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SOME RESULTS ON 2^{n-k} FRACTIONAL FACTORIAL DESIGNS AND SEARCH FOR MINIMUM ABERRATION DESIGNS[1]

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In this paper we find several interesting properties of 2^{n-k} fractional factorial designs. An upper bound is given for the length of the longest word in the defining contrasts subgroup. We obtain minimum aberration 2^{n-k} designs for $k = 5$ and any n. Furthermore, we give a method to test the equivalence of fractional factorial designs and prove that minimum aberration 2^{n-k} designs for $k \leq 4$ are unique.

1. Introduction and definitions. When n two-level factors are to be studied in an experiment, one may choose a full factorial design which investigates all possible level combinations of n factors. However, it involves a large number of runs which may be too expensive and unnecessary. A 2^{n-k} fractional factorial design is a 2^{-k}th fraction of the 2^n full factorial design. So when the run size economy is of primary concern, a fractional factorial design is often used. A good choice of fractional factorial design allows us to study many factors with relatively small run size, and yet enables us to estimate a large number of effects. See Box, Hunter and Hunter (1978) for examples. However, with given n and k, there are many 2^{n-k} designs. We should choose a design that fits our need best. When specific knowledge about these factors and their interactions is available, one may be able to find a particular design that suites the need. When there is little knowledge about the factors, some optimality criteria are used to select a design. So it is important to supply as many different designs as possible and also give optimal designs under some criteria.

In the following, we introduce some notation and definitions of optimality. To save space, we refer to Fries and Hunter (1980) and Franklin (1984) for detailed discussions on these optimality criteria.

Suppose in an experiment, we have six factors at two levels to be studied and only 16 runs can be performed. Denote the six factors by *letters* 1, 2, 3, 4, 5 and 6. A fractional factorial 2^{6-2} design can be arranged. We refer to Chen and Wu (1991) for the detailed arrangement. Here, however, we only point out that the design can be characterized by

$$I = 125 = 2346 = 13456.$$

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The elements 125, 2346 and 13456 are called *words*. Especially, 125 and 2346 are called *generators* because the other word 13456 is their symbolic multiplication. All the words under this multiplication form a *defining contrasts subgroup* with I being its identity. The number of letters in a word is called the length of the word or *wordlength*. Let $A_i(d)$ be the number of words of length i in design d. The *wordlength pattern* is given by the relations

$$W(d) = (A_1(d), A_2(d), A_3(d), \dots).$$

The resolution of a design is the smallest r such that $A_r(d) \neq 0$. The ith moment is defined by

$$M_i(d) = \sum_{j=1}^{\infty} j^i A_j(d).$$

For convenience, we may use W, A_i, M_i and so on directly without d.

DEFINITIONS. A 2^{n-k} fractional factorial design has maximum resolution, if no other 2^{n-k} fractional factorial design has larger resolution.

Let d_1 and d_2 be two 2^{n-k} fractional factorial designs and r be the smallest i such that $A_i(d_1) \neq A_i(d_2)$. d_1 has less aberration than d_2 if $A_r(d_1) < A_r(d_2)$. A 2^{n-k} fractional factorial design has minimum aberration, if no other 2^{n-k} fractional factorial design has less aberration.

Let m be the first i such that $M_i(d_1) \neq M_i(d_2)$. If m is odd and $M_m(d_1) < M_m(d_2)$, d_2 has better moments; if m is even and $M_m(d_1) < M_m(d_2)$, d_1 has better moments. A 2^{n-k} fractional factorial design has optimal moments, if no other 2^{n-k} fractional factorial design has better moments. A 2^{n-k} fractional factorial design is a minimum-variance design if it has maximum first moment and minimizes the second moment $M_2(d)$.

Without loss of generality, we assume throughout the paper that each of the n letters in a 2^{n-k} design must appear in the defining contrasts subgroup. This is equivalent to maximizing the first moment.

Optimal designs in the sense defined have been studied extensively, though their theoretical properties remain largely unexplored. On the other hand, when fractional factorial designs are interpreted as codes, there are plenty of theoretical results in coding theory [see MacWilliam and Sloane (1977) and Verhoeff (1987)]. The present study has benefited from the ideas in coding theory. The main contribution of this paper is on the minimum aberration designs which, because of its experimental design context, is irrelevant in coding theory.

We will use a new method to present defining contrasts subgroups. With a new representation, we prove that the second moment of a 2^{n-k} design is divisible by 2^{k-1} (Theorems 1 and 2). We use variance to bound the length of the words in a defining contrasts subgroup (Theorem 3). A relation between 2^{n-k} and $2^{(n+1)-k}$ designs is given in Theorem 4. In Section 3 we obtain minimum aberration 2^{n-5} designs for all n. The integer linear programming method and the results of Section 2 are used to prove these designs have minimum aberration. In Section 4, we study the uniqueness of the minimum aberration designs. We give a method to test the equivalence of two 2^{n-k} designs (Theorem 5). Using this method, we prove the uniqueness of 2^{n-k} minimum aberration designs with $k \le 4$. The techniques developed in this paper are apparently new. The uniqueness result is the first one in the literature and the minimum aberration 2^{n-5} designs were only known for $n \le 16$ [Franklin (1984)].

2. Properties of 2^{n-k} fractional factorial designs. The defining contrasts subgroup can also be presented in another way [see also Chen and Wu (1991)]. Let us construct a matrix

$$H = \begin{bmatrix} I_k & B \\ B^t & B^tB \end{bmatrix}, \tag{1}$$

where I_k is a $k \times k$ identity matrix; B is a $k \times (2^k - k - 1)$ matrix which contains all distinct and nonzero linear combinations (modulo 2) of column vectors of I_k; B^t is its transpose. By adding a 0 row and a 0 column to it, we obtain a Hadamard matrix H_{2^k} (when 0's are replaced by -1's) whose rows form a group under summation modulo 2 (so do its columns).

To define a 2^{n-k} fractional factorial design, let us divide the n letters into $2^k - 1$ subsets. Let f_i be the number of letters in the ith subset, such that $\sum_{i=1}^{2^k-1} f_i = n$. For each row vector $\mathbf{u}_j$ of H, form a word w_j by combining all the letters in those subsets for which the component of $\mathbf{u}_j$ is 1.

Hereafter, we regard $(H, \mathbf{f})$ as a design, where $\mathbf{f} = (f_1, f_2, \dots, f_{2^k-1})$ is called the frequency vector of the design.

Clearly, $\langle \mathbf{v}_i, \mathbf{f} \rangle$ equals the length of the ith word, where $\mathbf{v}_j$'s are column vectors of H. Thus all moments of the design can be calculated easily, that is,

$$M_m = \left\| \sum_{j=1}^{2^k-1} f_j \mathbf{v}_j \right\|_m,$$

with $\|\mathbf{v}\|_m = \sum_{i=1}^{2^k-1} v_i^m$, when $\mathbf{v} = (v_1, v_2, \dots, v_{2^k-1})^t$.

From the property of Hadamard matrix, we have $\|\mathbf{v}_j\|_2 = 2^{k-1}$ and $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 2^{k-2}$ for any $i \ne j$. Therefore, when $\sum f_i = n$ is enforced,

$$M_1 = n 2^{k-1}$$

and

$$M_2 = 2^{k-1}\left[\sum f_j^2 + \sum_{i<j} f_i f_j\right] = 2^{k-2} n^2 + 2^{k-2} \sum f_j^2. \tag{2}$$

The formula for M_1 is well known [Brownlee, Kelly and Loraine (1948)]. There are also other formulae for M_2 [Burton and Connor (1957)]. However, our formula relates the design and its second moment more clearly. It is easy to see that in order to minimize M_2, we should choose the values of f_j's as close to being equal as possible. This leads to the following result given in Chen and Wu (1989).

THEOREM 1. *A 2^{n-k} fractional factorial design $(H, \mathbf{f})$ has minimum variance if and only if $f_j = q$ or $q+1$ for all j, where q is determined by $n = q(2^k - 1) + r$, where $0 \le r < 2^k - 1$.*

From (2) we obtain another property of the second moment of a 2^{n-k} fractional factorial design as follows.

THEOREM 2. *For any 2^{n-k} fractional factorial design, its second moment is divisible by 2^{k-1}.*

More properties of the wordlength patterns can be found under our representation. Since searching for optimal designs is essentially searching for optimal wordlength patterns, the following results are helpful.

As is found in Chen and Wu (1991), when we consider all minimum aberration 2^{n-k} designs with fixed k and general n, the range of the wordlengths is finite regardless of the size of n. This is also true for minimum variance designs. In general, a minimum variance design has a relatively short longest word. As variance increases, it enables its longest word to be longer. More precisely, we have:

THEOREM 3. *Let v be the second moment of a minimum variance 2^{n-k} design, M_2 be the second moment of any 2^{n-k} design d and let*

$$M_2 - v = m 2^{k-1}. \tag{3}$$

Then, the length L of the longest word of d satisfies

$$L \le q 2^{k-1} + r' + m, \tag{4}$$

where $n = q(2^k - 1) + r$ with $0 \le r < 2^k - 1$ and $r' = \min\{r, 2^{k-1}\}$.

PROOF. Let $\mathbf{f}$ be the frequency vector of a 2^{n-k} design. We have

$$\mathbf{f}^t H = 2^{k-1} q \mathbf{1}^t + (\mathbf{f}^t - q\mathbf{1}^t)H. \tag{5}$$

Clearly, the largest component of (5) is at most the sum of the positive components of $\mathbf{f}^t - q\mathbf{1}^t$ plus $q2^{k-1}$. Using the formula for M_2, Theorem 1 and condition (3), we have

$$\sum (f_i - q)^2 = 2m + r.$$

Since $\Sigma(f_i - q) = r$, we have

$$\sum (f_i - q)^+ = \Big(r + \sum (f_i - q)^- + \sum (f_i - q)^+\Big)/2$$
$$= \Big(r + \sum |f_i - q|\Big)/2 \le (r + 2m + r)/2 = m + r,$$

where super indices $\pm$ mean positive part or negative part. So the theorem is proved for $r \le 2^{k-1}$.

For $r > 2^{k-1}$, we use mathematical induction. When $m = 0$, d is a minimum variance design. By Theorem 1, $f_i - q = 0$ or 1 for any i. Since each column of H has 2^{k-1} components of 1, all components of $(\mathbf{f}^t - q\mathbf{1}^t)H \le 2^{k-1}$. So (4) is true for $m = 0$. Now suppose (4) is true for $m \le \alpha$, we prove (4) for $m = \alpha + 1$. Let $\mathbf{f}$ be the frequency vector for a design of $m = \alpha + 1$. By Theorem 1, there is at least one pair of (i, j) such that $f_i - f_j \ge 2$. Consider $\mathbf{f}'$ which is the same as $\mathbf{f}$ but with $f_i' = f_i - 1$ and $f_j' = f_j + 1$. It is easy to show that for design $(H, \mathbf{f}')$

$$M_2 - v = m'2^{k-1}, \qquad m' \le m - 1.$$

By induction, its longest words have length at most $q2^{k-1} + 2^{k-1} + m'$. From the relation of $\mathbf{f}$ and $\mathbf{f}'$, the longest words of $\mathbf{f}'$ have length at most $q2^{k-1} + 2^{k-1} + m' + 1 \le q2^{k-1} + 2^{k-1} + m$. This proves (4). □

We have seen that 2^{n-k} and $2^{(n+1)-k}$ designs are closely related. The knowledge of one helps to understand the other. The following theorem focuses on the number of words of the shortest length in the defining contrasts subgroup.

THEOREM 4. *Suppose a 2^{n-k} fractional factorial design d_1 has resolution R, and $A_R(d_1)$ is the first nonzero component of its wordlength pattern. Then, there exists a $2^{(n+1)-k}$ design d_2 with $A_R(d_2) < (1/2)A_R(d_1)$, and $A_R(d_2)$ is the first possible nonzero component of its wordlength pattern.*

PROOF. Let $\mathbf{g}$ be a $(2^k - 1)$-dimensional column vector of 0 and 1. Since

$$\|\mathbf{g}^t H\|_2 = \mathbf{g}^t H H^t \mathbf{g} = 2^{k-2}\|\mathbf{g}\|_1^2 + 2^{k-2}\|\mathbf{g}\|_2,$$

there must be a component of $\mathbf{g}^t H$ which is larger than the square root of $2^{k-2}\|\mathbf{g}\|_1^2/2^k$. That is, there is a column $\mathbf{v}_i$ of H, such that

$$\langle \mathbf{v}_i, \mathbf{g}\rangle > \tfrac{1}{2}\|\mathbf{g}\|_1. \tag{6}$$

We will use (6) to prove this theorem.

Let $\mathbf{f} = (f_1, f_2, \dots, f_{2^k-1})$ and $(H, \mathbf{f})$ be a 2^{n-k} design. Recall that the words in the defining contrast subgroup have a one-to-one correspondence with the rows of H. Let $\mathbf{g}$ be a vector which shows the position of the words of length $A_R(d_1)$, that is, its ith component is 1 when the ith row of H corresponds to a length $A_R(d_1)$ word of $(H, \mathbf{f})$, and is 0 otherwise. By (6), there is a row vector $\mathbf{v}_i$ of H such that $\langle \mathbf{v}_i, \mathbf{g}\rangle > (1/2)\|\mathbf{g}\|_1$. Let $\mathbf{f}'$ be the same as $\mathbf{f}$ but with ith component $f_i + 1$ instead of f_i. Then $(H, \mathbf{f}')$ defines a $2^{(n+1)-k}$ design with $A_R(d_2) < (1/2)A_R(d_1)$. This is because more than half of the words with length R in $(H, \mathbf{f})$ have length $R + 1$ in $(H, \mathbf{f}')$ now. □

An interesting special case is when $A_R(d_1) = 2$. The preceding theorem concludes that $A_R(d_2) = 0$ which implies the latter design has higher resolution.

3. Minimum aberration 2^{n-5} designs. All minimum aberration 2^{n-k} designs with $k \le 4$ are given in Chen and Wu (1989). It is also found that minimum aberration designs have a nice periodicity property when n is large and k is fixed. That is, for fixed k, we can give optimal designs of large n via optimal designs with relatively small n. Precisely, it is proved that:

THEOREM A. *For any fixed k, there exists a positive integer M_k such that for $n \geq M_k$, if a minimum aberration 2^{n-k} design has the wordlength pattern W, then there exists a minimum aberration design $2^{(n+2^k-1)-k}$ with the wordlength pattern* $\mathrm{lag}(W, 2^{k-1})$, *where*

$$\mathrm{lag}(W, 2^{k-1}) = \Big(\overbrace{0, \dots, 0}^{2^{k-1}}, W\Big).$$

REMARK. If $(H, \mathbf{f})$ is a minimum aberration 2^{n-k} design and $n \geq M_k$, then $(H, \mathbf{f} + \mathbf{1})$ is a minimum aberration $2^{(n+2^k-1)-k}$ design.

It is known that $M_k = 1$ when $k < 5$. In this section we find that $M_5 = 14$. In other words, minimum aberration 2^{n-5} designs for $n \geq 45$ can be obtained from those of $14 \leq n \leq 44$. The minimum aberration designs for $1 \leq n \leq 31$, $38 \leq n \leq 40$ and $42 \leq n \leq 44$ and their wordlength patterns are given in Tables 1 to 4. Table 1 specifies the matrix H by presenting the first five row vectors. The other row vectors can be generated from these. Table 2 are the frequency vectors. Each row in the table defines a 2^{n-5} design together with the Hadamard matrix in Table 1. For any n that is not in the previously mentioned range, the minimum aberration 2^{n-5} design can be constructed from that of $2^{(n-31m)-5}$ design. The method is given in the preceding remark. When $n \leq 5$, the designs in the tables are given solely for constructing minimum aberration designs with large n's. They are not necessarily meaningful designs.

An interesting observation can be obtained from our result. The 2^{38-5} design presented has minimum aberration but not minimum variance. This contradicts a conjecture made by Franklin (1984).

We prove that these designs have minimum aberration by ruling out possibilities of less aberration. We have seen that for a vector to be a wordlength pattern of a fractional factorial design, it must satisfy:

RESTRICTION 0. $\Sigma A_i = 2^k - 1$.

RESTRICTION 1. $\Sigma A_{2i-1} = 2^{k-1}$ or 0.

RESTRICTION 2. $\Sigma i A_i = n 2^{k-1}$.

RESTRICTION 3. $\Sigma i^2 A_i \geq 2^{k-2}[n^2 + q^2(2^k - 1) + 2qr + r]$, where $n = q(2^k - 1) + r$.

RESTRICTION 4. $\Sigma i^2 A_i$ is divisible by 2^{k-1}.

RESTRICTION 5. The maximum length of words in a 2^{n-k} design is no more than $q2^{k-1} + r' + m$, which is defined in Theorem 3.

TABLE 1

The H matrix for minimum aberration 2^{n-5} designs (first 5 rows)

1	0	0	0	0	1	1	1	1	1	0	1	0	0	0	1	0	1	1	0	1	0	0	1	1	0	0	1	0	1	1
0	1	0	0	0	1	1	1	0	0	1	0	1	1	0	1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
0	0	1	0	0	1	0	0	1	1	0	0	1	1	1	1	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0
0	0	0	1	0	0	1	0	1	0	1	1	1	0	1	1	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0
0	0	0	0	1	0	0	1	0	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0

TABLE 2

Frequency vectors of minimum aberration 2^{n-5} designs

1	1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
2	1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
3	1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
4	1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
5	1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
6	1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
7	1 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0
8	1 1 1 1 1 0 0 0 0 0 1 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0
9	1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 1 0 0 0 0 0 0 0 0 0 0
10	1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 1 0 0 0 1 0 0 0 0 0 0
11	1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 1 0 0 0 1 0 0 0 0 0 1
12	1 1 1 1 1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
13	1 1 1 1 1 1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
14	1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
15	1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
16	1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
17	1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0
18	1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0
19	1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0
20	1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 1 0 0 0 0 0 0 0 0 0 0
21	1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 1 0 0 0 1 0 0 0 0 0 0
22	1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0 0
23	1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0
24	1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 0 0 0 0 0 0
25	1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 0 0 0 0 0
26	1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 0 0 0 0
27	1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 0 0 0
28	1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 0 0
29	1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 0
30	1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0
31	1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
38	1 2 1 1 2 2 1 1 1 2 1 1 1 1 1 1 1 2 1 2 1 1 1 1 1 2 1 2 1 0 1
39	1 2 1 1 2 2 1 1 1 2 1 1 1 1 1 1 1 2 1 2 1 1 1 1 1 2 1 2 2 0 1
40	1 2 1 1 2 2 1 1 1 2 1 1 1 1 1 1 1 2 1 2 1 1 1 1 1 2 1 2 2 0 2
42	1 2 1 2 2 2 1 1 2 2 2 1 1 1 1 2 1 1 1 1 1 2 1 2 1 2 1 2 1 0 1
43	1 2 1 2 2 2 1 1 2 2 2 1 1 1 1 2 1 1 1 1 1 2 1 2 1 2 1 2 2 0 1
44	1 2 1 2 2 2 1 1 2 2 2 1 1 1 1 2 1 1 1 1 1 2 1 2 1 2 1 2 2 0 2

TABLE 3

Wordlength patterns of minimum aberration 2^{n-5} *designs* $(1 \le n \le 31)$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	15	16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	7	16	8	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	3	12	12	4	0	0	0	0	0	0	0	0	0	0	0	0	0
4	1	8	12	8	2	0	0	0	0	0	0	0	0	0	0	0	0
5	0	5	10	10	5	1	0	0	0	0	0	0	0	0	0	0	0
6	0	0	15	0	15	0	1	0	0	0	0	0	0	0	0	0	0
7	0	0	5	12	7	4	3	0	0	0	0	0	0	0	0	0	0
8	0	0	1	10	11	4	3	2	0	0	0	0	0	0	0	0	0
9	0	0	0	4	14	8	0	4	1	0	0	0	0	0	0	0	0
10	0	0	0	0	10	16	0	0	5	0	0	0	0	0	0	0	0
11	0	0	0	0	4	14	8	0	3	2	0	0	0	0	0	0	0
12	0	0	0	0	1	8	12	8	1	0	0	0	1	0	0	0	0
13	0	0	0	0	0	3	12	12	3	0	0	0	0	1	0	0	0
14	0	0	0	0	0	0	7	16	7	0	0	0	0	0	1	0	0
15	0	0	0	0	0	0	0	15	15	0	0	0	0	0	0	1	0
16	0	0	0	0	0	0	0	0	30	0	0	0	0	0	0	0	1
17	0	0	0	0	0	0	0	0	14	16	0	0	0	0	0	0	1
18	0	0	0	0	0	0	0	0	6	16	8	0	0	0	0	0	1
19	0	0	0	0	0	0	0	0	2	12	12	4	0	0	0	0	1
20	0	0	0	0	0	0	0	0	0	8	12	8	2	0	0	0	1
21	0	0	0	0	0	0	0	0	0	0	20	0	10	0	0	0	1
22	0	0	0	0	0	0	0	0	0	0	6	16	6	0	2	0	1
23	0	0	0	0	0	0	0	0	0	0	0	14	14	0	0	2	1
24	0	0	0	0	0	0	0	0	0	0	0	0	28	0	0	0	3
25	0	0	0	0	0	0	0	0	0	0	0	0	12	16	0	0	3
26	0	0	0	0	0	0	0	0	0	0	0	0	4	16	8	0	3
27	0	0	0	0	0	0	0	0	0	0	0	0	0	12	12	4	3
28	0	0	0	0	0	0	0	0	0	0	0	0	0	0	24	0	7
29	0	0	0	0	0	0	0	0	0	0	0	0	0	0	8	16	7
30	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	16	15
31	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	31

TABLE 4

Wordlength patterns of minimum aberration 2^{n-5} *designs*
$(n = 38, 39, 40, 42, 43, 44)$

	17	18	19	20	21	22	23	24	25	26	27	28	29	30
38	0	4	16	6	0	4	0	1	0	0	0	0	0	0
39	0	0	12	14	0	0	4	1	0	0	0	0	0	0
40	0	0	0	26	0	0	0	5	0	0	0	0	0	0
42	0	0	0	3	16	8	0	3	0	0	0	1	0	0
43	0	0	0	0	11	12	4	3	0	0	0	0	1	0
44	0	0	0	0	0	23	0	7	0	0	0	0	0	1

The first three restrictions are from the literature [see Burton and Connor (1957) or Fries and Hunter (1984)]. The others are discussed in this paper earlier.

In addition, any subgroup of a defining contrasts subgroup defines a design and satisfies Restrictions 0–5 (possibly with different n and k). In particular, we have:

RESTRICTION 6. The first moment of even length words is divisible by 2^{k-2}.

The following lemma is useful in conjunction with the above arguments. This is a standard result in group theory.

LEMMA 1. *Let G be a defining contrasts subgroup of a 2^{n-k} design with $k \geq 2$, w_1, w_2 are any two nonidentity elements. There always exists a subgroup $G_1 \subset G$ of size 2^{k-1}, such that $w_1, w_2 \notin G_1$.*

With the help of the integer linear programming method and these necessary conditions, we prove that no other designs may have less aberration. In the next section, we briefly introduce the integer linear programming method. We will also give a detailed example for illustration.

3.1. *The integer linear programming method.* In a linear programming problem, a linear function is minimized subject to some linear constraints,

$$\begin{aligned} \text{minimize:} \quad & a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \\ \text{subject to:} \quad & a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \leq b_1 \\ & a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \leq b_2 \\ & \qquad \vdots \\ & a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \leq b_m . \end{aligned} \tag{7}$$

When all x_i's are restricted to be integers, it is called integer linear programming. For more details on linear programming see Srinath (1982).

In our problem, we optimize wordlength pattern $W = (x_1, x_2, \dots)$. By the definition of minimum aberration, x_1, x_2 and so on are minimized sequentially. The corresponding constraints are Restrictions 0–6. Thus, the existing linear integer programming method can be used except that:

1. For each n, the length of W is n. As n grows, it is impossible for any program to accommodate so many variables. This should be fixed before a program can be applied. In the next example we will show how this problem can be avoided.
2. We may get a W which is not a wordlength pattern. Restrictions 0–6 are necessary but not sufficient.

So our procedure is:

1. For each n and k, find a 2^{n-k} design which is likely to be a minimum aberration design.
2. Use the integer linear programming method to search for W's with less aberration and satisfying Restrictions 0–6. We call these W's *feasible solutions*.
3. If there is no feasible solution, the design has minimum aberration. Otherwise, either we prove none of the feasible solutions are wordlength pattern of any design or we look for a design with less aberration from the feasible solutions.

As an example, we prove the $2^{(31m+14)-5}$ designs given in Table 1 and 2 have minimum aberration by using the integer linear programming method. The designs we give have the following wordlength pattern:

length($16m+$)	6	7	8	9	10	11	12	13	14
# of words	7	16	7	0	0	0	0	0	1

We will omit "length($16m+$)" and "# of words" hereafter. A vector $W=(x_1,x_2,\dots)$ will have less aberration if $x_1=x_2=\cdots=x_5=0$ and $x_6<7$. If there is no odd length word, the proof is straightforward. Otherwise there are 16 odd length words present, the problem becomes:

$$
(8)\qquad
\begin{aligned}
\text{minimize:}\quad & x_6\\
\text{subject to:}\quad & x_6 \le 6\\
x_6+x_8+x_{10}\ \cdots &= 15\\
x_7+x_9+x_{11}\ \cdots &= 16\\
x_7+2x_8+3x_9+\cdots &= 38\\
x_7+4x_8+9x_9+\cdots &\ge 108.
\end{aligned}
$$

According to the first constraint we look for solutions with $x_1<7$ only. The second and the third count the number of odd and even length words. The last two take care of the first and second moments and have been simplified according to the following relationships:

$$\sum (i-c)A_i=\sum iA_i+cn2^{k-1}=M_1+cM_0,$$

$$\sum (i-c)^2A_i=\sum i^2A_i-2c\sum iA_i+c^2n2^{k-1}=M_2-2cM_1+c^2M_0.$$

We choose $c=16m+6$ to eliminate m in (8). Thus the solution of (8) is for all m.

To apply the NAG (numerical algorithms group) computer software, we need to limit the number of x's in (8). This is done by noticing that the first four constraints imply all $x_i=0$ for $i\ge 13$. The computer finds that there is no feasible solution for (8). Next, we set $x_6=7$ and minimize x_7 and so on. There is no feasible solution found and hence the $2^{(31m+14)-5}$ design in Table 2 has minimum aberration.

We have completed this example by using Restrictions 0–3 only. However, Restrictions 4–6 are useful in other cases. We may find a feasible solution which has the second moment not divisible by 2^{k-1}. To avoid such solutions, for example, we can replace the last constraint in (8) by

$$x_7 + 4x_8 + 9x_9 + \cdots = 108 + 16 \times j$$

and use Restriction 5 to control the number of x_i's in the problem. If there are still unwanted possibilities, Restriction 6 becomes our last systematic method to rule them out.

We will go over all the designs in Table 1 and 2 in the next section.

3.2. *Brief proofs*. In this section, we use the integer linear program provided by NAG to search for feasible solutions. If as in the example given in the last section, there is no feasible solution, we will give the conclusion without details.

For $n = 31m + 1$, $31m + 2$, $31m + 3$, $31m + 4$, there is no feasible solution.

For $n = 31m + 5$, there is no feasible solution when an extra constraint $A_{16m+1} + A_{16m+2} \geq 15$ is used. If this constraint is not true, we will obtain a $2^{(31m+6)-5}$ design with less aberration than the one in Table 2. Here, we preassume that the $2^{(31m+6)-5}$ designs in Table 2 have minimum aberration. This is shown in the following discussion.

For $n = 31m + 6$, our designs have wordlength pattern

1	2	3	4	5	6
0	15	0	15	0	1

It is straightforward to see there is no feasible solution without odd length words. Thus, a feasible solution $W = (x_1, x_2, \ldots)$ must satisfy

$$x_2 + x_4 + x_6 + \cdots = 15,$$
$$x_3 + x_5 + x_7 + \cdots = 16,$$
$$2x_2 + 3x_3 + 4x_4 + \cdots = 96.$$

These imply

$$2x_2 + 4x_4 + 2x_5 + 6x_6 + \cdots = 48,$$

and hence

$$2x_2 + 4x_4 + 6x_6 + \cdots \leq 48.$$

If a feasible solution W exists and is indeed a wordlength pattern, all the even words will define a $2^{n'-4}$ design. The first moment of this design is

$$\sum (2i + 16m)x_{2i} = \sum (2i)x_{2i} + 16m(2^4 - 1) \leq (30m + 6) \times 2^{4-1}.$$

That is, according to Restriction 1, $n' \leq 30m + 6 < n$ unless $m = 0$. The case of $m = 0$ is trivial. When $n' < n$, all the odd length words share a common letter. When this letter is removed from each of the words in W, the set of new words form a new defining contrasts subgroup for a $2^{(31m+5)-5}$ design. The new design has wordlength pattern $(y_1, y_2, \dots)$ with $y_{2i} = x_{2i} + x_{2i+1}$ and $y_{2i-1} = 0$. Thus, this design has resolution $16m + 2$ which is impossible.

For $n = 31m + 7$ and $m = 0$, its proof is straightforward. When $m \geq 1$, it is clear from Theorem 2 that there must be at least three words of length $16m + 2$. Otherwise, it contradicts the result for $n = 31m + 8$. Again, we preassume the result for $n = 31m + 8$. It is to be proved later.

Using the integer linear programming method, we find a feasible solution as follows:

2	3	4	5	6	7	8
4	14	10	0	0	2	1

For this solution, note that the upper bound of the longest word given by Theorem 3 is reached. If the solution is a wordlength pattern of some design, then from (5) there must be two columns of H, $\mathbf{v}_1$ and $\mathbf{v}_2$, such that

$$\langle (\mathbf{f}^t - q), \mathbf{v}_1 \rangle = 8, \qquad \langle (\mathbf{f}^t - q), \mathbf{v}_2 \rangle = 7.$$

Also, from the second moment formula, we find that $f_i - q = 1$ for eight i's and $f_i - q \leq 0$ for others. So the preceding equations imply that $\mathbf{v}_1(i) = 1$ whenever $f_i - q = 1$, and so does $\mathbf{v}_2$ with one exception. Therefore, $\mathbf{v}_1(i) + \mathbf{v}_2(i) = 0 \pmod 2$ whenever $f_i - q = 1$ with only one exception. So if $\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2 \pmod 2$,

$$\langle (\mathbf{f}^t - q), \mathbf{v}_3 \rangle = 1.$$

This implies that there is a word with length $16m + 1$, which contradicts the solution itself.

For $n = 31m + 8$ and $m = 0$, the proof is straightforward. When $m \geq 1$, let x_{19} and x_{20} be the words of length $16m + 19$ and $16m + 20$. Since minimum aberration $2^{(31m+9)-5}$ design has 26 words of length $16m + 20$. We must have $x_{19} + x_{20} \geq 26$. Under this restriction and Restrictions 0–6, we find no solutions.

For $n = 31m + 9$ and $m = 0$, the proof is straightforward. When $m \geq 1$, we have one feasible solution:

4	5	6	7	8	9	10
25	0	3	0	2	0	1

Using a similar method to the one for $n = 31m + 7$, we can show if a defining contrasts subgroup has words of length $16m + 8$ and $16m + 10$ at the same time and its variance is the same as the variance of this solution, then it must have a word of length $16m + 2$. So this solution is not a wordlength pattern.

For $n = 31m + 10$, there is only one feasible solution:

4	5	6	7	8	9	10
9	16	3	0	2	0	1

We find that this solution has minimum variance, so if this is a wordlength pattern of some m, it must also be a wordlength pattern of $m = 0$ because of periodicity. This is impossible.

For $n = 31m + 11$ and $m = 0$, the proof is straightforward. When $m \geq 1$, assumed that our result for $n = 31m + 12$ has been proved. By Theorem 4, we can see that the first nonzero component of its wordlength pattern must be at least 3. From Table 2, we know it can be as small as 4. If the smallest possible is 4, there will be no feasible solutions by using the integer linear programming method. Let us assume that it can be 3. Under this assumption, we find one feasible solution:

4	5	6	7	8	9	10	11	12	13
3	15	11	0	0	0	1	0	0	1

The proof of impossibility of this solution is similar to that for $n = 31m + 7$.

For $n = 31m + 12$ and $m = 0$, the proof is straightforward. When $m \geq 1$, we find one feasible solution:

5	6	7	8	9	10	11	12
10	14	5	0	0	0	1	1

The proof of impossibility of this solution is again similar to that for $n = 31m + 7$.

For $n = 31m + 13$, when $m = 0$ and $m \geq 1$ are considered separately, there is no feasible solution for either of them.

For $n = 31m + 14$, $31m + 15$, $31m + 16$, $31m + 17$ and $n = 31m + 18$, there is no feasible solution.

For $n = 31m + 19$, we find seven feasible solutions:

8	9	10	11	12	13	14	15	16	17	18	19	20
1	16	12	0	0	0	0	0	2	0	0	0	0
1	16	12	0	1	0	0	0	0	0	0	0	1
1	16	11	0	1	0	1	0	1	0	0	0	0
1	16	11	0	2	0	0	0	0	0	1	0	0
1	16	10	0	2	0	2	0	0	0	0	0	0
1	15	12	0	2	0	0	0	0	1	0	0	0
1	14	13	1	0	0	1	1	0	0	0	0	0

Note that all the feasible solutions share a common feature: There are a few words which are much longer than the other ones. From Lemma 1, for the first two solutions, we can find a subgroup of size 15 (not counting the identity) whose elements have length 8, 9 and 10 only. According to Restriction 1, each has wordlength pattern as one of the following:

8	9	10
1	8	6
0	8	7

which violates Restriction 2. Similarly, for the rest of the feasible solutions, we can first find a subgroup of size 15, then a subgroup of this subgroup of size 7, whose elements have words of length 9 and 10 only. According to Restriction 1, each has wordlength patterns as:

9	10
4	3

which violates Restriction 2.

For $n = 31m + 20, 31m + 21, \dots, 31m + 31$, there is no feasible solution.

4. Uniqueness of the minimum aberration designs. A lot of designs with practical usage or theoretical interest have been found. However, for fractional factorial designs, many optimality criteria are based on the wordlength pattern. There are examples where the wordlength pattern does not uniquely determine the design. It is possible that for given n and k, there are two different minimum aberration designs. It is of interest to know whether we have found all minimum aberration designs for each n and k.

When we say different designs, we should note that designs can appear in different ways. Two designs are said to be *equivalent* if one can be obtained from the other via sign changes in columns, rearrangement of runs and rearrangement of columns. When two designs are nonequivalent, they are different.

The following two designs given by Draper and Mitchell (1968) have the same wordlength pattern:

$$\begin{aligned}(1)\qquad I &= 12367t_0 = 12389t_1 = 6789t_0t_1 = 14569t_2\\ &= 234579t_0t_2 = 234568t_1t_2 = 14578t_0t_1t_2,\end{aligned}$$

$$\begin{aligned}(2)\qquad I &= 13469t_0 = 13578t_1 = 456789t_0t_1 = 1234568t_2\\ &= 2589t_0t_2 = 2467t_1t_2 = 12379t_0t_1t_2,\end{aligned}$$

where t_0, t_1, t_2 are used for 10, 11, 12. They are found to be nonequivalent because of different *letter patterns*. The letter pattern is defined as a matrix whose (i, j)th entry is the frequency of the letter i in length j words. The letter pattern does not uniquely determine a fractional factorial design either, see Chen and Lin (1991).

In order to study this problem, we first need a practical way to test whether two designs are equivalent. With the help of the frequency representation, we suggest the following testing method.

THEOREM 5. *Let* $\mathbf{f} = (f_1, f_2, \dots, f_{2^k-1})^t$ *and* $\mathbf{g} = (g_1, g_2, \dots, g_{2^k-1})^t$ *be two frequency vectors, and* H *be given by* (1), *such that* $(H, \mathbf{f})$ *and* $(H, \mathbf{g})$ *are two* 2^{n-k} *fractional factorial designs. If there exists a relabelling map* ψ *for* $(1, 2, \dots, 2^k - 1)$, *such that for any* i *and* j,

1. $f_i = g_{\psi(i)}$;
2. $\mathbf{v}_{\psi(i)} * \mathbf{v}_{\psi(j)} = \mathbf{v}_{\psi(l)}$, *when* $\mathbf{v}_i * \mathbf{v}_j = \mathbf{v}_l$,

where $\mathbf{v}_i, \mathbf{v}_j$ *are row vectors of* H, $*$ *stands for sum of modulo* 2, *then* $\mathbf{f}$ *and* $\mathbf{g}$ *are equivalent. Otherwise, they are not.*

PROOF. The second conclusion is obvious.

The design $(H, \mathbf{g})$ is equivalent to $(H', \mathbf{g}')$ where

$$H' = (\mathbf{v}_{\psi(1)}, \mathbf{v}_{\psi(2)}, \dots, \mathbf{v}_{\psi(2^k-1)}),$$

$$\mathbf{g}' = (g_{\psi(1)}, g_{\psi(2)}, \dots, g_{\psi(2^k-1)})^t = \mathbf{f}.$$

Although H' is not the matrix specified in (1), $(H', \mathbf{g}')$ can determine a defining contrasts subgroup in the same way as $(H, \mathbf{g})$ does. We therefore name it a design for the sake of convenience.

If H' can be obtained from H by rearranging the rows of H, then $(H', \mathbf{g}')$ has the same words as $(H, \mathbf{f})$. Therefore, they are equivalent. Since $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ generate all the other columns, by condition 2, $\mathbf{v}_{\psi(1)}, \mathbf{v}_{\psi(2)}, \dots, \mathbf{v}_{\psi(k)}$ generate all the other columns in the same way. So if these two sets of vectors can be obtained from each other by rearranging rows, so can H' be obtained from H. Note that there are only 2^k different k-dimensional vectors of 0 and 1, and both $[\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k]$ and $[\mathbf{v}_{\psi(1)}, \mathbf{v}_{\psi(2)}, \dots, \mathbf{v}_{\psi(k)}]$ must contain all of them except the 0 vector. This shows that they can be obtained from each other by rearranging rows. Hence the theorem is proved. □

With the help of this theorem, we prove that all minimum aberration designs of $k \le 4$ are unique up to equivalence.

Consider the 2^{n-k} designs. For $k = 1$ and 2, we have the following stronger result.

THEOREM 6. *Any* 2^{n-k} *fractional factorial design with* $k = 1, 2$ *is uniquely determined by its wordlength pattern.*

PROOF. It is obvious when $k = 1$. For $k = 2$, let $\mathbf{f} = (f_1, f_2, f_3)$ and $\mathbf{g} = (g_1, g_2, g_3)$, be two frequency vectors. If they have the same wordlength pattern, then using moments relations we find

$$f_1 + f_2 + f_3 = g_1 + g_2 + g_3,$$
$$f_1 f_2 + f_1 f_3 + f_2 f_3 = g_1 g_2 + g_1 g_3 + g_2 g_3,$$
$$f_1 f_2 f_3 = g_1 g_2 g_3.$$

This shows that $\mathbf{g}$ is only a rearrangement of $\mathbf{f}$. Since any two of the nonzero columns of H generate the third nonzero column, this rearrangement map satisfies the conditions in Theorem 5. Therefore, $\mathbf{f}$ and $\mathbf{g}$ are equivalent. □

The same conclusion is not true when $k = 3$ or 4. However, we have:

THEOREM 7. *Any* 2^{n-k} *fractional factorial design with minimum aberration is uniquely determined by its wordlength pattern when* $k = 3, 4$.

We give a lemma first.

LEMMA 2. *Suppose A and B are two subsets of a group G which satisfies $a * a = I$ (identity) for any $a \in G$, where $*$ is the operation of G. If there exists a bijective map ψ: $A \to B$, such that for $a_i \in A$, $i = 1, 2, \dots, n$,*

$$(9)\quad \psi(a_1) * \psi(a_2) * \cdots * \psi(a_n) = I, \quad \text{if and only if} \quad a_1 * a_2 * \cdots * a_n = I,$$

then ψ can be extended to an isomorphic map of $G \to G$.

REMARK. For formal definition of isomorphism, see Pinter (1982).

The proof of this lemma is straightforward. It uses standard methods to extend ψ from A to the subgroup generated by A, and so on. Note, it is important that $a * a = I$ for any $a \in G$. Without this condition, (9) might not be satisfied after ψ is extended.

If such a ψ exists, we say A and B are isomorphic subsets.

PROOF OF THEOREM 7. Under our new representation, we show any two minimum aberration designs $(H, \mathbf{f}_1)$ and $(H, \mathbf{f}_2)$ with the same n and k ($k \le 4$), are equivalent.

From Chen and Wu (1991), all minimum aberration designs for $k = 3$ or 4 have minimum variance, and minimum variance designs are fully periodic. Thus we need only prove the uniqueness for $n \le 2^k - 1$. In addition, $\mathbf{f}$'s can only have components 0 or 1. To prove uniqueness, we need only examine the isomorphism of the column sets determined by different $\mathbf{f}$'s.

We first consider $k = 3$. Let us name the columns of H as $\{a, b, c, ab, ac, bc, abc\}$ with, as usual, $a + b \pmod 2 = ab$ and so on.

For $n = 1$, any two columns of H are clearly isomorphic with each other. Thus there is only one minimum aberration design. The same is true for $n = 2$.

For $n = 3$, we have two nonisomorphic subsets of columns, $\{a, b, c\}$ and $\{a, b, ab\}$. The first one gives the minimum aberration design and the second one determines a design with more aberration.

For $n = 4$, a subset of four nonzero elements is a complement of a subset of three nonzero elements. So the proofs is the same as $n = 3$. For $n = 5, 6$, the proofs are similar to that for $n = 4$. For $n = 7$, it is obvious.

Next we consider $k = 4$. Let the columns of H be

$$(a, b, ab, c, ac, bc, abc, d, ad, bd, abd, cd, acd, bcd, abcd).$$

For $n = 1, 2, 3, 12, 13, 14, 15$, it is clear that the proofs for $k = 3$ can also be used for $k = 4$.

For $n = 4, 11$, there are three nonisomorphic subsets:

$$\{a, b, c, d\}, \qquad \{a, b, c, abc\}, \qquad \{a, b, c, ab\}.$$

For $n = 5, 10$, there are four nonisomorphic subsets:

$$\{a, b, c, d, abcd\}, \qquad \{a, b, c, d, abc\}, \qquad \{a, b, c, d, ab\}, \qquad \{a, b, c, ab, abc\}.$$

For $n = 6, 9$, there are at most eight nonisomorphic subsets:

1. $\{a, b, c, d, abc, abcd\}$
2. $\{a, b, c, d, ab, abcd\}$
3. $\{a, b, c, d, abc, bcd\}$
4. $\{a, b, c, d, ab, abc\}$
5. $\{a, b, c, d, ab, bcd\}$
6. $\{a, b, c, d, ab, cd\}$
7. $\{a, b, c, d, ab, ac\}$
8. $\{a, b, c, ab, ac, bc\}$.

Calculations show that only one structure for each n has minimum aberration. For $n = 7$, the proof is similar. □

REMARK. As pointed out by a referee, in the case of $n = 6, 9$, subsets $1, 2, 5$ are isomorphic, $3, 7$ are isomorphic. However, by including all eight sets, it is clearer that we miss nothing.

REMARK. The same technique can also be applied to find the number of nonequivalent fractional factorial designs for each n and k. However, it becomes more complicated as n and k get large.

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