implies that if $p_i \geq -M + kC/2$, $i = 1, \dots, n$ and $n \geq n_k$ then

$$-M + \frac{k+1}{2}C \leq V(F|\tilde{p}_n) + C.$$

If $p_n \leq -M + (k+1)C/2$, it is optimal to accept p_n.

The following corollary establishes that for any price, say p, if enough price quotations greater than p have been obtained then p would be accepted if observed.

COROLLARY 2.2. *Let $p = p_n$ and assume $p_i \geq p$, $i = 1, \dots, n-1$. There exists an $n^* = n^*(p)$ such that if $n \geq n^*$ then it is optimal to accept p_n.*

PROOF. For some k we have $p \in [-M + kC/2, -M + (k+1)C/2)$. Clearly the conditions of Lemma 2.1 are satisfied. Let $n^* = n_k$. If $n \geq n^*$, the cost of continued sampling is at least $-M + (k+1)C/2 > p = p_n$. □

The corollary can be strengthened as follows.

COROLLARY 2.3. *Let $p_i \geq p$, $i = 1, \dots, n-1$ and $p_n \leq p$. There exists $n^* = n^*(p)$ such that if $n \geq n^*$, then it is optimal to accept p_n.*

PROOF. Take n^* as in Corollary 2.2. If $p_n = p$ then it is optimal to stop. The reservation price property indicates that if $p_n < p$ it is optimal to accept p_n. □

We now show that all optimal search procedures are bounded with probability 1.

THEOREM 2.4. *Let $F \sim \mathscr{D}(\alpha)$. There exists an integer N such that any optimal procedure stops sampling no later than the* Nth *stage of the search with probability* 1.

PROOF. Since every optimal procedure has a version that stops sampling whenever $p_n < V(F|\tilde{p}_n) + C$, it is enough to show the result for such procedures.

Suppose that no such N exists; then for any S there exists an optimal strategy, τ, and observations $p_1, p_2, \dots, p_S$ for which $\tau(p_1, p_2, \dots, p_S) > S$. That is,

$$p_i \geq V(F|\tilde{p}_i) + C, \qquad i = 1, 2, \dots, S.$$

It will be shown that for large enough S, $p_S > M$, which is a contradiction to $\mathscr{X} = [-M, M]$. An inductive argument is used to show that in order to have a sequence of prices that avoids stopping, the prices must get systematically larger. Eventually they exceed M. An upper bound N is found as a number such that if $S \geq N$, the contradiction results.

(a) The initial step. $-M + C$ is always a lower bound for the cost of continued sampling, so any price less than $-M + C$ would be accepted. Therefore, we can assume $p_i \geq -M + C$, $i = 1, 2, \ldots, S$.

(b) The inductive step. We want to show that if the sequence $p_1, \ldots, p_S$ never stops and if the observations are eventually above some constant, then the observations must eventually be larger than another, larger, constant. In particular, suppose that $p_i \geq -M + kC/2$ for $i = N_k, N_k + 1, \ldots, S$; then there exists $N_{k+1} \geq N_k$ so that $p_i \geq -M + (k+1)C/2$ for $i = N_{k+1}, N_{k+1} + 1, \ldots, S$ if $S \geq N_{k+1}$. Note that for $k = 1$, $N_1 = 1$.

After $N_k - 1$ observations have been obtained, the updated Dirichlet process is

$$F | \tilde{p}_{N_k - 1} \sim \mathscr{D}\left(\alpha(\cdot) + \sum_{i=1}^{N_k - 1} \delta_{p_i}(\cdot) \right).$$

Lemma 2.1 can be applied to the updated Dirichlet process. The lemma says that since $p_{i+N_k-1} \geq -M + kC/2$ for $i = 1, 2, \ldots, S - N_k + 1$ there exists n_k (depending on $\alpha(\cdot) + \sum_{i=1}^{N_k-1} \delta_{p_i}(\cdot)$) such that if $n \geq n_k$ the cost of continued sampling when $\tilde{p}_{n+N_k-1}$ has been observed is at least $-M + (k+1)C/2$.

Ignoring, for the moment, that n_k depends on $p_1, \ldots, p_{N_k-1}$, we complete the inductive argument. If $S - N_k + 1 \geq n_k$ then for $i = n_k, n_{k+1}, \ldots, S - N_k + 1$ it is optimal to stop for any $p_{i+N_k-1} < -M + (k+1)C/2$. Since $p_1, \ldots, p_S$ does not stop, it must be that $p_{i+N_k-1} \geq -M + (k+1)C/2$ for $i = n_k, n_{k+1}, \ldots, S - N_k + 1$. Rewriting, we must have $p_i \geq -M + (k+1)C/2$ for $i = N_k + n_k - 1, N_k + n_k, \ldots, S$. Letting $N_{k+1} = N_k + n_k - 1$ the inductive step is proven.

Since N will later be chosen as a function of the N_k's it will not do to have N_{k+1} depend on $p_1, \ldots, p_{N_k-1}$. In the proof of Lemma 2.1, n_k was taken so that for $n \geq n_k$, $(1 - k\beta_{n,k}/2) \geq \frac{1}{2}$ where

$$\beta_{n,k} = \frac{\alpha([-M, -M + kC/2))}{W + n}.$$

This condition is equivalent to picking n_k so that for $n \geq n_k$, $\beta_{n,k} \leq 1/k$. In the current instance we are applying the theorem to $F|(\tilde{p}_{N_k-1})$. If we pick n_k so that for $n \geq n_k$

$$\frac{\alpha([-M, -M + kC/2)) + N_k - 1}{W + N_k - 1 + n} \leq \frac{1}{k},$$

then we automatically have

$$\frac{\alpha([-M, -M + kC/2)) + \sum_{i=1}^{N_k-1} \delta_{p_i}([-M, -M + kC/2))}{W + N_k - 1 + n} \leq \frac{1}{k}$$

and now n_k does not depend on $p_1, \ldots, p_{N_k-1}$.

(c) The contradiction. Let $N = N_{k_0}$ where $k_0 = [4M/C] + 1$. ($[a]$ is the greatest integer of a.) By the induction result, if $S \geq N$ and stopping has not occurred for $p_1, \dots, p_S$ then

$$p_i \geq -M + (k_0/2)C \quad \text{for } i = N, N+1, \dots, S.$$

Thus $p_S \geq -M + ([4M/C] + 1)C/2 > -M + (4M/C)C/2 = M$, but $p_S > M$, a contradiction; therefore $p_1, \dots, p_S$ must have stopped previous to the Nth stage. □

Although a closed form solution for N is not available, computation of N is easy: $N_1 = 1$, $N_{k+1} = N_k + n_k - 1$, and n_k can be taken as the greatest integer in

$$(k)\alpha\left(\left[-M, -M + \frac{k}{2}C\right)\right) + (k-1)(N_k - 1) + W + 1.$$

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