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INADMISSIBILITY OF STUDENTIZED TESTS FOR NORMAL ORDER RESTRICTED MODELS[1]

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Consider the model where X_{ij}, $i = 1, \dots, k$; $j = 1, 2, \dots, n_i$; $n_i \geq 2$, are observed. Here X_{ij} are independent $N(\theta_i, \sigma^2)$, θ_i, σ^2 unknown. Let $X_i = \sum_{j=1}^{n_i} X_{ij}/n_i$, $\mathbf{X}' = (X_1, \dots, X_k)$, $\boldsymbol{\theta}' = (\theta_1, \dots, \theta_k)$, $V = \sum_{i=1}^{k} \sum_{j=1}^{n_i} X_{ij}^2 - n \sum_{i=1}^{k} X_i^2$. Let $\mathbf{A}_1$ be a $(k-m) \times k$ matrix of rank $(k-m) \geq 2$ and test H: $\mathbf{A}_1 \boldsymbol{\theta} = \mathbf{0}$ versus $K - H$ where K: $\mathbf{A}_1 \boldsymbol{\theta} \geq \mathbf{0}$.

Suppose we assume σ^2 known and consider a constant size α test ($\alpha < 1/2$) which is admissible for H versus $K - H$ based on $\mathbf{X}$. Next assume σ^2 is unknown. Consider the same test but now as a function of $\mathbf{X}/V^{1/2}$ (i.e., Studentize the test). The resulting test is inadmissible. Examples are noted.

1. Introduction and summary. Consider the model where X_{ij}, $i = 1, 2, \dots, k$; $j = 1, 2, \dots, n_i$ are observed. Here X_{ij} are independent normal random variables with unknown means θ_i and unknown variance σ^2. For ease of exposition only, we take $n_i = n$ and require $n \geq 2$. Let $X_i = \sum_{j=1}^{n} X_{ij}/n$, $\mathbf{X}' = (X_1, \dots, X_k)$, $\boldsymbol{\theta}' = (\theta_1, \dots, \theta_k)$, $U = \sum_{i=1}^{k} \sum_{j=1}^{n} X_{ij}^2$, $V = U - n\sum_{i=1}^{k} X_i^2$, $\bar{X} = \sum_{i=1}^{k} X_i/k$. Sufficient statistics are equivalently $(\mathbf{X}', U)$ and $(\mathbf{X}', V)$.

Let $\mathbf{A}$ be a $k \times k$ nonsingular matrix partitioned as $\begin{pmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{pmatrix}$, where $\mathbf{A}_1$ is $(k-m) \times k$ and $\mathbf{A}_2$ is $m \times k$, $0 \leq m \leq k-1$, with the rows of $\mathbf{A}_2$ orthogonal to the rows of $\mathbf{A}_1$. We wish to test H: $\mathbf{A}_1\boldsymbol{\theta} = \mathbf{0}$ versus $K - H$ where K: $\mathbf{A}_1\boldsymbol{\theta} \geq \mathbf{0}$. It will be convenient to regard H as the linear subspace of vectors $\{\boldsymbol{\theta}$: $\boldsymbol{\theta} \in \mathbb{R}^k$, $\mathbf{A}_1\boldsymbol{\theta} = \mathbf{0}\}$ and K as the polyhedral cone $\{\boldsymbol{\theta}$: $\boldsymbol{\theta} \in \mathbb{R}^k$, $\mathbf{A}_1\boldsymbol{\theta} \geq \mathbf{0}\}$. See Cohen, Kemperman and Sackrowitz (1993) for a wide variety of problems in which the above hypotheses are appropriate.

Now suppose a constant size α test ($\alpha < 1/2$), that depends on $\mathbf{X}$, is given for the problem of testing H versus $K - H$ when σ^2 is known. Assume the given test is admissible when σ^2 is known. For σ^2 unknown, consider the same test function but now as a function of $\mathbf{X}/V^{1/2}$. We call this Studentizing. For $(k-m) \geq 2$, the resulting test is then inadmissible. Examples of some popular tests which are inadmissible by virtue of the above finding appear in Hayter (1990), Marcus (1976), Williams (1977), Robertson and Wright (1985), where Dunnett's test is discussed, and Mukerjee, Robertson and Wright (1985). Cohen and Sacrkowitz (1992) discuss improved tests for some specific problems and offer better tests, new tests, and do a Monte Carlo study of

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amounts of improvement. In specific problems, the likelihood ratio test (LRT) does not divide $\mathbf{X}$ by $V^{1/2}$, but by $(U - kn\overline{X}^2)^{1/2}$. In specific examples the LRT is admissible.

We prove the result in the next section.

2. Inadmissibility of Studentized tests. Let $\phi(\mathbf{x}', u)$ be a test function. Let $\mathbf{w}$ be a $k \times 1$ vector.

DEFINITION. A test $\phi(\mathbf{x}', u)$ is said to be monotone as a function of $(\mathbf{x}', u)$ with respect to (w.r.t.) $\mathbf{w}$ (in the direction $\mathbf{w}$) if

$$\phi(\mathbf{x}', u) \le \phi(\mathbf{x}' + \lambda\mathbf{w}', u) \quad \text{for all } \lambda \ge 0, \text{ all } u. \tag{2.1}$$

Let $\mathbf{w}_i'$, $i = 1, 2, \dots, k - m$, be the rows of $\mathbf{A}_1$. Then $K = \{\boldsymbol{\theta} \in \mathbb{R}^k\colon \mathbf{w}_i'\boldsymbol{\theta} \ge \mathbf{0}$, $i = 1, 2, \dots, k - m\}$. Also let Γ denote the space spanned by $\mathbf{w}_1, \dots, \mathbf{w}_{k-m}$. Finally, let $\mathbf{Y}(\mathbf{X}) = \mathbf{B}'(\mathbf{X})$ with $\mathbf{B} = \mathbf{A}^{-1}$. Let $\mathbf{e}_i$ be the ith unit vector, $i = 1, 2, \dots, k - m$.

LEMMA 2.1. *Let $1 \le i \le k - m$ be fixed. Then a test function is monotone as a function ϕ of $(\mathbf{x}', u)$ w.r.t. $\mathbf{w}_i$ (in the direction $\mathbf{w}_i$) if and only if as a function $\phi_1(\mathbf{y}', u)$ it is monotone w.r.t. $\mathbf{e}_i$ (in the direction $\mathbf{e}_i$).*

PROOF. Note that $\mathbf{Y}(\mathbf{X} + \lambda\mathbf{w}_i) = \mathbf{Y}(\mathbf{X}) + \lambda\mathbf{e}_i$. Hence the lemma follows. □

The transformation to $\mathbf{Y}$ is handy in establishing the following complete class result. Let $\mathbf{Y} = (\mathbf{Y}^{(1)}, \mathbf{Y}^{(2)})$ where $\mathbf{Y}^{(1)}$ is $(k - m) \times 1$.

THEOREM 2.2. *A complete class of tests for H versus $K - H$ are those tests $\psi(\mathbf{y}', u)$ such that*

$$\psi(\mathbf{y}', u) \text{ is monotone as a function of } (\mathbf{y}', u) \text{ w.r.t. } \mathbf{e}_i, \quad i = 1, 2, \dots, k - m, \tag{2.2}$$

and

(2.3) *For fixed $(\mathbf{y}^{(2)}, u)$, the acceptance sections of the tests are convex.*

PROOF. Consider the joint density of $(\mathbf{x}', u)$. For $n\sum_{i=1}^k x_i^2 \le u$, it is

$$\begin{aligned} f_{\boldsymbol{\theta}, \sigma^2} &= K(\boldsymbol{\theta}, \sigma^2) h(u) \exp - \frac{1}{2\sigma^2}\left[n(\mathbf{x} - \boldsymbol{\theta})'(\mathbf{x} - \boldsymbol{\theta}) + u - n\sum_{i=1}^k x_i^2\right] \\ &= K^*(\boldsymbol{\theta}, \sigma^2) h(u) \exp\left[\left(n\mathbf{x}'\frac{\boldsymbol{\theta}}{\sigma^2}\right) - \frac{u}{2\sigma^2}\right] \\ &= K^*(\boldsymbol{\theta}, \sigma^2) h(u) \exp\left[\left(n\mathbf{x}'\mathbf{B}\mathbf{A}\frac{\boldsymbol{\theta}}{\sigma^2}\right) - \frac{u}{2\sigma^2}\right] \\ &= K^*(\boldsymbol{\theta}, \sigma^2) h(u) \exp\left[\mathbf{y}'\boldsymbol{\nu} - \frac{u}{2\sigma^2}\right] \\ &= K^*(\boldsymbol{\theta}, \sigma^2) h(u) \exp\left[\mathbf{y}^{(1)\prime}\boldsymbol{\nu}^{(1)} + \mathbf{y}^{(2)\prime}\boldsymbol{\nu}^{(2)} - \frac{u}{2\sigma^2}\right], \end{aligned} \tag{2.4}$$

where $\boldsymbol{\nu} = n\mathbf{A}\boldsymbol{\theta}/\sigma^2$. Note that under H, $(\mathbf{y}^{(2)\prime}, u)$ are sufficient complete statistics and now the theorem follows from Eaton (1970). □

REMARK 2.3. Condition (2.2) could also be described as $\phi^*(\mathbf{x}', u)$ is monotone as a function of $(\mathbf{x}', u)$ w.r.t. $\mathbf{w}_i$, $i = 1, 2, \ldots, k - m$. Here u is kept fixed. Also observe that there is an important distinction between the expressions of a test function as a function $\phi(\mathbf{x}', v)$ of the variables $(\mathbf{x}, v)$ and as a function $\phi'(\mathbf{x}', u)$ of the variables $(\mathbf{x}', u)$ since v is a function of both u and $\mathbf{x}$.

REMARK 2.4. For the same model discussed above (σ^2 unknown), if σ^2 is known the complete class theorem version of Theorem 2.2 is the same as Theorem 2.2 save u is erased whenever it appears.

Now we proceed with the result on Studentizing. To avoid confusion in what follows, we call attention to the fact that sets such as H, K and Γ, defined earlier, that were viewed as subsets of the parameter space are subsets of $\mathbb{R}^k$. As such, they can be (and often will be) viewed as subsets of the sample space. We begin with consideration of the variance known ($\sigma^2 = 1$) case for size $\alpha < 1/2$. Say we decide to use the test $\phi_R(\mathbf{x})$ which we may assume has the form

$$\phi_R(\mathbf{x}) = \begin{cases} 0, & \text{if } \mathbf{x} \in R, \\ 1, & \text{if } \mathbf{x} \notin R, \end{cases}$$

where R is a closed subset of $\mathbb{R}^k$. We will study the behavior of $\phi_R(\mathbf{x})$ when it is restricted to subspaces where $\mathbf{A}_2\mathbf{x}$ is fixed. Since the rows of $\mathbf{A}_1$ are orthogonal to those of $\mathbf{A}_2$,

$$\{\mathbf{x}\colon A_2\mathbf{x} = \mathbf{b}\} = \left\{\mathbf{x}\colon \mathbf{x} = \mathbf{b}^* + \sum_{k=1}^{k-m} \lambda_i \mathbf{w}_i\right\} = \Gamma_{\mathbf{b}}, \quad \text{say},$$

where $\mathbf{b}^* \in H$ is uniquely determined by $\mathbf{A}_2\mathbf{b}^* = \mathbf{b}$.

For notational simplicity, we will take $\mathbf{b} = \mathbf{0}$ (and so $\mathbf{b}^* = \mathbf{0}$). This is done without loss of generality as the results we are seeking concerning acceptance sections can be obtained by considering the projection of an acceptance section on $\Gamma_0 = \Gamma$ and the fact that this projection will have the same geometric properties as the original section.

We will say that, on section Γ_0, ϕ_R has:

PROPERTY 1. $E_{\boldsymbol{\theta}_0}\{\phi_R(\mathbf{x}) | \mathbf{A}_2\mathbf{X} = \mathbf{0}\} = \alpha$, for $\boldsymbol{\theta}_0 \in H$;

PROPERTY 2. $R_0 = R \cap \Gamma_0$ is convex;

PROPERTY 3. $\mathbf{x} \in R^c \cap \Gamma_0$ implies $\mathbf{x} + \lambda\mathbf{w}_i \in R^c \cap \Gamma_0$ for all $\lambda > 0$, $i = 1, \ldots, k - m$.

Before going through the details of our argument, we will describe the basic ideas in words. Recall that knowledge of $\mathbf{Y}^{(2)}$ is equivalent to knowledge of

$\mathbf{A}_2\mathbf{X}$. Thus $\mathbf{A}_2\mathbf{X}$ and $(\mathbf{A}_2\mathbf{X}, U)$, respectively, are complete sufficient statistics, under H, when the variance is known and unknown. The complete class results, Theorem 2.2 and Remark 2.4, require that any admissible test be monotone and convex on almost all sections. That is, they have Properties 2 and 3 above for almost all $(\mathbf{A}_2\mathbf{X})$ or $(\mathbf{A}_2\mathbf{X}, U)$, respectively, in the variance known or unknown case. Also, if a test has size α, completeness of the sufficient statistic implies conditional size α, that is, Property 1. The thrust of the proof of the main result of this section is to demonstrate that if the test $\phi(\mathbf{x})$ has Properties 1, 2 and 3 on some section $\mathbf{A}_2\mathbf{x} = \mathbf{0}$, then the test $\phi(\mathbf{x}/v^{1/2})$ will *not* be monotone on the sections $(\mathbf{A}_2\mathbf{x}, U) = (\mathbf{0}, \mathbf{u})$. Thus $\phi(\mathbf{x})$ and $\phi(\mathbf{x}/\sqrt{v})$ cannot both be admissible in their respective problems.

LEMMA 2.5. *If ϕ_R has Properties* 1 *and* 2 *on section Γ_0, then there exists an $\varepsilon > 0$ such that* $\{\mathbf{x}: \|\mathbf{x}\|^2 \le \varepsilon\} \cap \Gamma_0 \subseteq R_0$.

PROOF. The lemma asserts that $\mathbf{0}$ is an interior point of R_0. If this is not the case, then since R_0 is convex there exists a hyperplane $\boldsymbol{\gamma}'\mathbf{x} = 0$ through $\mathbf{0}$ such that R_0 is on one side of this hyperplane. That is, $\boldsymbol{\gamma}'\mathbf{x} \ge 0$, say, all $\mathbf{x} \in R_0$. Recall that $\mathbf{0} \in \mathbf{H}$ so that, by Property 1,

$$\alpha = E_0\{\phi_R(\mathbf{X})|\mathbf{A}_2\mathbf{X} = \mathbf{0}\} = P_0(R^c|\mathbf{A}_2\mathbf{X} = \mathbf{0})$$
$$= \frac{P_0(R^c \cap \Gamma_0)}{P_0(\Gamma_0)} \ge \frac{P_0(\boldsymbol{\gamma}'\mathbf{x} \le 0, \Gamma_0)}{P_0(\Gamma_0)} = \frac{1}{2},$$

by symmetry of the normal distribution. This contradicts the assumption that $\alpha < 1/2$. □

LEMMA 2.6. *Assume ϕ_R has Properties* 1 *and* 2 *on section Γ. For any boundary point $\boldsymbol{\xi}$ of R_0, there exists a $\boldsymbol{\gamma} \in \Gamma$ such that* (i) $\mathbf{x} \in R_0 \Rightarrow \boldsymbol{\gamma}'\mathbf{x} \le c_{\gamma}$, (ii) $\boldsymbol{\gamma}'\mathbf{x} > c_{\gamma} \Rightarrow \mathbf{x} \in R_0^c$ *and* (iii) $c_{\gamma} = \boldsymbol{\gamma}'\boldsymbol{\xi} \ge \|\boldsymbol{\gamma}\|\varepsilon$, *where ε is defined by Lemma* 2.5.

PROOF. Since R_0 is convex, the supporting hyperplane theorem guarantees the existence of such a hyperplane, through $\boldsymbol{\xi}$, for some $\boldsymbol{\gamma} \in \Gamma$. To establish $c_{\gamma} \ge \|\boldsymbol{\gamma}\|\varepsilon$, note that by Lemma 2.5, $(\varepsilon/\|\boldsymbol{\gamma}\|)\boldsymbol{\gamma} \in R_0$ and so $\boldsymbol{\gamma}'((\varepsilon/\|\boldsymbol{\gamma}\|)\boldsymbol{\gamma}) \le c_{\gamma}$, which completes the proof. □

LEMMA 2.7. *Assume ϕ_R has Properties* 1, 2 *and* 3 *on section Γ. Then* $-\lambda\mathbf{w}_i \in R_0$ *for all* $\lambda > 0$, $i = 1, \dots, k - m$.

PROOF. Immediate since ϕ_{R_0} is monotone w.r.t. $\mathbf{w}_i$ and $\mathbf{0} \in R_0$. □

LEMMA 2.8. *Assume ϕ_R has Properties* 1, 2 *and* 3 *on section Γ. If $\mathbf{x} \in R_0^c$, then $\lambda\mathbf{x} \in R_0^c$ for all $\lambda > 1$.*

PROOF. Follows from Lemma 2.6 as $\boldsymbol{\gamma}'(\lambda\mathbf{x}) = \lambda\boldsymbol{\gamma}'\mathbf{x} \ge c_{\gamma}$ for $\lambda > 1$. □

LEMMA 2.9. *Assume ϕ_R has Properties* 1, 2 *and* 3 *on section* Γ. *There exists a point* $\boldsymbol{\xi} \in R_0^c$ *such that* $\mathbf{w}'\boldsymbol{\xi} < 0$ *for some* $\mathbf{w} = \mathbf{w}_1, \dots, \mathbf{w}_{k-m}$.

PROOF. If not, $\mathbf{w}_i'\boldsymbol{\xi} \geq 0$ for all $i = 1, \dots, k - m$ and all $\boldsymbol{\xi} \in R_0^c$. This would imply $R_0^c \subseteq K$ and so $K^c \subseteq R_0$. But R_0 is convex and (in two or more dimensions) the convex hull of the complement of a polyhedral cone is the entire space. Therefore, $\mathbb{R}^{k-m} \subseteq R_0$ and the size of the test would be 0 which is a contradiction. □

Now let $\boldsymbol{\xi} \in R_0^c$ be a reject point and $\mathbf{w}$ one of the vectors $\mathbf{w}_1, \dots, \mathbf{w}_{k-m}$ such that (as guaranteed by Lemma 2.9)

$$\mathbf{w}'\boldsymbol{\xi} < 0. \tag{2.5}$$

By Lemma 2.8, $\lambda\boldsymbol{\xi} \in R_0^c$ for all $\lambda > 1$ and by Lemma 2.7, $-\lambda\mathbf{w} \in R_0$ for all $\lambda > 0$. Thus, for each $\lambda > 1$, there exists a $0 \leq \tau_\lambda \leq 1$ such that

$$\boldsymbol{\xi}_\lambda = \lambda((1 - \tau_\lambda)\boldsymbol{\xi} - \tau_\lambda\mathbf{w}) \tag{2.6}$$

is a boundary point of R_0. By Lemma 2.6, there exists a $\boldsymbol{\gamma}_\lambda \in \Gamma$ such that

$$\mathbf{x} \in R_0 \quad \Rightarrow \quad \boldsymbol{\gamma}_\lambda'\mathbf{x} \leq \boldsymbol{\gamma}_\lambda'\boldsymbol{\xi}_\lambda, \tag{2.7}$$

$$\boldsymbol{\gamma}_\lambda'\mathbf{x} > \boldsymbol{\gamma}_\lambda'\boldsymbol{\xi}_\lambda \quad \Rightarrow \quad \mathbf{x} \in R_0^c, \tag{2.8}$$

$$\boldsymbol{\gamma}_\lambda'\boldsymbol{\xi}_\lambda \geq \|\boldsymbol{\gamma}_\lambda\|\varepsilon. \tag{2.9}$$

We can now consider the variance unknown problem with sufficient statistics $X_1, \dots, X_k, U$. We will study the test $\phi^*(\mathbf{x}', u) = \phi_R(\mathbf{x}'/\sqrt{v})$ where $v = u - n\|\mathbf{x}\|^2$. Consider the section where $\mathbf{A}_2\mathbf{X} = \mathbf{0}$ and $U = u$. We begin with the points $(\mathbf{x}_\lambda, u)$ which map into the points $\boldsymbol{\xi}_\lambda$. That is, $\mathbf{x}_\lambda/\sqrt{v_\lambda} = \boldsymbol{\xi}_\lambda$ where $v_\lambda = u - n\|\mathbf{x}_\lambda\|^2$. The points $(\mathbf{x}_\lambda, u)$ are boundary points of the acceptance region of ϕ^*. Next we look at points of the form $(\mathbf{x}_{\lambda,a}, u)$ where $\mathbf{x}_{\lambda,a} = \mathbf{x}_\lambda - a\mathbf{w}$, $a > 0$. All such points must be in the acceptance region of ϕ^* if ϕ^* is to be monotone w.r.t. $\mathbf{w}$. Equivalently, $\boldsymbol{\xi}_{\lambda,a} = \mathbf{x}_{\lambda,a}/\sqrt{v_{\lambda,a}}$ where $v_{\lambda,a} = u - n\|\mathbf{x}_{\lambda,a}\|^2$, must be in R_0. We will show that, for sufficiently large λ and small a, this is not the case. Figures 1 and 2 reflect the situation.

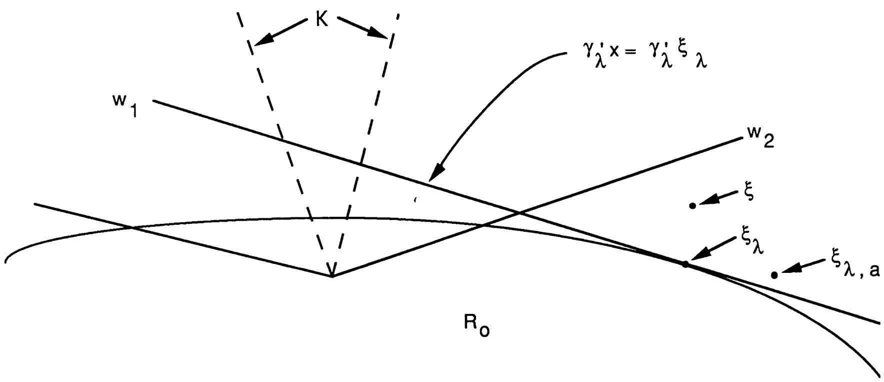

FIG. 1. $\{\mathbf{x}: \mathbf{A}_2\mathbf{x} = \mathbf{0}\}$.

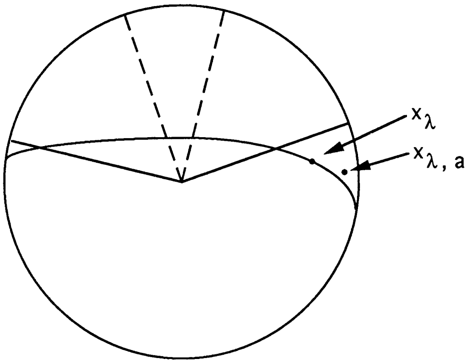

FIG. 2. $\{\mathbf{x}: \mathbf{A}_2\mathbf{x} = \mathbf{0}, n\|\mathbf{x}\|^2 \le u\}$.

LEMMA 2.10. *Assume ϕ_R has Properties* 1, 2 *and* 3 *on section* Γ. *There exists* $\lambda > 1$ *and* $a > 0$ *such that* $\boldsymbol{\xi}_{\lambda, a} \in R_0^c$.

PROOF. By (2.8), we only need show that $\boldsymbol{\gamma}_\lambda'\boldsymbol{\xi}_{\lambda,a} > \boldsymbol{\gamma}_\lambda'\boldsymbol{\xi}_\lambda$ for some $\lambda > 1$, $a > 0$. Note

$$\begin{aligned}\boldsymbol{\gamma}_\lambda'\boldsymbol{\xi}_{\lambda,a} - \boldsymbol{\gamma}_\lambda'\boldsymbol{\xi}_\lambda &= \frac{\boldsymbol{\gamma}_\lambda'(\mathbf{x}_\lambda - a\mathbf{w})}{\sqrt{v_{\lambda,a}}} - \frac{\boldsymbol{\gamma}_\lambda'\mathbf{x}_\lambda}{\sqrt{v_\lambda}}\\ &= \left(\frac{1}{\sqrt{v_{\lambda,a}}} - \frac{1}{\sqrt{v_\lambda}}\right)\boldsymbol{\gamma}_\lambda'\mathbf{x}_\lambda - \frac{a\boldsymbol{\gamma}_\lambda'\mathbf{w}}{\sqrt{v_{\lambda,a}}} \qquad (2.10)\\ &= \frac{a}{\sqrt{v_{\lambda,a}}}\left[\frac{\left(\sqrt{v_\lambda} - \sqrt{v_{\lambda,a}}\right)}{a}\frac{\boldsymbol{\gamma}_\lambda'\mathbf{x}_\lambda}{\sqrt{v_\lambda}} - \boldsymbol{\gamma}_\lambda'\mathbf{w}\right].\end{aligned}$$

Now

$$\begin{aligned}\frac{\sqrt{v_\lambda} - \sqrt{v_{\lambda,a}}}{a} &= \frac{v_\lambda - v_{\lambda,a}}{a\left(\sqrt{v_\lambda} + \sqrt{v_{\lambda,a}}\right)} = \frac{(-2\mathbf{x}_\lambda'\mathbf{w} + a\mathbf{w}'\mathbf{w})}{\sqrt{v_\lambda} + \sqrt{v_{\lambda,a}}} \cdot n\\ &\to_{a\to 0} \frac{-2\mathbf{x}_\lambda'\mathbf{w}}{2\sqrt{v_\lambda}} \cdot n = -\boldsymbol{\xi}_\lambda'\mathbf{w} \cdot n.\end{aligned}$$

Thus, (2.10) will be greater than 0 for some small $a > 0$ if

$$n(-\boldsymbol{\xi}_\lambda'\mathbf{w})(\boldsymbol{\gamma}_\lambda'\boldsymbol{\xi}_\lambda) - \boldsymbol{\gamma}_\lambda'\mathbf{w} > 0. \qquad (2.11)$$

Using (2.9) and the Cauchy–Schwarz inequality we have

$$n(-\boldsymbol{\xi}_\lambda'\mathbf{w})(\boldsymbol{\gamma}_\lambda'\boldsymbol{\xi}_\lambda) - \boldsymbol{\gamma}_\lambda'\mathbf{w} \ge n(-\boldsymbol{\xi}_\lambda'\mathbf{w})\|\boldsymbol{\gamma}_\lambda\|\varepsilon - \|\boldsymbol{\gamma}_\lambda\|\,\|\mathbf{w}\|.$$

The result now follows as (2.5) and (2.6) imply that $-\boldsymbol{\xi}_\lambda'\mathbf{w} \to \infty$ (uniformly in τ_λ) as $\lambda \to \infty$. □

THEOREM 2.11. *If $\phi_R(\mathbf{X}')$ is admissible size α test, where $0 < \alpha < 1/2$, for the variance known case, then $\phi_R^*(\mathbf{X}', U) = \phi_R(\mathbf{X}'/\sqrt{V})$ is inadmissible in the unknown variance case.*

PROOF. As discussed earlier, the results obtained for R_0 also hold for all $R_b = R \cap \Gamma_b$. By Remark 2.4, ϕ_R has Properties 1, 2 and 3 almost everywhere $\mathbf{A}_2\mathbf{X}$ (i.e., on almost all sections R_b). But by Lemma 2.10, this means that ϕ_R^* is *not* monotone on almost all sections $(\mathbf{A}_2\mathbf{X}, U)$. Thus, use of Theorem 2.2 completes the proof. □

REMARK 2.12. The result of Theorem 2.11 holds even if ϕ_R is not exact but is of level α.

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