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ON SOME FILTRATION PROCEDURE FOR JUMP MARKOV PROCESS OBSERVED IN WHITE GAUSSIAN NOISE

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The importance of optimal filtration problem for Markov chain with two states observed in Gaussian white noise (GWN) for a lot of concrete technical problems is well known. The equation for a posteriori probability $\pi(t)$ of one of the states was obtained many years ago. The aim of this paper is to study a simple filtration method. It is shown that this simplified filtration is asymptotically efficient in some sense if the diffusion constant of the GWN goes to 0. Some advantages of this procedure are discussed.

1. Let $X(t)$ be a Markov process with the states 0 and 1, which is characterized by the transition densities λ (from 0 to 1) and μ (from 1 to 0) and the initial condition $P(X(0) = 0) = \xi$. The observed process $Y(t)$ has the form

$$Y(t) = \int_0^t X(s)\,ds + \sigma W(t), \tag{1}$$

$W(t)$ is a standard Wiener process, σ is a constant.

Let us denote by $\mathscr{F}_0^t$ σ field of the events generated by $Y(s)$, $0 \le s \le t$. It is well known that the estimator

$$\hat{X}(t) = \begin{cases} 1, & \text{if } \pi(t) \ge 1/2, \\ 0, & \text{if } \pi(t) < 1/2, \end{cases} \tag{2}$$

$$\pi(t) = P\big(X(t) = 1 \big| \mathscr{F}_0^t\big)$$

minimizes the filtration probability error. It is well known [Liptser and Shiryayev (1977)] also that the function $\pi(t)$ is the solution of the stochastic differential equation,

$$\begin{aligned} d\pi(t) &= (\lambda - (\lambda + \mu)\pi(t))\,dt \\ &\quad + \sigma^{-1}\pi(t)(1 - \pi(t))(dY(t) - \pi(t)\,dt), \\ \pi(0) &= 1 - \xi. \end{aligned} \tag{3}$$

The mean square error of the optimal in mean square sense filter for this model was obtained by Wonham (1965). In this paper we propose a "simplified" filter and compare its probability error to those of (2).

It was known that $\pi(t)$ is also a Markov process because the expression $\sigma^{-1}(dY(t) - \pi(t)\,dt)$ is equal to the differential of new Wiener process

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$(\overline{W}(t), \mathscr{F}_0^t)$ (innovation process). So the infinitesimal generator of π has the form

$$L = (\lambda - (\lambda + \mu)x)\frac{d}{dx} + \frac{1}{2}\sigma^{-2}x^2(1-x)^2\frac{d^2}{dx^2}$$

and the Fokker–Plank–Kolmogorov equation for the stationary density $q(x)$ of this process is

$$L^*q = 0.$$

Here L^* is the formal adjoint operator for L.

This equation together with the boundary conditions

$$q(x) \to 0, \qquad q'(x)x^2(1-x)^2 \to 0, \qquad x \to +0 \text{ and } x \to 1-0$$

has the unique solution [see Liptser and Shiryaev (1977)]

$$q(x) = c\left(\frac{1-x}{x}\right)^{2(\beta-\alpha)} x^{-2}(1-x)^{-2}\exp\left(-\frac{2\alpha(1-x)}{x} - \frac{2\beta x}{1-x}\right),$$

where the constant c is defined by the condition

$$\int_0^1 q(x)\,dx = 1.$$

We use the notation $\alpha = \lambda\sigma^2$, $\beta = \mu\sigma^2$. Note that α and β are the dimensionless parameters of the transition densities for $X(t)$.

The process $(\pi(t), X(t))$ is an ergodic Markov process for $\lambda > 0$, $\mu > 0$. Let $\mathbb{P}_{st}(\cdot)$, $\mathbb{E}_{st}(\cdot)$ denote the probability and expectation over the stationary initial distribution of this process. It is clear that the value

$$\begin{aligned} R_0(\lambda,\mu,\sigma) &= \lim_{T\to\infty}\frac{1}{T}\int_0^T 1\big(\hat{X}(t) \neq X(t)\big)\,dt \\ &= \mathbb{E}_{st}\left\{1\left(\pi(t) < \frac{1}{2}\right)1(X(t)=1) + 1\left(\pi(t) \geq \frac{1}{2}\right)1(X(t)=0)\right\} \\ &= \mathbb{E}_{st}\left\{\pi(t)1\left(\pi(t) < \frac{1}{2}\right) + (1-\pi(t))1\left(\pi(t) \geq \frac{1}{2}\right)\right\} \\ &= \int_0^{1/2} xq(x)\,dx + \int_{1/2}^1 (1-x)q(x)\,dx \end{aligned} \tag{4}$$

is equal to the average probability of error for the estimator (2). Note that

$$\lim_{t\to\infty}\mathbb{E}\big(\hat{X}(t) - X(t)\big)^2 = R_0(\lambda,\mu,\sigma)$$

and $R_0(\lambda,\mu,\sigma) = R_0(\alpha,\beta,1)$. It follows from (4) and the explicit formula for $q(x)$ that for fixed $\lambda > 0$, $\mu > 0$ and $\sigma \to 0$

$$R_0(\lambda,\mu,\sigma) = -\frac{2\alpha\beta}{\alpha+\beta}\ln(\alpha\beta) + O(\sigma^2). \tag{5}$$

But the optimal filtration procedure requires the solution of a rather complicated stochastic differential equation (3). The aim of the presented paper is to propose a simpler filtration method which is asymptotically (as $\sigma \to 0$) equivalent to (2).

2. Equation (3) and Itô's formula imply the following equation for

$$Z(t) = \ln(\pi(t)/(1 - \pi(t)),$$

$$dZ(t) = \sigma^{-2}(dY(t) - 1/2\, dt) + (\lambda - \mu + \lambda \exp(-Z(t)) - \mu \exp Z(t))\, dt. \tag{6}$$

The process $Z(t)$ is $\mathscr{F}_0^t$ measurable, $Z(0) = \ln((1 - \xi)/\xi)$ and the decision rule (2) in terms of $Z(t)$ has the form

$$\hat{X}(t) = \begin{cases} 1, & \text{if } Z(t) \geq 0, \\ 0, & \text{if } Z(t) < 0. \end{cases}$$

The principal part of the right-hand side in (6) if $\sigma \to 0$ is $\sigma^{-2}(dY(t) - 1/2\, dt)$, provided the value $|Z(t)|$ is not too large. In addition, if $Z(t) \gg 1$.

$$P(X(t) = 1 | \mathscr{F}_0^t) = \pi(t) = \frac{\exp Z(t)}{1 + \exp Z(t)} \approx 1 - \exp(-Z(t))$$

and the drift coefficient for the process $Z(t)$ is approximately equal to the value $(2\sigma^2)^{-1} - \mu \exp Z(t)$.

This drift changes sign in the vicinity of the point $Z_+^0 = -\ln(2\beta)$ and is of the order of σ^{-2} for $Z > Z_+^0 + \varepsilon$, for any constant $\varepsilon > 0$.

So we see that $Z(t)$ is changing direction in the vicinity of the point Z_+^0, where $X(t) = 1$. The analogous heuristic arguments leads to a conclusion that a similar point for $Z(t)$ while $X(t) = 0$ is the point

$$Z_-^0 = \ln(2\alpha).$$

These considerations make natural the following simplified filtration method. Denote by $Z_0(t)$ the diffusion process with reflecting barriers in the points Z_-, Z_+, $Z_- < Z_+$, satisfying for $Z_0(t) \in [Z_-, Z_+]$ the equation

$$\begin{aligned} dZ_0(t) &= \sigma^{-2}(dY(t) - (1/2)\, dt), \\ Z_0(0) &= \ln((1 - \xi)/\xi). \end{aligned} \tag{7}$$

Our goal is to study the asymptotic properties of the filtration procedure

$$\overline{X}(t) = \begin{cases} 1, & \text{if } Z_0(t) \geq 0, \\ 0, & \text{if } Z_0(t) < 0, \end{cases} \tag{8}$$

for different $Z_- < 0 < Z_+$ and $t \to \infty$, $\sigma \to 0$.

3. It follows from the law of large numbers that the value

$$\frac{1}{T}\int_0^T 1(\overline{X}(t) \neq X(t))\,dt$$

converges a.s. to the nonrandom limit $R(\lambda, \mu, \sigma)$ if $T \to \infty$. This limit is equal to the filtration probability error if the initial conditions are stationary:

$$R(\lambda, \mu, \sigma) = \mathbb{P}_{st}(\overline{X}(t) \neq X(t)).$$

The main result of this paper is the following theorem.

THEOREM 1. *The formula*

$$\begin{aligned} R(\lambda, \mu, \sigma) &= R(\alpha, \beta, 1) \\ &= \sum_{i=1}^{3} c_i\{k_i^{-1}(\exp(k_i z_+) - \exp(k_i z_-)) \\ &\qquad -\exp(k_i z_+) - \exp(k_i z_-) + 2\} \end{aligned} \tag{9}$$

is true for $Z_- < 0 < Z_+$. *Here* $k_1 < -1 < k_2 < 1 < k_3$ *are the roots of the cubic equation*

$$k^3 - (1 + 2\alpha + 2\beta)k + 2(\alpha - \beta) = 0 \tag{10}$$

and the constants c_1, c_2, c_3 *are given by the following expressions* ($\Delta Z = Z_+ - Z_-$):

$$\begin{aligned} c_2 = \frac{1}{2}\Bigg\{&\frac{\exp(k_2 Z_+) - \exp(k_2 Z_-)}{k_2} \\ &+ \frac{(1 - k_2^2)\exp(k_2 Z_+)(\exp(-k_1\,\Delta Z) - 1)(\exp((k_3 - k_2)\,\Delta Z) - 1)}{k_2(1 - k_1^2)(\exp((k_3 - k_1)\,\Delta Z) - 1)} \\ &+ \frac{(1 - k_2^2)\exp(k_2 Z_-)(\exp(k_3\,\Delta Z) - 1)(\exp((k_2 - k_1)\,\Delta Z) - 1)}{k_3(k_3^2 - 1)(\exp((k_3 - k_1)\,\Delta Z) - 1)}\Bigg\}^{-1}, \end{aligned} \tag{11}$$

$$c_1 = \frac{(1 - k_2^2)\exp((k_2 - k_1)Z_+)(\exp((k_3 - k_2)\,\Delta Z) - 1)}{(k_1^2 - 1)(\exp((k_3 - K_1)\,\Delta Z) - 1)}c_2,$$

$$c_3 = \frac{(1 - k_2^2)\exp((k_2 - k_3)Z_-)(\exp((k_2 - k_1)\,\Delta Z) - 1)}{(k_3^2 - 1)(\exp((k_3 - k_1)\,\Delta Z) - 1)}c_2.$$

For barriers $Z_- = \ln\sigma^2 - c_-$; $Z_+ = -\ln\sigma^2 + c_+$, $\sigma \to 0$ *and fixed* λ, μ, c_-, c_+ *the relation*

$$R(\lambda, \mu, \sigma) = \mathbb{P}_{st}(X(t) \neq X(t)) = -\frac{2\alpha\beta}{\alpha + \beta}\ln(\alpha\beta) + O(\sigma^2) \tag{12}$$

is valid.

REMARK 1. It is clear from (10) that $k_2 = 0$ if $\lambda = \mu$. It is necessary to put

$$k_i^{-1}(\exp(k_i Z_+) - \exp(k_i Z_-)) = Z_+ - Z_-$$

in (9) and (11) for this case.

REMARK 2. The optimal filtration method (2), (3) depends on the true values of λ, μ. On the contrary the procedure (7), (8) for fixed c_-, c_+ does not need a knowledge of λ, μ. It is interesting to notice that the quality of this procedure in the sense of (9) asymptotically coincides with the corresponding value of the optimal one.

REMARK 3. It follows from computations that the optimal filtration (2), (3) is practically useless if λ and μ have the different exponents. Since if for instance $\lambda \gg \mu$, the trivial estimator $X^*(t) \equiv 1$ is rather good. Therefore it is natural to consider another extreme case $\lambda = \mu$. The following formula is a result of the computations in (9)–(11) for $c_- = c_+ = -\ln(2\lambda)$ with regard to Remark 1:

$$\begin{aligned} R(\lambda, \lambda, \sigma) &= R(\alpha, \alpha, 1) \\ &= \frac{1}{2} \frac{b^2 - 1 - (b-1)^2\sqrt{1+4\alpha} + 4\alpha(b^2+1)\ln b}{b^2 - 1 + 4\alpha(b^2+1)\ln b}, \qquad (13) \\ b &= (2\alpha)^{-\sqrt{1+4\alpha}}. \end{aligned}$$

It follows from (13) that the procedure (7), (8) is surprisingly close to the optimal one for $\lambda = \mu$, $\alpha \leq 0.1$. In more detail let

$$R(\alpha) = R(\alpha, \alpha, 1), \qquad R_0(\alpha) = R_0(\alpha, \alpha, 1)$$

be the probability error for the procedures (2), (3) and (7), (8) correspondingly. Based on (13) and (4) computations give the results:

$$R(\alpha) - R_0(\alpha) \leq 10^{-3} \quad \text{if } \alpha \leq 10^{-2};$$

$$R(0.1) \approx 0.256; \qquad R(0.05) \approx 0.191;$$

$$R(0.1) - R_0(0.1) \approx 0.01;$$

$$R(0.05) - R_0(0.05) \approx 0.004.$$

The procedure (7), (8) with $c_\pm = -\ln(2\lambda)$ is essentially worse if α approaches 0.5 and it has no sense for $\alpha \geq 0.5$.

PROOF OF THEOREM 1. The couple $(X(t), Z_0(t))$ is a Markov process with the two identical segments $[Z_-, Z_+]$ as the states space [first of these segments corresponds to the states with $X(t) = 0$ and the other one to the states with $X(t) = 1$]. It is clear that this process is an ergodic one for $\lambda > 0$, $\mu > 0$. It is easy to verify that its unique stationary distribution $\mathbb{P}_{st}(X(t) = i,\ Z(t) \in A)$

has a density. Let us denote it by $p_0(z), p_1(z)$ so that

$$p_i(z) = \lim_{\Delta\to 0} \Delta^{-1}\mathbb{P}_{st}(X(t) = i, Z(t) \in [z, z+\Delta]).$$

The strong law of large numbers for the process $(X(t), Z_0(t))$ implies

$$R(\lambda,\mu,\sigma) = \int_{Z_-}^{0} p_1(z)\,dz + \int_0^{Z_+} p_0(z)\,dz. \tag{14}$$

The functions $p_i(z)$ can be found in a standard way. It is known that the twice continuously differentiable in $z \in [Z_-, Z_+]$ functions $F(i,z)$, $i = 0, 1$, satisfying conditions

$$\left.\frac{dF(i,z)}{dz}\right|_{z=z_-} = \left.\frac{dF(i,z)}{dz}\right|_{z=z_+} = 0 \tag{15}$$

belong to the domain of definition of the infinitesimal generator L of the process $(X(t), Z_0(t))$. For such functions

$$LF(0,z) = (2\sigma^2)^{-1}(F''(0,z) - F'(0,z)) + \lambda(F(1,z) - F(0,z)),$$

$$LF(1,z) = (2\sigma^2)^{-1}(F''(1,z) - F'(1,z)) + \mu(F(0,z) - F(1,z)).$$

In a standard way we obtain that the functions p_0, p_1 satisfy the system of equations

$$\begin{aligned} &(2\sigma^2)^{-1}(p_0'' + p_0') + \mu p_1 - \lambda p_0 = 0,\\ &(2\sigma^2)^{-1}(p_1'' + p_1') + \lambda p_0 - \mu p_1 = 0, \end{aligned} \tag{16}$$

which is adjoint to the system $LF = 0$. The boundary conditions for $p_0(z), p_1(z)$ are adjoint to (15), that is,

$$\begin{aligned} &(p_0' + p_0)|_{z=Z_-} = (p_0' + p_0)|_{z=Z_+} = 0,\\ &(p_1' - p_1)|_{z=Z_-} = (p_1' - p_1)|_{z=Z_+} = 0. \end{aligned} \tag{17}$$

A more precise assertion is: Any pair of nonnegative functions $p_0(z), p_1(z)$ satisfying (16), (17) and the normalizing condition

$$\sum_{i=0}^{1} \int_{Z_-}^{Z_+} p_i(z)\,dz = 1 \tag{18}$$

is the stationary distribution density of the process $(X(t), Z_0(t))$. An ergodicity of this process implies the uniqueness of a nonnegative solution of the problem (16)–(18).

Introducing the notation

$$q_1(z) = p_0(z) + p_1(z), \qquad q_2(z) = p_0(z) - p_1(z) \tag{19}$$

we obtain from (16) the following system of equations

$$\begin{aligned} q_1'' + q_2' &= 0,\\ q_2'' + q_1' + 2\beta(q_1 - q_2) - 2\alpha(q_1 + q_2) &= 0. \end{aligned} \tag{20}$$

Equations (20) and (17) imply

$$q_2 = -q_1' \tag{21}$$

$$q_1''' - q_1'(1 + 2\alpha + 2\beta) + 2q_1(\alpha - \beta) = 0. \tag{22}$$

The general solution of (22) is

$$q_1(z) = 2\sum_{i=1}^{3} c_i \exp(k_i z).$$

Here $k_1 < -1 < k_2 < 1 < k_3$ are the roots of the characteristic equation (10). Equations (19) and (21) imply

$$p_0(z) = \sum_{i=1}^{3} c_i(1 - k_i^2)\exp(k_i z), \qquad p_1(z) = \sum_{i=1}^{3} c_i(1 + k_i^2)\exp(k_i z). \tag{23}$$

Let us use the first pair of boundary conditions (17) and (18) for determination of the constants c_1, c_2, c_3. We obtain

$$\sum_{i=1}^{3} c_i(1 - k_i^2)\exp(k_i z_-) = 0, \qquad \sum_{i=1}^{3} c_i(1 - k_i^2)\exp(k_i z_+) = 0,$$
$$\sum_{i=1}^{3} c_i k_i^{-1}(\exp(k_i z_+) - \exp(k_i z_-)) = 1/2. \tag{24}$$

Solving the first two equations of this system with respect to c_1, c_3 and substituting these expressions in the third equation (24) we obtain the formulas (11). Equation (9) is a consequence of (23) and (14).

At last, using the expressions ($\sigma^2 \to 0$)

$$k_1 = -1 - 2\alpha + o(\sigma^2), \qquad k_2 = 2(\alpha - \beta) + o(\sigma^2),$$
$$k_3 = 1 + 2\beta + o(\sigma^2)$$

we get the assertion of Theorem 1 concerning asymptotics of $R(\lambda, \mu, \sigma)$ if $\sigma \to 0$ which follows from the explicit formula (9). Indeed, (24) and (11) imply the asymptotic formulas

$$c_2 = \frac{2\alpha\beta}{\alpha + \beta}(1 + o(1)),$$

$$c_1 = c_2 \exp(-c_-)(4\lambda)^{-1}(1 + o(1)), \qquad c_3 = c_2 \exp(-c_+)(4\mu)^{-1}(1 + o(1)).$$

Substituting this value in (9) we obtain (12). Theorem 1 is proved. □

REMARK 4. It was noticed in Remark 2 that the procedure (7), (8) is still asymptotically optimal ($T \to \infty$, $\sigma \to 0$) for unknown λ and μ. But sometimes the estimation of λ, μ is of the main interest. The proposed procedure can be applied for this purpose.

Let $Z^0(t)$ be a process with the reflecting barrier at the points $Z^0_{\pm} = \pm \ln \sigma^{-2}$. We call a cycle a segment of the process trajectory between two successive visits of Z^0_- if an attainment of Z^0_+ takes place within it.

Let ν_T be a number of cycles of $Z^0(t)$ in $[0, T]$. Then it is easily seen that the statistics

$$\hat{\lambda} = \nu_T \left(\int_0^T 1(Z^0(t) < 0)\, dt \right)^{-1}, \qquad \hat{\mu} = \nu_T \left(\int_0^T 1(Z^0(t) > 0)\, dt \right)^{-1}$$

are the consistent estimators of λ and μ as $T \to \infty$, $\sigma \to 0$. These estimators turn into MLE (maximum likelihood estimator) of λ, μ if $\sigma = 0$ [$X(t)$ is observed without noise].

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