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ASYMPTOTIC PROPERTIES OF SELF-CONSISTENT ESTIMATORS BASED ON DOUBLY CENSORED DATA

BY M. G. GU[1] AND C.-H. ZHANG[2]

McGill University and Rutgers University

This paper concerns self-consistent estimators for survival functions based on doubly censored data. We establish strong uniform consistency, asymptotic normality and asymptotic efficiency of the estimators under mild conditions on the distributions of the censoring variables.

1. Introduction. In biometry and reliability studies, the distribution function of the underlying lifetime is often of special interest. It is common in these problems that the observations are incomplete, therefore the empirical distribution function is not appropriate in estimating the distribution function of the lifetime. In the right-censoring case the product limit estimate of Kaplan and Meier (1958) has been generally accepted as a substitute for the empirical distribution function, since it is the nonparametric maximum likelihood estimator (NPMLE) [Cox and Oakes (1984), page 48] and possesses the properties of self-consistency [Efron (1967)] asymptotic normality [Breslow and Crowley (1974)], and asymptotic efficiency [Wellner (1982)]. In cases other than right censoring, it is natural to search for estimates which possess similar properties. For detailed discussions, see Tsai and Crowley (1985) and Gill (1989).

In the case where observations have the possibility of being censored either from right or left, Turnbull (1974) proposed a self-consistent algorithm to find the NPMLE for the underlying distribution. Chang and Yang (1987) proved the asymptotic consistency of self-consistent estimates under mild conditions on the censoring for continuous distributions with support $(0, \infty)$, and Chang (1990) obtained the asymptotic normality on compact intervals under a quite strong additional condition on the censoring. Some related models were studied by Ayer, Brunk, Ewing, Reid and Silverman (1955), Groeneboom (1987) and Samuelsen (1989). In this paper, we shall generalize the consistency result to noncontinuous distributions with an arbitrary support, establish the asymptotic normality on the entire range of the lifetime under weaker conditions on censoring and show that Turnbull's estimator is asymptotically efficient.

2. Main results. Let X_i, $i \geq 1$, be independent identically distributed (i.i.d.) random variables with a common survival function S_X. Independent of

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the X_i's let $Y_i \geq Z_i$, $i \geq 1$, be i.i.d. pairs of censoring times with possibly defective marginal survival functions S_Z and S_Y. For $i \geq 1$ set

$$\tag{2.1} V_i = \max(\min(X_i, Y_i), Z_i), \qquad \delta_i = \begin{cases} 1, & \text{if } Z_i < X_i \leq Y_i, \\ 2, & \text{if } V_i = Y_i < X_i \text{ (right censoring)}, \\ 3, & \text{if } V_i = Z_i \geq X_i \text{ (left censoring)}. \end{cases}$$

We study asymptotic properties of self-consistent estimators for S_X based on (V_i, δ_i), $1 \leq i \leq n$.

Let $Q_n^{(j)}$, $1 \leq j \leq 3$, Q_n and $Q_n^{(0)}$ be the empirical version of

$$\tag{2.2} \begin{aligned} Q^{(j)}(t) &= \Pr\{V > t, \delta = j\}, \qquad 1 \leq j \leq 3, \\ Q &= (Q^{(1)}, Q^{(2)}, Q^{(3)}), \qquad Q^{(0)} = \sum_{j=1}^{3} Q^{(j)}. \end{aligned}$$

Set $K = S_Y - S_Z$. It follows from (2.1) that $\Pr\{\delta = 1 | X = t\} = K(t-)$ and

$$\tag{2.3} dQ^{(1)}(t) = K(t-)\, dS_X(t), \qquad dQ^{(2)} = S_X\, dS_Y, \qquad dQ^{(3)} = (1 - S_X)\, dS_Z.$$

By Tsai and Crowley [(1985), page 1328], the estimate S_n is self-consistent if

$$\tag{2.4} S_n(t) = Q_n^{(0)}(t) - \int_{u \leq t} \frac{S_n(t)}{S_n(u)} dQ_n^{(2)}(u) + \int_{t<u} \frac{1 - S_n(t)}{1 - S_n(u)} dQ_n^{(3)}(u),$$

where $\int_{u \leq t} = 0$ ($\int_{t<u} = 0$) if $S_n(t) = 0$ [$S_n(t) = 1$]. The NPMLE of S_X is self-consistent [Turnbull (1974)], but a self-consistent estimate is not necessarily an NPMLE. For example, with four observations $(1,1)$, $(2,2)$, $(3,3)$ and $(4,3)$ from (V, δ), S_n defined by $dS_n(1) = -2/3$ and $dS_n(4) = -1/3$, is self-consistent but not an NPMLE, which essentially puts mass $1/2$ at 1 and 3. Among other things, an NPMLE must satisfy

$$\tag{2.5} \begin{aligned} Q_n^{(2)}(-\infty) - Q_n^{(2)}(a_n -) &= -\int_{u<a_n} \frac{dQ_n^{(2)}(u)}{S_n(u)}, \\ Q_n^{(3)}(b_n) &= -\int_{b_n<u} \frac{dQ_n^{(3)}(u)}{1 - S_n(u)}, \end{aligned}$$

where $a_n = \min\{V_i\colon \delta_i = 1 \text{ or } 3\}$ and $b_n = \max\{V_i\colon \delta_i = 1 \text{ or } 2\}$. This constraint is useful later.

Let $\|h\| = \sup_t |h(t)|$ be the supremum norm throughout the paper. Chang and Yang (1987) proved $\|S_n - S_X\| \to 0$ a.s. for S_n satisfying (2.4) under the assumptions that

$$\tag{2.6} K(t-) = \Pr\{\delta = 1 | X = t\} > 0$$

for all $t > 0$, that S_X, S_Y and S_Z are all continuous, and that the support of

X is $(0, \infty)$. Our first result gives the strong uniform consistency under a single condition on the censoring.

THEOREM 1. *Suppose* (2.6) *holds on* $\{t: S_X(t) < 1$ *and* $S_X(t-) > 0\}$. *Then*, $\|S_n - S_X\| \to 0$ *a.s.*

Since $Q_n \to Q$ uniformly and S_n satisfies (2.4), $S_{n_k}(t) \to S(t)$ for each t as $n_k \to \infty$ implies

$$(2.7)\quad S(t) = Q^{(0)}(t) - \int_{u \le t} \frac{S(t)}{S(u)} dQ^{(2)}(u) + \int_{t<u} \frac{1-S(t)}{1-S(u)} dQ^{(3)}(u).$$

Theorem 1 is proved in Section 3 via the uniqueness of the solution of (2.7). The method of Chang and Yang [(1987), (2.5) and Lemma 4.1] cannot be used, because their fundamental integral equation system is not continuous under pointwise convergence.

A sequence of estimators $\{S_n\}$ is asymptotically normal if $\sqrt{n}(S_n - S_X)$ converges in distribution to a Gaussian process in a suitable metric space of functions. In this paper, a Banach space is always equipped with its ball σ-algebra, and random elements and convergence in distribution are defined as in Pollard [(1984), page 65]. For any survival function S define linear operators A_S, R_S, K and B_S by

$$(2.8)\quad \begin{aligned}(A_S h)(t) = &\int_{u \le t} \frac{S(t)}{S(u)} h(u)\, dS_Y(u)\\ &+ \int_{t<u} \frac{1-S(t)}{1-S(u)} h(u)\, dS_Z(u),\end{aligned}$$

$$(2.9)\quad R_S = K - A_S, \quad (Kh)(t) = K(t)h(t), \quad K(t) = S_Y(t) - S_Z(t),$$

$$(2.10)\quad \begin{aligned}\big(B_S(h^{(1)}, h^{(2)}, h^{(3)})\big)(t) = &\sum_{j=1}^{3} h^{(j)}(t) - \int_{u \le t} \frac{S(t)}{S(u)} dh^{(2)}(u)\\ &+ \int_{t<u} \frac{1-S(t)}{1-S(u)} dh^{(3)}(u),\end{aligned}$$

where integrating by parts should be used in (2.10) whenever necessary. Domains of these operators include all bounded measurable functions, and we shall extend those of A_S and R_S under the conditions of Theorem 2. By (2.3) and (2.8), $A_{S_n}(S_n - S_X) = B_{S_n}Q - Q^{(0)} + S_n(S_Y - 1) + (1 - S_n)S_Z$. Since $S_n = B_{S_n}Q_n$ by (2.4) and $Q^{(0)} = KS_X + S_Z$ by (2.1),

$$(2.11)\quad R_{S_n}\xi_n = B_{S_n}W_n, \qquad \xi_n = \sqrt{n}(S_n - S_X), \qquad W_n = \sqrt{n}(Q_n - Q).$$

Let $(D, \|\cdot\|)$ be the Banach space of all real-valued functions defined on $(-\infty, \infty)$ which are right-continuous and have left limits at $x \le \infty$ and right limit at $-\infty$, and $(D[a,b], \|\cdot\|)$ the restrictions of $h \in D$ on $[a, b]$, where

$a = \sup\{t\colon S_X(t) = 1\}$ and $b = \inf\{t\colon S_X(t) = 0\}$. Define Banach spaces

$$(D_0[a,b], \|\cdot\|) = \big\{h \in D[a,b]\colon S_X(x) = 1 \Rightarrow h(x) = 0, \\ S_X(x-) = 0 \Rightarrow h(x-) = 0\big\},$$

$$(D_K[a,b], \|\cdot\|_K) = \{h\colon Kh \in D[a,b]\}, \qquad \|h\|_K = \|Kh\|,$$

$$(D_0^3, \|\cdot\|_3) = \big\{h \in D \otimes D \otimes D\colon B_{S_X}h \in D_0[a,b]\big\},$$

$$\|(h^{(1)}, h^{(2)}, h^{(3)})\|_3 = \sum_{j=1}^{3} \|h^{(j)}\|.$$

Since Q_n is the empirical version of Q and by (2.10) $B_{S_X}(Q_n - Q) \in D_0[a,b]$, it follows from (2.2) that

$$W_n \to_{\mathscr{D}} W = (W^{(1)}, W^{(2)}, W^{(3)}) \quad \text{in } D_0^3 \tag{2.12}$$

with $EW^{(j)}(t) = 0$ and $EW^{(j)}(t)W^{(k)}(s) = Q^{(j)}(\max(t,s))I\{j = k\} - Q^{(j)}(t)Q^{(k)}(s)$.

THEOREM 2. *For each n let S_n be a solution of* (2.4) *such that either* (2.5) *holds or $S_n - S_X \in D_0[a,b]$. Suppose* (2.6) *holds on the set $\{t\colon S_X(t) < 1, S_X(t-) > 0\}$ and*

$$\int_{\tau < S_X(u) < 1} \frac{-dS_Y(u)}{S_Y(u) - S_Z(u)} + \int_{0 < S_X(u) < \tau} \frac{-dS_Z(u)}{S_Y(u) - S_Z(u)} < \infty, \quad \forall\, 0 < \tau < 1. \tag{2.13}$$

Then $R_{S_X}^{-1}$, the inverse of R_{S_X} in (2.9), *exists as a bounded operator from $D_0[a,b]$ to $D_K[a,b]$, and*

$$\sqrt{n}\,(S_n - S_X) = \xi_n \to_{\mathscr{D}} \xi = R_{S_X}^{-1} B_{S_X} W \quad \text{in } D_K[a,b], \tag{2.14}$$

where W is the Gaussian process in (2.12) *and* $\Pr\{B_{S_X}W = R_{S_X}\xi \in D_0[a,b]\} = 1$.

COROLLARY 1. *Let S_n be a solution of* (2.4). *If* $\inf_{0 < S_X(t) < 1} K(t-) > 0$, *then $\xi_n \to_{\mathscr{D}} R_{S_X}^{-1} B_{S_X} W$ in D.*

Theorem 2 is proved in Section 4. Conditions (2.4) and (2.5) hold for all NPMLE's. The condition $S_n - S_X \in D_0[a,b]$ says that the support of S_n is contained in that of S_X, which holds if $S_n(0) = 1$ and the support of S_X is $(0,\infty)$. Here the invertibility of R_S means $R_S R_S^{-1} h = h$ on $D_0[a,b]$. By (2.9), R_{S_X} is defined on the entire space $D_K'[a,b]$ if and only if (2.13) holds. By (2.6) and (2.13),

$$\inf\{K(t)\colon \tau < S_X(t) \le 1 - \tau\} > 0, \qquad \forall\, 0 < \tau < 1/2 \tag{2.15}$$

and $\|h\|_K = 0 \Rightarrow \|h\| = 0$. Therefore, (2.14) is equivalent to $K\xi_n \to_{\mathscr{D}} KR_{S_X}^{-1}B_{S_X}W$ in $D[a,b]$.

Chang (1990) proved the asymptotic normality of $\xi_n(t)$ as processes on $[0,T]$ under the conditions of Chang and Yang (1987) for the asymptotic consistency and the additional conditions that

$$(2.16)\quad \Pr\{Z = 0\} > 0, \qquad \Pr\{0 < Z < \delta\} = 0 \quad \text{and} \quad \Pr\{Z > T\} = 0$$

for some $0 < \delta \le T < \infty$. Compared with his results, Theorem 2 gives the convergence on the entire support of X under weaker and more appealing conditions. The asymptotic normality of the product limit estimate on $(0, \infty)$ was proved by Gill (1983). Under the conditions of Chang (1990), $K(t)$ is continuous and (2.6) and (2.16) hold, so that $\inf_{0 \le t \le T} K(t) > 0$ and $S_Z(T) = 0$ for some $0 < S_X(T) < 1$, which imply (2.13). Chang [(1990), page 393] remarked that the purpose of his condition (2.16) was to avoid singularity of certain integral equations and it was not clear whether his results were valid without (2.16). In the following example we discuss a truncation-censoring model in which (2.13) is always satisfied when S_X is identifiable. Truncation models have been considered by Linden-Bell (1971), Woodroofe (1985) and Lagakos, Barraj and De Gruttola (1988) among others.

EXAMPLE 1. Let X^0, Y^0 and Z^0 be independent random variables. Suppose that the data is completely truncated (no observation) when $Y^0 < Z^0$, and the random variable of interest X^0 is doubly censored by (Y^0, Z^0) when $Y^0 \ge Z^0$. Then, the observations (V_i, δ_i), $1 \le i \le n$, are given by (2.1) with

$$\Pr\{X_i \in dx, Y_i \in dy, Z_i \in dz\} = \Pr\{X^0 \in dx\}\Pr\{Y^0 \in dy, Z^0 \in dz | Y^0 \ge Z^0\}.$$

Let S_Y^0 and S_Z^0 be the survival functions of Y^0 and Z^0, respectively. Then, $dS_Y(u) = \alpha(1 - S_Z^0(u))\, dS_Y^0(u)$, $dS_Z(u) = \alpha S_Y^0(u)\, dS_Z^0(u)$ and $S_Y(u) - S_Z(u) = \alpha(1 - S_Z^0(u))S_Y^0(u)$, where $\alpha = 1/\Pr\{Y^0 \ge Z^0\}$. Clearly, the survival function S_X is identifiable if and only if $S_Y^0 = 0 \Rightarrow S_X = 0$ and $S_Z^0 = 1 \Rightarrow S_X = 1$, and in this case conditions (2.6) and (2.13) are both satisfied and the asymptotic normality (2.14) holds for the NPMLE. But if $\Pr\{0 < Z^0 < \delta\} > 0$ for all $\delta > 0$ or $\Pr\{Z^0 = 0\} = 0$, then (2.16) does not hold and the result of Chang (1990) does not apply.

We shall also establish the asymptotic efficiency of self-consistent estimators here by proving a Hájek convolution theorem, which extends the results of Beran (1977) and Wellner (1982) to doubly censored data. Given a Banach space, a sequence of estimators $\{\tilde{S}_n\}$, $\tilde{S}_n$ based on (V_i, δ_i), $1 \le i \le n$, is regular if there exists a random element $\tilde{\xi}$ such that $\mathscr{L}(n^{1/2}(\tilde{S}_n - S_{X,n}); Q_{(n)})$ converges in distribution to $\mathscr{L}(\tilde{\xi}; Q)$ for all sequences of survival functions $\{Q_{(n)}\}$ of (V, δ) such that the joint distributions of $(V_1, \delta_1), \dots, (V_n, \delta_n)$ under $Q_{(n)}$ and those under Q are contiguous [Beran (1977)], where $\mathscr{L}(\cdot\,; Q)$ is the distribution under the probability measure corresponding to Q and $S_{X,n} \sim \mathscr{L}(X; Q_{(n)})$.

THEOREM 3. *Suppose the conditions of Theorem* 2 *hold. If* $\{\tilde{S}_n\}$ *is regular in* $D_K[a, b]$, *then*

$$\mathscr{L}\big(n^{1/2}(\tilde{S}_n - S_X); Q\big) \to_{\mathscr{D}} \mathscr{L}\big(\tilde{\xi}; Q\big) = \mathscr{L}(\xi + \zeta; Q) \quad \text{in } D_K[a, b],$$

where ξ *is as in Theorem* 2 *and* ζ *is some process independent of* ξ.

The proof of Theorem 3 is analogous to those in Beran (1977). We shall give a sketch in Section 5.

3. Proof of Theorem 1. Lemma 1 below, proved in the Appendix, will be used to show that S_X is completely determined by Q via (2.7), so that we can regard it as an identifiability result. It has two parts: Part (i) deals with functions S and h which are not necessarily right-continuous, while part (ii), used in Section 4, concerns functions h whose support is larger than that of S. A set J of real numbers does not have a limit point from left if every nonempty subset of J contains its maximum.

LEMMA 1. *Let S be a $[0,1]$-valued nonincreasing function and h be a function such that*

$$h(t)K(t) = \int_{u \le t} \frac{S(t)}{S(u)} h(u)\, dS_Y(u) + \int_{t<u} \frac{1 - S(t)}{1 - S(u)} h(u)\, dS_Z(u), \qquad \forall\, t. \tag{3.1}$$

(i) *Suppose* (2.6) *holds on the set* $\{t\colon 0 < S(t) < 1\}$. *Then*, $h(t) = 0$ *for all* t, *provided that*

$$h(t+) \ne h(t) \Rightarrow S(t+) < S(t) \quad \text{on } \{t\colon 0 < S(t) < 1\}, \tag{3.2}$$

$$h(t) = 0 \quad \text{on } \{t\colon S(t) = 0 \text{ or } S(t) = 1\}. \tag{3.3}$$

(ii) *Let* $J_X = \{t\colon\ 0 < S_X(t) < 1\}$. *Suppose* (2.6) *holds on* J_X *and the set* $J_X \cap \{t\colon\ S_Y(t) \le S_Z(t-)\}$ *does not have a limit point from left. If S is right-continuous with* $\{t\colon\ 0 < S(t) < 1\} \subset J_X$, *$h$ is right-continuous for* $t \in J_X$ *and* $h(t) = 0$ *for* $t \notin J_X$, *then* $h(t) = 0$ *for all* t.

PROOF OF THEOREM 1.

STEP 1 [Uniqueness of the solution of (2.7)]. We shall first prove $S = S_X$ under

$$S_X(t) = 0 \Rightarrow S(t) = 0 \quad \text{and} \quad S_X(t) = 1 \Rightarrow S(t) = 1. \tag{3.4}$$

Set $h = S - S_X$. We only need to verify the conditions of Lemma 1(i). It can be shown in the same manner as in the derivation of (2.11) that (2.7) implies (3.1). By (3.4) and the condition of Theorem 1, (2.6) holds on the set $\{t\colon 0 < S(t) < 1\}$. Since $S_X(t)$ is right-continuous, (3.2) is obvious. By (2.6) $dQ^{(1)}(t) = K(t-)\, dS_X(t)$ and $K(t-) > 0$, so that by (2.7) $S(t) = 0 \Rightarrow Q^{(1)}(t) = 0 \Rightarrow S_X(t) = 0$ and $S(t) = 1 \Rightarrow Q^{(1)}(t) = Q^{(1)}(-\infty) \Rightarrow S_X(t) = 1$, which implies (3.3). Therefore, $S = S_X$.

Step 1 is completed if we can drop (3.4). To this end we need

$$\int_{b \le u} \frac{dQ^{(3)}(u)}{1 - S(u)} = \int_{b \le u} dQ^{(3)}(u), \qquad \int_{u<a} \frac{dQ^{(2)}(u)}{S(u)} = \int_{u<a} dQ^{(2)}(u). \tag{3.5}$$

Define $K_S(t) = 1 + \int_{u \le t} S^{-1}(u)\, dQ^{(2)}(u) + \int_{t<u}(1 - S(u))^{-1}\, dQ^{(3)}(u)$. It follows from (2.7) that $K_S(t-)\,dS(t) = dQ^{(1)}(t)$ and $S(t)K_S(t) = Q^{(0)}(t) + \int_{t<u}(1 - S(u))^{-1}\, dQ^{(3)}(u)$. Since $Q^{(2)}(b-) = Q^{(1)}(b) = 0$, $K_S(b-)S(b) = Q^{(3)}(b-) + \int_{b \le u}(1 - S(u))^{-1}\, dQ^{(3)}(u) \le 0$. Since $dQ^{(1)}(t) < 0$ at b or near $b-$, $K_S(b-) \ge 0$. and we have the first equation of (3.5). The second one can be obtained in the same manner.

Let $S_0(t) = S(t)I\{0 < S_X(t) < 1\} + I\{S_X(t) = 1\}$. Since (2.7) holds for S, by (3.5) it also holds for S_0. Since S_0 satisfies (3.4), we have $S_0 = S_X$, which implies $S(t) = S_X(t)$ for $0 < S_X(t) < 1$. If $dS_X(b) = 0$, then $S(b-) = S_X(b-) = 0$. Otherwise, (2.7) and (3.5) imply $K_S(b-)S(b) = 0$, and $dQ^{(1)}(b) < 0$ ensures $K_S(b-) > 0$. In any case we have the first part of (3.4), and the proof for the second part is omitted.

Step 2 (Uniform consistency). By (2.4) all limit points of S_n must satisfy (2.7), so that by Step 1 and the Helly–Bray selection theorem we have $S_n(t) \to S_X(t)$ a.s. $\forall\, t$. If $dS_X(t) < 0$, then by (2.4) and (2.6),

$$\frac{dQ_n^{(1)}}{dS_n}(t) \le 1 + \int_{u<t-\varepsilon} \frac{dQ_n^{(2)}(u)}{S_n(u)} + \int_{t \le u} \frac{dQ_n^{(3)}(u)}{1 - S_n(u)} \to \frac{dQ^{(1)}}{dS_X}(t)$$

as $n \to \infty$ and then $\varepsilon \to 0+$, which implies $|dS_n(t)| \ge (1 - o(1))|dS_X(t)|$, since $dQ_n^{(1)}/dQ^{(1)}(t) \to 1$. Hence, $\|S_n - S_X\| \to 0$ a.s. $\square$

4. Proof of Theorem 2. Let $S_{X,m}, S_{Y,m}, S_{Z,m}$, $m \ge 1$, and S be survival functions such that

$$(4.1)\qquad \begin{gathered} S - S_X \in D_0[a,b], \qquad S(t) = 1 \Rightarrow S_{X,m}(t) = 1, \\ S(t-) = 0 \Rightarrow S_{X,m}(t-) = 0, \end{gathered}$$

$$(4.2)\qquad \begin{gathered} \|S_{X,m} - S\| \to 0, \qquad \|S_{Y,m} - S_Y\| \to 0, \qquad \|S_{Z,m} - S_Z\| \to 0, \\ K_m = S_{Y,m} - S_{Z,m} \ge 0. \end{gathered}$$

Lemma 2. *Let h_m, g_m, $m \ge 1$, and g be functions in $D_0[a,b]$ such that $\|g_m - g\| \to 0$ and $R_m h_m = g_m$, where $R_m = K_m - A_m$, and A_m, R_m, and K_m are defined as in (2.8)–(2.10) with (S_X, S_Y, S_Z) replaced by $(S_{X,m}, S_{Y,m}, S_{Z,m})$. Suppose that the conditions of Theorem 2 hold and*

$$(4.3)\qquad \lim_{\tau \to 0+} \sup_m \left[\int_{1-\tau<S_X(u)<1} \frac{-dS_{Y,m}(u)}{K_m(u)} + \int_{0<S_X(u)<\tau} \frac{-dS_{Z,m}(u)}{K_m(u)} \right] = 0.$$

Let R_S be given by (2.9). Then there exists $h \in D_K[a,b]$ such that $\|K_m h_m - Kh\| \to 0$ and $R_S h = g$.

Lemma 2 is proved in the Appendix. Theorem 2 is proved via strong continuity of linear operators indexed by survival functions in the metric space $\Theta = \{S\colon S - S_X \in D_0[a,b]\}$ with the distance $\|S - S'\|$. In the sequel, a function is said to be simple if it is a step function with finite number of jumps.

PROOF OF THEOREM 2. We consider the case $S_n \in \Theta$ in Steps 1–4 and (2.5) in Step 5.

STEP 1 (Existence of R_S^{-1} as a linear operator from $D_0[a,b]$ to $D_K[a,b]$, $\forall\, S \in \Theta$). Let $S_{X,m}$, $S_{Y,m}$, and $S_{Z,m}$ be finite discrete survival functions satisfying the conditions of Lemma 2 and $K_m(t-) > 0$ for $0 < S_X(t) < 1$. The existence of such survival functions is guaranteed by (2.13) and (2.6). For any $g \in D_0[a,b]$ let g_m be simple functions in $D_0[a,b]$ with $\|g_m - g\| \to 0$. Also, let Γ_m be the finite set of discontinuities of $S_{X,m}$, $S_{Y,m}$, $S_{Z,m}$ or g_m, and D_m the space of simple functions in $D_0[a,b]$ with jumps only at Γ_m. Since $R_m D_m \subset D_m$ and D_m is a finite-dimensional space, it follows from Lemma 1(ii) that R_m is a one-to-one mapping from D_m onto D_m, so that $R_m h_m = g_m$ has a solution $h_m \in D_m$. Since $\|g_m - g\| \to 0$, by Lemma 2 $R_S h = g$ has a unique solution $h \in D_K[a,b]$. Define $h = R_S^{-1} g$.

STEP 2 (Strong continuity of $\{R_S^{-1}: S \in \Theta\}$). Let $g'_m \in D_0[a,b]$ and S_m in Θ be such that $\|g'_m - g\| \to 0$ and $\|S_m - S\| \to 0$. It suffices to show $\|KR_{S_m}^{-1} g'_m - KR_S^{-1} g\| \to 0$, since $S_m = S$ gives the boundedness of R_S^{-1} and $g'_m = g$ gives the strong continuity. As in Step 1 we may obtain $S_{X,m}$, $S_{Y,m}$, $S_{Z,m}$, h_m, g_m and $\varepsilon_m \to 0$ such that $R_m h_m = g_m$, $\|K_m h_m - KR_{S_m}^{-1} g'_m\| \le \varepsilon_m$, $\|S_{X,m} - S_m\| \le \varepsilon_m$ and $\|g_m - g'_m\| \le \varepsilon_m$. Since $\|g_m - g\| \le \varepsilon_m + \|g'_m - g\| \to 0$ and $\|S_{X,m} - S\| \le \varepsilon_m + \|S_m - S\| \to 0$, it follows from Lemma 2 that $\|K_m h_m - KR_S^{-1} g\| \to 0$, which implies

$$\|KR_{S_m}^{-1} g'_m - KR_S^{-1} g\| \le \varepsilon_m + \|K_m h_m - KR^{-1} g\| \to 0.$$

STEP 3 (Strong continuity of $\{B_S,\ S \in \Theta\}$). Let h be a simple function in D_0^3. Since $S - S_X \in D_0[a,b]$, by (2.10) $B_S h \to B_{S_X} h$ in $D_0[a,b]$ as $\|S - S_X\| \to 0$. Since $\|B_S\| \le 2$ and the collection of simple functions is dense in D_0^3, we have the strong continuity.

STEP 4 (Conclusion). By Steps 2 and 3, $\{H_S = R_S^{-1} B_S,\ S \in \Theta\}$ is strongly continuous, so that by Theorem 1 and the Banach–Steinhaus theorem $\sup\{\|H_{S_n} h - H_S h\|_K : h \in C(\varepsilon)\} \to 0$ as $n \to \infty$ and then $\varepsilon \to 0+$ a.s. for all compact set $C \subset D_0^3$, where $C(\varepsilon) = \{h \in D_0^3: \|h - h'\|_3 < \varepsilon$ for some $h' \in C\}$. Since $W_n \to_D W$ in D_0^3, $\{W_n\}$ is uniformly tight [Pollard (1984), page 81], so that $\|\xi_n - H_{S_X} W_n\|_K = \|(H_{S_n} - H_{S_X}) W_n\|_K = o_P(1)$, which implies (2.14) by (2.11) and the continuous mapping theorem [Pollard (1984), page 70].

STEP 5 (Assume (2.5) holds for S_n). Let $a'_n \in [a, a_n)$ and $b'_n \in (b_n, b]$, where $(c,c]$ and $[c,c)$ are defined to be $\{c\}$. Set $S'_n(t) = S_n(t) I\{a'_n \le t < b'_n\} + I\{t < a'_n\}$. Then, (2.4) holds for S'_n and $S'_n - S_X \in \Theta$, so that $\sqrt{n}(S'_n - S_X)$ converges in distribution in $D_K[a,b]$. The conclusion follows if we choose a'_n and b'_n such that $2K(a'_n)|\xi_n(a'_n)| \ge \sup_{a \le t \le a'_n} K(t)|\xi_n(t)|$ and $2K(b'_n -)\,|\xi(b'_n -)| \ge \sup_{b'_n \le t \le b} K(t)|\xi_n(t)|$. □

PROOF OF COROLLARY 1. Let $a'_n = \min\{t: dQ_n^{(1)}(t) < 0\}$ and $b'_n = \max\{t: dQ_n^{(1)}(t) < 0\}$. Similar to Step 1 of the proof of Theorem 1 we can show that $S_n - S_X \in D_0[a, b]$ if $Q_n^{(2)}(b'_n -) = 0$ and $Q_n^{(3)}(a'_n -) = Q_n^{(3)}(-\infty)$. Under our assumption, it can be proved that $Q^{(2)}(b_n -) = o(1/n)$ and $Q^{(3)}(-\infty) - Q^{(3)}(a'_n -) = o(1/n)$. Since the size of jump is at least $1/n$, $\Pr\{S_n - S_X \in D_0[a, b]\} \to 1$. □

5. Proof of Theorem 3. Let μ be a bounded linear functional on $D_K[a, b]$. By the argument of Beran (1977), it suffices to prove the existence of random variables $\eta = \eta_\mu(V, \delta) \in L^2(Q)$ and normal random variables $\{Z(\alpha), \alpha \in L^2(Q), E\alpha = 0\}$ with $EZ(\alpha) = 0$ and $EZ(\alpha)Z(\alpha') = E\alpha\alpha'$ such that

$$E\exp\bigl[i\mu(\tilde{\xi})\bigr] = E\exp\bigl[i\mu(\tilde{\xi}) - iE\alpha\eta + Z(\alpha) - E\alpha^2/2\bigr], \qquad (5.1)$$
$$E|\mu(\xi)|^2 = \operatorname{Var}(\eta).$$

The covariance structure of $Z(\alpha)$ ensures that $Z(\alpha)$ is linear in α and $Z(\eta_{\mu'}) = \mu'(\xi)\ \forall \mu'$, for a version of ξ. The convolution is obtained by setting $Z(\alpha) = i\mu'(\xi)$ with $\alpha = i\eta_{\mu'}$. To prove (5.1), let $\alpha = \alpha(V, \delta)$ be a bounded function of (V, δ) with $E\alpha = 0$ such that $\alpha(t, j) = 0$ for $S_X(t) < \varepsilon$ or $S_X(t) > 1 - \varepsilon$. Define $dQ_{(n)}^{(j)}(t) = [1 + \alpha(t, j)/\sqrt{n}\,]\,dQ^{(j)}(t)$ for large n and $1_{(V,\delta)}^{(j)}(t) = I\{V > t,\ \delta = j\}$. Then, $Q = E1_{(V,\delta)}$, $1_{(V,\delta)} - Q \in D_0^3$, $\sqrt{n}\,(Q_{(n)} - Q) = E\alpha(V, \delta)(1_{(V,\delta)} - Q)$, and, as in the proof of Theorem 2, $\mu(\sqrt{n}\,(S_{X,n} - S_X)) \to \mu(H_{S_X}E\alpha(V, \delta)(1_{(V,\delta)} - Q)) = E\alpha\eta$, where $\eta = \mu(H_{S_X}(1_{(V,\delta)} - Q))$ and $H_S = R_S^{-1}B_S$. Since $\{Q_{(n)}\}$ and $\{Q\}$ are contiguous, we can show as in Beran (1977) the existence of $Z(\alpha)$ satisfying the first equation of (5.1). For the second one, we have $E|\mu(\xi)|^2 = E|\mu(H_{S_X}W)|^2 = E|\mu(H_{S_X}(1_{(V,\delta)} - Q))|^2 = \operatorname{Var}(\eta)$. □

APPENDIX

PROOF OF LEMMA 1. Proof of (i). Since $d(UV) = U\,dV + V_-\,dU$, by (3.1) and (3.3) for $0 < S(t) < 1$,

$$K(t)\,dh(t+) = g(t)\,dS(t+), \qquad K(t-)\,dh(t) = g(t-)\,dS(t), \qquad (A.1)$$

where $dh(t) = h(t) - h(t-)$ and $dh(t+) = h(t+) - h(t)$ at discontinuities for all h, and

$$g(t) = \int_{u \le t} \frac{h(u)}{S(u)}\,dS_Y(u) - \int_{t < u} \frac{h(u)}{1 - S(u)}\,dS_Z(u). \qquad (A.2)$$

Assume $h(t_0) > 0$ and $0 < S(t_0) < 1$ at some point t_0. Our goal is to establish a contradiction. Define

$$t_1 = \sup\{t \le t_0: h(t) \le 0\}, \qquad t_2 = \inf\{t \ge t_0: h(t) \le 0\},$$
$$J = \{t: h(t) > 0, t_1 \le t \le t_2\}.$$

Then, $t_0 \in J$ and $(t_1, t_2) \subset J \subset [t_1, t_2]$.

STEP 1. Show that $g(t) = g(t-) = 0$ on J. By (A.2) $g(t)$ is a right-continuous function with

$$dg(t) = h(t)\left[\frac{dS_Y(t)}{S(t)} + \frac{dS_Z(t)}{1 - S(t)}\right] \le 0 \quad \text{on } J. \tag{A.3}$$

To show that $g(t) \le 0$ and $g(t-) \le 0$ in J, we have four cases; the "greater than or equal to" part is similar and omitted.

CASE 1. $t_1 = -\infty$ or $S(t_1) = 1$. By (3.3) $\int_{u \le t}(h(u)/S(u))\, dS_Y(u) \le 0$ in J, which implies $\int_{t<u}(1 - S(u))^{-1}h(u)\, dS_Z(u) \ge 0$ by (3.1). Since $t_1 \notin J$, $g(t) \le 0$ and $g(t-) \le 0$ on J by (A.2).

For Cases 2–4, we show $g(t_1 -) \le 0$ for $t_1 \in J$ and $g(t_1) \le 0$ for $t_1 \notin J$, which will imply $g(t) \le 0$ and $g(t-) \le 0$ on J by (A.3).

CASE 2. $h(t_1) > 0$, $t_1 \in J$. Since $K(t_1 -)\, dh(t_1) > 0$, $g(t_1 -) < 0$ by (A.1).

CASE 3. $h(t_1) \le 0$, $h(t_1 +) > 0$, and $S(t_1) < 1$, $t_1 \notin J$. Here (3.2) implies $dS(t_1 +) < 0$, and (A.1) implies $g(t_1) \le 0$.

CASE 4. $h(t_1) \le 0$ and $h(t_1 +) = 0$, $t_1 \notin J$. Since $t_1 < t_0$, there exists $\{t_n\} \subset J$ with $t_n \downarrow t_1$ and $dh(t_n) > 0$ or $dh(t_n +) > 0$. Therefore, $g(t_n -) \le 0$ or $g(t_n) \le 0$ by (A.1), so that $g(t_1) \le 0$ by the right continuity of g.

STEP 2. Find a contradiction. Since $h > 0$ on J, by Step 1 and (A.3),

$$dS_Y(t) = dS_Z(t) = 0 \quad \text{and} \quad K(t) = K(t-) = \text{constant} > 0 \text{ on } J. \tag{A.4}$$

By (A.1), Step 1, and (A.4), we have $dh(t) = dh(t+) = 0$ on J, so that

$$h(t) = h(t_0) > 0 \quad \text{on } J \quad \text{and} \quad t_0 \in J = (t_1, t_2). \tag{A.5}$$

CASE 1. $S(t_1) < 1$. By (A.5) $h(t_1) \le 0$ and $h(t_1 +) = h(t_0) > 0$. Since $K(t_1) > 0$ by (A.4), it follows from (A.1) that $g(t_1 +) < 0$, which is a contradiction to Step 1.

CASE 2. $S(t_2) > 0$. By (A.5) $h(t_2) \le 0$ and $h(t_2 -) = h(t_0) > 0$, so that by (A.1) and (A.4) $g(t_2 -) > 0$, which is again a contradiction to Step 1.

CASE 3. $S(t_1) = 1$ and $S(t_2) = 0$. It follows from (3.3) that the right-hand side of (3.1) is nonpositive, so that $h(t)K(t) \le 0$ for all t. However, $h(t)K(t) > 0$ on J by (A.4) and (A.5).

Proof of (ii). It suffices to show that (3.3) holds. Let $t_1 = \max\{t\colon h(t) \ne 0, S_Y(t) - S_Z(t-) \le 0, S(t) = 0\}$ if the set is not empty and $t_1 = \inf\{t\colon S(t) = 0\}$ otherwise. Since $\{t\colon h(t) \ne 0, S_Y(t) - S_Z(t-) \le 0\}$ does not have a limit point from left, the maximum is reached for the nonempty case. For $t \ge t_1$ by (3.1)

$K(t)h(t) = \int_{t<u} h(u)\, dS_Z(u)$. Assume $h(t_2) > 0$ for some $t_2 > t_1$. We shall find a contradiction. Let $t_3 = \inf\{t\colon h(t) \le 0,\ t \ge t_2\}$. By the right continuity of h and (2.6), h has to change sign on the right of t_2, so that $S_X(t_3) > 0$. Since $K(t)h(t)$ is nondecreasing in $[t_2, t_3)$, by (3.1)

$$\begin{aligned} 0 < h(t_3 -)K(t_3 -) &= h(t_3)K(t_3) + h(t_3)\, dS_Z(t_3) \\ &= h(t_3)\left[S_Y(t_3) - S_Z(t_3 -)\right]. \end{aligned}$$

This gives a contradiction to the definition of t_1 and t_3. Therefore, $h(t) = 0$ for $t \ge t_1$ and $t_1 = \inf\{t\colon S(t) = 0\}$ by its definition, so that $S(t) = 0 \Rightarrow h(t) = 0$. Similarly, $S(t) = 1 \Rightarrow h(t) = 0$. □

Proof of Lemma 2. A sequence of functions is totally bounded if every subsequence contains a uniformly convergent further subsequence. Basically we shall establish the total boundedness of $K_m h_m$ and then prove that there exists exactly one cluster point Kh. Define

$$v_m^+(t) = \int_{t<u} \frac{1 - S_{X,m}(t)}{1 - S_{X,m}(u)} h_m(u)\, dS_{Z,m}(u),$$

$$v_m^-(t) = \int_{u \le t} \frac{S_{X,m}(t)}{S_{X,m}(u)} h_m(u)\, dS_{Y,m}(u).$$

Step 1. For a fixed $0 < \tau_0 < 1$ establish the total boundedness of $v_m^+(t)I\{S(t) \le \tau_0\}$ and $v_m^-(t)I\{S(t) > \tau_0)$ for the case $\|K_m h_m\| \le 1$. Since $S - S_X \in D_0[a, b]$, there exists $0 < \tau < 1$ such that

$$\{S(t) > \tau_0\} \subset \{S_X(t) > \tau\} \quad \text{and} \quad \{S(t) \le 1 - \tau_0\} \subset \{S_X(t) \le 1 - \tau\}. \tag{A.6}$$

It follows from (4.1)–(4.3) and (2.15) that

$$\sup_{S_X(t) > \tau} \left| \int_{u \le t,\, S_X(u) < 1} \frac{dS_{Y,m}(u)}{K_m(u)} - \int_{u \le t,\, S_X(u) < 1} \frac{dS_Y(u)}{K(u)} \right| = \varepsilon_m \to 0.$$

Let $s < t$, $S_X(s) < 1$ and $S(t) > \tau_0$. Since $\|K_m h_m\| \le 1$, by (A.6)

$$\begin{aligned} &|v_m^-(t)/S_{X,m}(t) - v_m^-(s)/S_{X,m}(s)| \\ &\quad = \left| \int_{s<u\le t} (S_{X,m}(u))^{-1} h_m(u)\, dS_{Y,m}(u) \right| \\ &\quad \le (\tau_0 - \|S_{X,m} - S\|)^{-1} \left[2\varepsilon_m - \int_{s<u\le t} (K(u))^{-1}\, dS_Y(u) \right]. \end{aligned}$$

Since $g_m(t) = h_m(t) = v_m^-(t) = 0$ for $S_X(t) = 1$, by (2.13) the sequence $[v_m^-(t)/S_{X,m}(t)]I\{S(t) > \tau_0\}$ is totally bounded. Since $\|S_{X,m} - S\| \to 0$, $v_m^-(t)I\{S(t) > \tau_0\}$ is the product of two totally bounded functions and therefore totally bounded itself. The total boundedness of $v_m^+(t)I\{S(t) \le \tau_0\}$ can be proved similarly.

STEP 2. Assume $\|K_m h_m\| \le 1$. Verify $\sup\{|v_m^+(t)|: S(t) > 1-\delta\} = o(1)$ and $\sup\{|v_m^-(t)|: S(t) \le \delta\} = o(1)$ as $m \to \infty$ and then $\delta \to 0+$. There are two cases for the first equation.

CASE 1. $\{S(t) < 1\} \subset \{S_X(t) \le 1-\tau\}$ for some $\tau > 0$. Since $1 - S_{X,m}(t) = v_m^+(t) = 0$ for $S(t) = 1$, by (4.3), (2.15) and the dominated convergence theorem we have

$$\begin{aligned}
&\sup\{|v_m^+(t)|: S(t) > 1-\delta\}\\
&\quad \le \sup_{1-\delta<S(t)<1} \int_{0<S_X(u)<1-\tau} \frac{1-S_{X,m}(t)}{1-S_{X,m}(u)} I\{t<u\} \frac{-dS_{Z_m}(u)}{K_m(u)}\\
&\quad \to \sup_{1-\delta<S(t)<1} \int_{0<S_X(u)<1-\tau} \frac{1-S(t)}{1-S(u)} I\{t<u\} \frac{-dS_Z(u)}{K(u)} \to 0.
\end{aligned}$$

CASE 2 (Not case 1). Since $S - S_X \in D_0[a,b]$, we have $S(a) = S_X(a) = 1$, so that $1-\delta < S(t) < 1$ for small δ implies that t is close to a. Given $\varepsilon > 0$, by Step 1 there exists $\tau > 0$ such that

$$\sup_{S(t)>1-\tau} |g_m(t)| + \sup_{S(t)>1-\tau} |v_m^-(t)| \le \varepsilon, \qquad \forall\, m, \tag{A.7}$$

since $g_m(t) = h_m(t) = v_m^-(t) = 0$ for $t \le a$ and $\|g_m - g\| \to 0$. By the definition of v_m^+ and $R_m h_m = g_m$,

$$v_m^+(t) = K_m(t)h_m(t) - v_m^-(t) - g_m(t). \tag{A.8}$$

Assume $v_m^+(t_0) > \varepsilon$ with $S(t_0) > 1-\tau$. Then, $h_m(t_0) > 0$ by (A.7) and (A.8). Let $t_1 = \inf\{t > t_0: h_m(t) \le 0\}$. We shall prove that $S(t_1) \le 1-\tau$. Similar to the derivation of (A.1) we have

$$\begin{aligned}
&K_m(t_1-)(h_m(t_1) - h_m(t_1-))\\
&\quad = \left[v_m^-(t_1-)/S_{X,m}(t_1-) - v_m^+(t_1-)/(1-S_{X,m}(t_1-))\right] dS_{X,m}(t_1) \qquad \text{(A.9)}\\
&\qquad + dg_m(t_1).
\end{aligned}$$

Since $v_m^+(t)$ is nondecreasing in $[t_0, t_1)$, $v_m^+(t_1-) \ge v_m^+(t_0) > \varepsilon$, so that by (A.9) and (A.8)

$$\begin{aligned}
0 &\ge K_m(t_1-)h_m(t_1)\\
&\ge K_m(t_1-)h_m(t_1-) + \left[v_m^-(t_1-)/S_{X,m}(t_1-)\right] dS_{X,m}(t_1) + dg_m(t_1)\\
&= v_m^+(t_1-) + v_m^-(t_1-)S_{X,m}(t_1)/S_{X,m}(t_1-) + g_m(t_1)\\
&> \varepsilon - |v_m^-(t_1-)| - |g_m(t_1)|,
\end{aligned}$$

which implies $S(t_1) \le 1-\tau$ by (A.7). This proves $h_m(t) > 0$ on the set $\{t_0 \le t,$

$S(t) > 1 - \tau\}$, and therefore

$$v_m^+(t_0) \le \int_{S(u)\le 1-\tau} \frac{1 - S_{X,m}(t_0)}{1 - S_{X,m}(u)} h_m(u)\, dS_{Z,m}(u)$$
$$\le -\int_{S(u)\le 1-\tau} \frac{1 - S_{X,m}(t_0)}{1 - S_{X,m}(u)} \frac{dS_{Z,m}(u)}{K_m(u)}.$$

Since t_0 is arbitrary, by (4.1)–(4.3) and (2.15), as $\delta \to 0+$ and then $\varepsilon \to 0+$,

$$\limsup_{m\to\infty} \sup\{v_m^+(t): S(t) > 1 - \delta\} \le \varepsilon - \int_{S(u)\le 1-\tau} \frac{(\delta/\tau)\, dS_Z(u)}{K(u)} \to \varepsilon \to 0.$$

For the second equation of Step 2, the proof for "Case 1" is similar to that of the first equation and omitted, but "Case 2" is slightly different. Let τ be such that $|v_m^+(t)| + |g_m(t)| \le \varepsilon$ for $S(t) \le \tau$ [see (A.7)]. Assume $v_m^-(t_0) > \varepsilon$ for some $S(t_0) \le \tau$. Then, $h_m(t_0) > 0$. Let $t_1 = \sup\{t < t_0: h_m(t) \le 0\}$. Then

$$v_m^-(t_1) - v_m^-(t_1 -) = \frac{dS_{X,m}(t_1)}{S_{X,m}(t_1)} v_m^-(t_1) + \frac{S_{X,m}(t_1 -)}{S_{X,m}(t_1)} h_m(t_1)\, dS_{Y,m}(t_1) \le 0,$$

so that $v_m^-(t_1 -) \ge v_m^-(t_1) \ge v_m^-(t_0) > \varepsilon$, which implies $S(t_1 -) > \tau$. The rest of the proof is omitted.

STEP 3. Prove that the equation $R_S h = g$ has at most one solution with $Kh \in D_0[a, b]$ for every g. Since R_S is a linear operator, we assume $g = 0$. It suffices to check the conditions of Lemma 1(ii). Condition (3.1) says $R_S h = 0$ and holds naturally. Since $S - S_X \in D_0[a, b]$, S is right-continuous and $\{0 < S(t) < 1\} \subset J_X = \{0 < S_X(t) < 1\}$. Since K and Kh are both right-continuous, h is right-continuous on J_X by (2.15). Since $Kh \in D_0[a, b]$, $h(t) = 0$ for $t \notin J_X$. It remains to show that the set $J_X \cap \{t: S_Y(t) \le S_Z(t -)\}$ does not have a limit point from left. If $S_Y(t) \le S_Z(t -)$ and $0 < S_X(t) < 1$, then by (2.6) $0 \le S_Y(u) - S_Z(u) \le S_Z(t -) - S_Z(u)$ for $u \ge t$ and $S_Z(t -) > S_Z(t + \varepsilon)$ for $\varepsilon > 0$, so that

$$-\int_{t\le u<t+\varepsilon} [S_Y(u) - S_Z(u)]^{-1} dS_Z(u)$$
$$\ge -\int_{t\le u<t+\varepsilon} [S_Z(t -) - S_Z(u)]^{-1} dS_Z(u) \ge 1, \qquad \forall\, \varepsilon > 0.$$

It follows from (2.13) that the set $\{S_Y(t) \le S_Z(t -),\ 0 < S_X(t) \le \tau\}$ is finite for all $0 < \tau < 1$.

STEP 4 (Conclusion). Let us consider the case $\|K_m h_m\| \le 1$ first. If Kh is a cluster point of the sequence $K_m h_m$, then by the conditions of this lemma and (2.15) we have $R_S h = g$. Since $K_m h_m = g_m + v_m^+ + v_m^-$ and $\|g_m - g\| \to 0$, Steps 1 and 2 imply that the sequence $K_m h_m$ is totally bounded. By Step 3, the sequence has only one cluster point. Hence $\|K_m h_m - Kh\| \to 0$

and $R_S h = g$. if $\sup_m \|K_m h_m\| = M < \infty$, then the conditions of this lemma hold for h_m/M and g_m/M, and we come back to the case $\|K_m h_m\| \le 1$. The proof is completed if we can find a contradiction to the case where $\|K_{m_k} h_{m_k}\| = M_k \to \infty$. Let $h'_k = h_{m_k}/M_k$, $g'_k = g_{m_k}/M_k$, $S'_{X,k} = S_{X,m_k}$, $S'_{Y,k} = S_{Y,m_k}$ and $S'_{Z,k} = S_{Z,m_k}$. Then by Lemma 2 for the case $\|K_m h_m\| \le 1$, we have $\|K'_k h'_k\| \to 0$, since $\|g'_k\| \to 0$. This is a contradiction to $\|K'_k h'_k\| = 1$. □

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DEPARTMENT OF MATHEMATICS AND STATISTICS
MCGILL UNIVERSITY
805 SHERBROOKE ST. WEST
MONTREAL, QUEBEC H3A 2K6
CANADA

DEPARTMENT OF STATISTICS
RUTGERS UNIVERSITY
HILL CENTER
BUSCH CAMPUS
NEW BRUNSWICK, NEW JERSEY 08903