The Annals of Statistics
1985, Vol. 13, No. 2, 583–591

ASYMPTOTIC SCORE-STATISTIC PROCESSES AND TESTS FOR CONSTANT HAZARD AGAINST A CHANGE-POINT ALTERNATIVE

BY D. E. MATTHEWS, V. T. FAREWELL AND R. PYKE

University of Waterloo, University of Washington and University of Washington

The problem of testing for a constant failure rate against alternatives with failure rates involving a single change-point is considered. The asymptotic significance level for tests based on maximal score statistics are shown to involve the solution to a first passage time problem for an Ornstein-Uhlenbeck process. An example illustrates the methodology.

1. Introduction. Let T be a random variable representing the time to some event, for example, the time-to-relapse after remission induction for patients with leukemia. Matthews and Farewell (1982) considered a model for the distribution of T specified by the failure rate or hazard function

$$\lambda(t) = \begin{cases} \lambda & \text{if } 0 \le t < \tau \\ (1-\xi)\lambda, & \text{if } t \ge \tau \end{cases} \tag{1}$$

in which the two parameters satisfy $0 \le \xi < 1$ and $\tau \ge 0$. Thus, τ is a change-point parameter for the hazard rate at which time it changes from the constant hazard λ to $(1-\xi)\lambda$. If $\xi = 0$, then $\lambda(t)$ is constant for any τ and the corresponding failure times are exponential with mean $1/\lambda$. In this case, the change-point parameter is not really part of the model. On the other hand, if $\xi > 0$, then T is a combination of exponentials with density given by

$$f(t) = \begin{cases} \lambda e^{-\lambda t}, & t < \tau \\ \rho\lambda \exp\{-\lambda\tau - \rho\lambda(t-\tau)\}, & t \ge \tau \end{cases} \tag{2}$$

where $\rho = 1 - \xi$. In this case of $\rho < 1$, questions of inference concerning τ are much more interesting.

Matthews and Farewell (1982) consider the problem of testing the hypothesis $\xi = 0$, discussing, in particular, the appropriate likelihood ratio test statistic. The distribution of this statistic was examined there by simulation. More generally, Davies (1977) discusses the broader issue of hypothesis testing when a nuisance parameter, such as the hazard change-point τ in model (1) above, is present *only under the alternative*. Davies recommends the use of either the normalized score, the normalized maximum likelihood estimator for ξ, or the signed square root of the generalized likelihood ratio as a test statistic. He establishes that these test statistics, as functions of the nuisance parameter, converge weakly to Gaussian

Received April 1984; revised January 1985.
AMS 1970 *subject classifications.* Primary 62E20; secondary 62F05.
Key words and phrases. Testing exponentiality, maximal score statistic, Ornstein-Uhlenbeck process, weak convergence, empirical process, nuisance parameter.

processes under suitable regularity conditions. This result can be used to investigate the asymptotic distribution of the test statistics and thus to obtain upper bounds on the significance levels of the corresponding tests.

The work of Davies (1977) cannot, however, be used directly to specify a test of the hypothesis $\xi = 0$ in the model specified by (1) because of the basic discontinuity present at τ. In this paper, we consider the test statistic which is the normalized score statistic based on model (1). We then show, in Section 2, that, for known λ, the score process indexed by τ converges weakly to an Ornstein-Uhlenbeck (O-U) process which is a stationary Gaussian Markov process with mean zero and unit variance (c.f., Cox and Miller, 1965, Chapter 5). The asymptotic significance level of a test for no reduction in the hazard rate is then shown in Section 3 to be the solution to a first passage time problem for this O-U process. Section 4 extends these results to the case of λ unknown, and the application of the test to a particular data set is discussed in Section 5.

2. The score-statistic process and the limiting Ornstein-Uhlenbeck process. In this section we first define the appropriate score statistic as a process in τ, demonstrate the weak convergence of this process, and then in the third subsection we identify the limiting process as the Ornstein-Uhlenbeck process described above.

2.1 *The score-statistic process.* Let $T_1, \cdots, T_n$ be independent, identically distributed (iid) random variables with hazard function specified by (1). Under H_0: $\xi = 0$, $T_1, \cdots, T_n$ have an exponential distribution with mean $1/\lambda$, while under H_1: $\xi \neq 0$, the probability density function for the T_i's is given by (2). The resulting likelihood function, L, when λ is known, yields a normalized score statistic $(\partial \log L/\partial\xi)\{E(-\partial^2 \log L/\partial\xi^2)\}^{-1/2}$, evaluated at $\xi = 0$ for a fixed value of τ, that can be written as

$$(3) \qquad Z_n(\tau) = n^{-1/2} \sum_{i=1}^{n} e^{\lambda\tau/2}\{(T_i - \tau)\lambda - 1\}H(T_i - \tau), \quad \tau \geq 0,$$

where $H(x) = 1$ or 0 according to whether $x \geq 0$ or not. The score-statistic process defined by $\{Z_n(\tau); \tau > 0\}$ has $E\{Z_n(\tau)\} = 0$, $\mathrm{Var}\{Z_n(\tau)\} = 1$ and for $\tau_1, \tau_2 \geq 0$, $\mathrm{Cov}\{Z_n(\tau_1), Z_n(\tau_2)\} = \exp\{-\frac{1}{2}\lambda \mid \tau_1 - \tau_2 \mid\}$, where all expectations are computed under the null hypothesis, $\xi = 0$.

The asymptotic distribution of Z_n cannot be specified using the results of Davies (1977) because his results require that the asymptotic covariance function of Z_n be twice differentiable with respect to τ_1 at $\tau_1 = \tau_2 = \tau$. However, it follows from the classical central limit theorem for iid random variables that, for each $\tau > 0$, Z_n has an asymptotic normal distribution with mean 0 and variance 1. Moreover, it can also be shown that all of the finite dimensional distributions are asymptotically normal with covariance $\exp\{-\frac{1}{2}\lambda \mid \tau_1 - \tau_2 \mid\}$ for each fixed τ_1, τ_2. This suggests, of course, that the processes Z_n converge weakly to an O-U process.

2.2 *The weak convergence of Z_n.* Let F_n be the empirical distribution function of $T_1, \cdots, T_n$, independent observations from an exponential distribution F

with mean $1/\lambda$. Then the Z_n-process may be rewritten as

$$(4)\qquad Z_n(\tau) = (ne^{\lambda\tau})^{1/2} \int_\tau^\infty \{(x-\tau)\lambda - 1\}\, dF_n(x).$$

Let $U_n^F(x) = n^{1/2}\{F_n(x) - F(x)\}$, $x \in R^1$, denote the empirical process based on n independent random variables from F. Then, since the integral in (4) is zero if F_n is replaced by the exponential F, (4) may be rewritten in this case as

$$(5)\qquad Z_n(\tau) = e^{\lambda\tau/2} \int_\tau^\infty \{(x-\tau)\lambda - 1\}\, dU_n^F(x).$$

(For general F, a nonzero centering function is required.) Thus, the score-statistic process can be viewed as the exponential empirical process indexed by a particular family of functions, $g_\tau(x) = \{\exp(\frac{1}{2}\lambda\tau)\}\{(x-\tau)\lambda - 1\}H(x-\tau)$, $\tau \ge 0$. The unboundedness of these functions necessitates some care in studying the weak convergence of Z_n. Our approach is to use strongly convergent versions of the empirical processes (for example, c.f., Pyke, 1969) after applying integration by parts to (5) to represent Z_n by

$$(6)\qquad Z_n(\tau) = -e^{\lambda\tau/2}\left\{\lambda \int_\tau^\infty U_n^F(x)\, dx - U_n^F(\tau)\right\}.$$

Let U_n denote the uniform empirical process, $U_n(u) = n^{1/2}\{F_n(u) - u\}$, $0 \le u \le 1$, based on n independent uniform random variables on [0, 1]. It is known that possible representations for U_n^F in terms of the uniform empirical process U_n include $U_n^F =_L U_n \circ F =_L U_n \circ (1-F)$ (c.f., Pyke, 1972), since both $F^{-1}(U)$ and $F^{-1}(1-U)$ have distribution F if U is uniform on [0, 1]. Without any loss of generality, define $U_n^F = U_n \circ (1-F)$ and introduce in (6) the changes of variable $1 - F(x) = u$ and $1 - F(\tau) = t$. Then, for $Z_n^*(t) := Z_n(\tau)$, we obtain

$$(7)\qquad Z_n^*(t) = Z_n\left(-\frac{1}{\lambda}\ln t\right) = t^{-1/2}\left\{U_n(t) - \int_0^t u^{-1}U_n(u)\, du\right\}, \quad 0 \le t \le 1.$$

Let $U \equiv \{U(u)\colon 0 \le u \le 1\}$ be Brownian bridge. It is known that $U_n \to_L U$, where the convergence in law is with respect to certain weighted supremum metrics ρ_q on $D([0, 1])$ (c.f., Pyke and Shorack, 1968; and O'Reilly, 1974). Specifically, let $\rho(f, g) = \sup_{0\le u\le 1} |f(u) - g(u)|$ define the supremum metric on $D([0, 1])$ and let ρ_q denote the weighted supremum metric defined by $\rho_q(f, g) = \rho(f/q, g/q)$, whenever finite. For our purposes here, it suffices to take the special case of $q(u) = u^\beta$ for any $0 < \beta < \frac{1}{2}$, and make use of the fact that as a consequence of the convergence $U_n \to_L U$ with respect to ρ_q, there exist equivalent versions of the processes U_n and U for which $\rho_q(U_n, U) \to_{\text{a.s.}} 0$. Since for any $t_0 > 0$,

$$\begin{aligned}
&\sup_{t_0\le t\le 1} t^{-1/2} \left| \int_0^t u^{-1}\{U_n(u) - U(u)\}\, du \right| \\
&\le \rho_q(U_n, U) \sup_{t_0\le t\le 1} t^{-1/2} \int_0^t u^{-1+\beta}\, du \\
&= \rho_q(U_n, U)\beta^{-1} t_0^{\beta-(1/2)} \to_{\text{a.s.}} 0,
\end{aligned}$$

it follows that for these versions,

$$\sup_{t_0 \le t \le 1} | Z_n^*(t) - Z^*(t) | \to_{\text{a.s.}} 0,$$

where

$$Z^*(t) := t^{-1/2}\left\{U(t) - \int_0^t u^{-1}U(u)\,du\right\}. \tag{8}$$

This shows that $Z_n^* \to_L Z^*$ under the supremum metric ρ when the processes are restricted to $[t_0, 1]$ for any $t_0 > 0$. This, in turn, implies that $\to_L$ holds with respect to the usual Skorokhod topology of $D([t_0, 1])$. For our purposes, this is sufficient, although the above approach would also yield convergence results for Z_n^* over the full interval $[0, 1]$ with respect to weighted supremum metrics. In this case, the weight functions would have to offset the factor $t^{-1/2}$ which, for the uniform metric ρ, is what necessitates the restriction above to an interval $[t_0, 1]$ bounded away from 0.

2.3 *The identification of* Z^*. The finite dimensional discussion in subsection 2.1 above suffices to identify the limiting process. However, in view of the above constructive proof of weak convergence, it is desirable to identify the limit directly from representation (8). Write

$$W(t) = U(t) - \int_0^t u^{-1}U(u)\,du, \tag{9}$$

so $Z^*(t) = t^{-1/2}W(t)$. Since U is Brownian bridge, it is well known that U can be represented by $\{B(t) - tB(1): 0 \le t \le 1\}$, where $\{B(t): t \ge 0\}$ is standard Brownian motion. Consequently, (9) may be replaced by

$$W(t) = B(t) - \int_0^t u^{-1}B(u)\,du.$$

From this it is clear that W is Gaussian, continuous, has mean 0, and has independent increments. Furthermore, for $0 \le s \le t$, straightforward calculations yield $\text{Cov}\{W(s), W(t)\} = s$. It then follows (c.f., Billingsley, 1968, Theorem 19.1) that W is a standard Brownian motion. Therefore, Z^* is a normalization of B with constant variance, and so Z^* is an O-U process when the parameter t is suitably transformed. In particular, for $t_1 < t_2$, $\text{Cov}\{Z^*(t_1), Z^*(t_2)\} = (t_1/t_2)^{1/2}$. Thus, since $1 - F(\tau) = t$ it follows that the covariance function of $Z(\tau) := Z^*(e^{-\lambda\tau})$ is $\exp\{-\tfrac{1}{2}\lambda\,|\,\tau_1 - \tau_2\,|\}$, which completes the proof of

THEOREM 1. *For any* $\tau_0 < \infty$, *the score-statistic process* $\{Z_n(\tau): 0 \le \tau \le \tau_0\}$, *based on independent exponential random variables with mean* $1/\lambda$, *converges weakly to the Ornstein-Uhlenbeck process* $\{Z(\tau): 0 \le \tau \le \tau_0\}$ *with mean* 0 *and covariance* $\exp\{-\tfrac{1}{2}\lambda\,|\,\tau_1 - \tau_2\,|\}$. *The convergence is with respect to the Skorokhod topology on* $D([0, \tau_0])$.

The proof of this result that we give above is direct and instructive, but other approaches are, of course, also possible. As mentioned following (5), central limit theorems for function-indexed empirical processes would be applicable (c.f., Giné and Zinn, 1984). On the other hand, one may calculate the moments of successive differences to permit the use of tightness results as in Billingsley (1968). For these exponential cases, a fourth approach is possible using the fact that the excesses $\{T_i - \tau\}$ for those $T_i > \tau$ are independent exponentials, independent of the number, $N_n(\tau)$ say, of T_i's exceeding τ. This permits the partial-sum representation

$$Z_n(\tau) =_L n^{-1/2} \sum_{i=1}^{N_n(\tau)} e^{\lambda\tau/2}(\lambda T_i - 1),$$

to which similar weak convergence methods can be applied.

3. Testing H: $\xi = 0$. If τ is confined to an interval $[\tau_l, \tau_u]$, then paralleling Davies' (1977) work, a suitable test of the hypothesis H_0: $\xi = 0$ is one which rejects for large values of statistics of the form

$$M_n(\tau_l, \tau_u) := \sup_{\tau_l \le \tau \le \tau_u} Z_n(\tau),$$

the supremum being taken over the nuisance parameter τ. To obtain the asymptotic distribution and asymptotic critical values for this statistic, observe that by Theorem 1,

$$M_n(\tau_l, \tau_u) \to_L M(\tau_l, \tau_u) := \sup_{\tau_l \le \tau \le \tau_u} Z(\tau),$$

where Z is the appropriate O-U process. Note that in view of the stationarity of Z, the distribution of $M(\tau_l, \tau_u)$ depends only on $\tau_u - \tau_l$. We, therefore, set $\tau_l = 0$ and write simply $M(\tau_u)$ for the corresponding statistic.

For $c > 0$, introduce the first passage times, $T(c) = \inf\{\tau \ge 0: Z(\tau) \ge c\}$. Clearly, $M(\tau_u) \ge c$ if and only if $T(c) \le \tau_u$; therefore,

$$\Pr\{M(\tau_u) \ge c\} = \Pr\{T(c) \le \tau_u\}.$$

Mandl (1962) provides formulae and tables for calculating such probabilities. In his notation, $P(t, A)$ denotes the probability that a stationary O-U process with correlation function $\exp(-\beta\,|\,t\,|)$ and initial distribution equal to the stationary distribution does not exceed the value A during an interval of length t. He obtains the approximation

$$P(t, A) \sim Ze^{-n_0\beta t}$$

where the values of $n_0 = n_0(A)$ and $Z = Z(A)$ may be obtained from Mandl (1962, Table 1). The rows in his table are indexed by $a = (A - \mu)/\sigma$, where μ is the mean and σ^2 the variance of the stationary distribution.

In our context, we have a stationary O-U process, $Z(\tau)$, with $\beta = ½\lambda$ so that

$$\Pr\{M(\tau_u) \ge c\} = \Pr\{T(c) \le \tau_u\} = 1 - P(\tau_u, c).$$

Thus, it follows that the results of Mandl (1962) can be used to approximate the asymptotic significance levels for this test of H_0: $\xi = 0$.

Although Mandl (1962) is ideally suited for the problem under consideration here, it is worthwhile to mention the tables of Keilson and Ross (1975). Their tables differ from Mandl's in that they provide probabilities for first passage times, *conditional* on initial values of an O-U process. These tables could be used to approximate the asymptotic significance levels required here although Mandl's table is more directly utilized. In other applications, however, the tables of Keilson and Ross may be quite useful.

4. The case of λ unknown. If λ is unknown, the normalized score-statistic $Z_n(\tau)$ specified by (3) is no longer appropriate for testing H_0: $\xi = 0$. Instead, it is appropriate to standardize $\partial \log L/\partial\xi$ by

$$[\partial^2 \log L/\partial\xi^2 - \{(\partial^2 \log L/\partial\xi\partial\lambda)^2/\partial^2 \log L/\partial\lambda^2\}]$$

to generate the partial score-statistic process (Lawless, 1982, page 524)

$$\begin{aligned}\hat{Z}_n(\tau) &= \{ne^{-\lambda\tau}(1 - e^{-\lambda\tau})\}^{-1/2}\{\partial \log L/\partial\xi\}\,|_{\xi=0,\lambda=\hat{\lambda}_n} \\ &= (1 - e^{-\lambda\tau})^{-1/2} Z_n(\tau, \lambda)\,|_{\lambda=\hat{\lambda}_n},\end{aligned} \tag{10}$$

in which $\hat{\lambda}_n = (n/\sum_{i=1}^n T_i)$ is the maximum likelihood estimator of λ under the null hypothesis, and $Z_n(\tau, \lambda) = Z_n(\tau)$ is as in (3)–(6), but the new notation emphasizes the presence of the unknown parameter λ in its functional form.

As in (4)–(6), write

$$\begin{aligned}&\int_\tau^\infty \{(x - \tau)\hat{\lambda}_n - 1\}\, dF_n(x) \\ &\quad = \int_\tau^\infty \{(x - \tau)\lambda - 1\}\, dF_n(x) + (\hat{\lambda}_n - \lambda) \int_\tau^\infty (x - \tau)\, dF_n(x) \\ &\quad = (ne^{\lambda\tau})^{-1/2} Z_n(\tau, \lambda) + \hat{\lambda}_n(1 - F(\tau)) \int_0^\infty n^{-1/2} U_n^F(x)\, dx \\ &\quad\quad - (\hat{\lambda}_n - \lambda) \int_\tau^\infty n^{-1/2} U_n^F(x)\, dx,\end{aligned}$$

using $n^{1/2}(\hat{\lambda}_n^{-1} - \lambda^{-1}) = -\int_0^\infty U_n^F(x)\, dx$. Consequently,

$$\begin{aligned}&\hat{Z}_n(\tau) = (1 - e^{-\hat{\lambda}_n\tau})^{-1/2} e^{1/2(\hat{\lambda}_n - \lambda)\tau} \\ &\quad \cdot \left\{Z_n(\tau, \lambda) + \hat{\lambda}_n e^{-1/2\lambda\tau} \int_0^\infty U_n^F(x)\, dx + (\hat{\lambda}_n - \lambda)\lambda^{-1}(Z_n(\tau, \lambda) - e^{1/2\lambda\tau} U_n^F(\tau))\right\}.\end{aligned} \tag{11}$$

Set $\hat{Z}_n^*(t) := \hat{Z}_n(\tau)$ with the changes of variables $t = 1 - F(\tau) = e^{-\lambda\tau}$ and $u = 1 - F(x)$, and then apply (8) and (9) to (11) to obtain that, for the same strongly convergent versions of Section 2.3, $\hat{Z}_n^*$ converges uniformly over $[t_0, t_1]$, $0 < t_0 \le t_1 < 1$, with probability one to

$$\begin{aligned}\hat{Z}^*(t) &:= (1 - t)^{-1/2}\left\{Z^*(t) + \sqrt{t} \int_0^1 u^{-1} U(u)\, du\right\} \\ &= (1 - t)^{-1/2} t^{-1/2}\left\{U(t) - \int_0^t u^{-1} U(u)\, du + t \int_0^1 u^{-1} U(u)\, du\right\} \\ &= (W(t) - tW(1))/\{t(1 - t)\}^{1/2},\end{aligned} \tag{12}$$

where W, defined in (9), is standard Brownian motion. This suffices to show that $\hat{Z}_n^* \to_L \hat{Z}^*$ with respect to the Skorokhod topology on $D([t_0, t_1])$, although the above representation would, as before, yield suitable convergence results over the full interval [0, 1] with respect to weighted supremum metrics.

Whereas, when λ was known, the limiting process was an O-U process, which is a Brownian motion normalized to give constant variance, in this case of unknown λ it is Brownian bridge (c.f., (12)) that is normalized to have a constant variance. To apply a test statistic such as $\sup_{\tau_l \le \tau \le \tau_u} \hat{Z}(\tau)$ to test H: $\xi = 0$, one may transform this tied-down case to fit the context of Mandl (1962). The asymptotic significance level of such a test would be given by

$$\Pr\{\sup_{\tau_l \le \tau \le \tau_u} \hat{Z}(\tau) \ge c\}. \tag{13}$$

Under the transformation $t = 1 - F(\tau)$, and in view of (12), (13) equals

$$\Pr\{\sup_{t_0 \le t \le t_1} [t(1-t)]^{-1/2} U(t) \ge c\}, \tag{14}$$

with $t_0 = 1 - F(\tau_u)$ and $t_1 = 1 - F(\tau_l)$. If Doob's transformation is used, this can, in turn, be shown to equal $1 - \Pr\{Z(s) < c;\ 0 \le s \le \ln[t_1(1 - t_0)/(1 - t_1)t_0]\}$ for appropriate c, where Z is the O-U process of Section 3.

It should be noted that Kendall and Kendall (1980) propose the test given by (14) for a problem that arises in the context of deriving a satisfactory method for testing whether a collection of n points in the plane contains too many triads that are approximately collinear. They assume that a Poisson process with intensity specified by (1), restricted to the finite section $[\tau_0, \tau_1]$ containing τ, is a suitable random mechanism for the generation of the angles determined by triads of points in the plane. They then develop a test of the null hypothesis of $\rho = 1$ against the alternative, $\rho < 1$, which asymptotically is based on the supremum of the same tied-down O-U process described above.

5. An example. Table 1 gives times from diagnosis to death for 31 individuals with advanced non-Hodgkin's lymphoma and presenting with clinical symptoms. Since 11 of the times are censored, because the patients were alive at the last time of follow-up, the results of the previous sections are not directly applicable. However, Matthews and Farewell (1982) show that moderate censoring has little impact on the distribution of the likelihood ratio statistic for testing $\xi = 0$ in model (1). For illustration, therefore, we assume that the results of Section 4 are similarly applicable if the likelihood function (3) is allowed to incorporate contributions

$$\exp\left[-\int_0^t \{\lambda - \lambda\xi H(u - \tau)\}\, du\right]$$

for individuals censored at a time t.

For the particular data set described above, the observed supremum of $\hat{Z}_n(\tau)$ is 6.35. If we arbitrarily fix the bounds for τ at $\tau_l = 10$ and $\tau_u = 35$ and use $\hat{\lambda} = 0.02441$ to estimate λ in the transformation $t = e^{-\lambda\tau}$, which is required to convert the tied-down case to fit the context of Mandl (1962), then the corre-

TABLE 1
Survival times, in months, for 31 *lymphoma patients*

2.5, 4.1, 4.6, 6.4, 6.7, 7.4, 7.6, 7.7, 7.8, 8.8, 13.3, 13.4, 18.3, 19.7, 21.9, 24.7, 27.5, 29.7, 30.1*, 32.9, 33.5, 35.4*, 37.7*, 40.9*, 42.6*, 45.4*, 48.5*, 48.9*, 60.4*, 64.4*, 66.4*

* indicates a censored observation.

TABLE 2
Values of $1 - P(1.5854, c)$ *derived from Mandl* (1962)

c	2.0	2.5	3.0	3.5	4.0	4.5	5.0
$1 - P(1.5854, c)$	0.13	0.05	0.01	0.003	0.0005	0.0001	0.00001

sponding values of $t_0 = e^{-\lambda\tau_u}$ and $t_1 = e^{-\lambda\tau_l}$ are $e^{-0.8544} = 0.4256$ and $e^{-0.2441} = 0.7834$, respectively. Therefore, the asymptotic significance level of the data is $1 - P(1.5854, 6.35)$.

Since the set of values of c is limited to the cases $c = 2.00(0.05)5.00$, the results of Mandl (1962) do not directly provide the value of $P(1.5854, 6.35)$. However, Table 2 which gives the values of $1 - P(1.5854, c)$ for $c = 2.00(0.50)5.00$ indicates that the value we require is quite small.

The asymptotic significance levels suggested above should be sufficiently accurate for practical applications. Simulations in Matthews and Farewell (1982) and Kendall and Kendall (1980) indicate the applicability of asymptotic distributions even for small sample sizes.

REFERENCES

BILLINGSLEY, P. (1968). *Convergence of Probability Measures.* Wiley, New York.

COX, D. R. and MILLER, H. D (1965). *The Theory of Stochastic Processes.* Methuen, London.

DAVIES, R. B. (1977). Hypothesis testing when a nuisance parameter is present only under the alternative. *Biometrika* **64** 247–254.

GINÉ, E. and ZINN, J. (1984). Some limit theorems for empirical processes. *Ann. Probab.* **12** 929–989.

KEILSON, J. and ROSS, H. F. (1975). Passage Time Distributions for Gaussian Markov (Ornstein-Uhlenbeck) Statistical Processes. In *Selected Tables in Mathematical Statistics*, **3** 233–328 (Institute of Mathematical Statistics, ed.) American Mathematical Society, Providence.

KENDALL, D. G. and KENDALL, W. S. (1980). Alignments in two-dimensional random sets of points. *Adv. Appl. Probab.* **12** 380–424.

LAWLESS, J. F. (1982). *Statistical Models and Methods for Lifetime Data.* Wiley, New York.

MANDL, P. (1962). On the distribution of the time which the Uhlenbeck process requires to exceed a boundary (in Czechoslovakian with Russian and German summaries). *Apl. Mat.* **7** 141–148.

MATTHEWS, D. E. and FAREWELL, V. T. (1982). On testing for a constant hazard against a change-point alternative. *Biometrics* **38** 463–468.

O'REILLY, N. E. (1974). On the weak convergence of empirical processes in sup-norm metrics. *Ann. Probab.* **2** 642–651.

PYKE, R. (1969). Applications of almost surely convergent constructions of weakly convergent processes. *Proc. Int. Symp. Prob. Inform. Th. Lecture Notes in Mathematics* **89** Springer-Verlag, New York.

PYKE, R. (1972). Empirical Processes. In *Jeffery-Williams Lectures: 1968–1972.* Canadian Mathematical Congress, Montreal.

PYKE, R. and SHORACK, G. R. (1968). Weak convergence of a two-sample empirical process and a new approach to Chernoff-Savage theorems. *Ann. Math. Statist.* **39** 755–771.

D. E. MATTHEWS
DEPARTMENT OF STATISTICS
AND ACTUARIAL SCIENCE
UNIVERSITY OF WATERLOO
WATERLOO, ONTARIO
CANADA N2L 3G1

V. T. FAREWELL
R. PYKE
DEPARTMENTS OF BIOSTATISTICS
AND MATHEMATICS
UNIVERSITY OF WASHINGTON
SEATTLE, WA 98195