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Consider the following simplified version of Freedman's (1963) example showing inconsistency of the posterior distribution for countable parameters.

Let $\theta_\infty$ denote the geometric distribution

$$p(i) = 3/4^{i+1}, \qquad i = 0, 1 \dots .$$

Let $\theta_k$ denote the truncated geometric distribution

$$\begin{aligned} p(i) &= 3/4^{i+1}, && i = 0, 1, \dots, k, \\ p(i) &= 0, && i = k+1, \dots, \\ p(i) &= 1/4^{k+1}, && i = -1. \end{aligned}$$

Let $\theta_0$ denote the geometric distribution

$$p(i) = 3^i/4^{i+1}, \qquad i = 0, 1, \dots .$$

A prior distribution gives probability $\Pi_k$ to $\theta_k$. The likelihood of $X_1, X_2, \dots, X_n$ at $\theta_k$ is $(\frac{3}{4})^n(\frac{1}{4})^{S_n}$ for $0 < k$, $k \geq \max X_i$, and is $(\frac{1}{4})^n \cdot (\frac{3}{4})^{S_n}$ at $k = 0$, where $S_n = \sum X_i$.

The posterior probability of $\theta_0$ is

$$\Pi_0 3^{S_n - n} \Big/ \Big( \Pi_0 3^{S_n - n} + \sum_{k \geq \max X_i} \Pi_k \Big).$$

When $\theta_\infty$ is true, $S_n = \frac{1}{3}n + O(\sqrt{n})$, $\max X_i \sim \log_4 n$. Let $Q_k = \sum_{j=k} \Pi_j$. Choose the prior $\Pi$ so that $\Pi_k > 0$ all $k$, $3^n Q(\log_4 n) \to 0$ as $n \to \infty$. Thus $\theta_\infty$ is true, and every neighborhood of $\theta_\infty$ has positive probability in the weak-star topology, yet the posterior probability of $\theta_0$ converges to 1.

From this and other arguments the authors conclude that posteriors are usually inconsistent when the parameters are countably dimensional. But note that the parameter space here is just the integers $k = 0, 1, \dots, \infty$ (and in Freedman's example it is a closed interval). The counterexample applies equally well in finite dimensional cases if one induces a topology on the parameters by the weak-star topology on distributions!

The weak-star topology gives little weight to the tails, but small differences in the tails can have very large relative effects on the likelihood, and therefore on the posterior probabilities. I wonder if it wouldn't be possible to escape from the inconsistency by using a likelihood-friendly topology such as one based on the distance

$$\rho(P, Q) = \sum_{i=0}^{\infty} (P_i - Q_i) \ln \frac{P_i}{Q_i}?$$

In this topology, the $\theta_i$ of the example are isolated points, and $\theta_\infty$ has zero probability, so there is no surprise in finding the posterior inconsistent.

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