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DOMINANCE OF LIKELIHOOD RATIO TESTS UNDER CONE CONSTRAINTS[1]

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It is well known that anomalies are sometimes observed when using the likelihood ratio test (LRT) for testing restricted hypotheses in a normal model. This paper considers a general framework for these anomalies to occur. We provide a condition, that relates the null and the alternative hypotheses, under which the dominance of the LRT is obtained. Conditions are also given which guarantee the equivalence between the LRT and a simpler test. The situations of known and unknown variances are considered and examples are given to illustrate the results.

1. Introduction. In this paper we consider a testing problem where both the null and the alternative hypotheses impose restrictions on the mean of a normal population. A number of papers have considered this problem, dealing among others with restrictions such as homogeneity, monotonicity, symmetry or unimodality. The book by Robertson, Wright and Dykstra (1988) should be mentioned as perhaps the most complete reference for these and other topics on restricted inference.

In a general sense, the hypotheses under consideration can be considered as cones of the parameter set. Before we present the aim and scope of the present paper, let us describe briefly a few simple geometrical facts associated with closed convex cones, which we will use throughout the paper. A reference book for the corresponding theory is Stoer and Witzgall (1970). Consider R^k with an inner product $\langle \cdot , \cdot \rangle$ and associated norm $\| \cdot \|$. Given a closed and convex cone C of R^k, the orthogonal projection $p(\cdot|C)$ onto C exists, is unique and verifies $\|x\|^2 = \|p(x|C)\|^2 + \|x - p(x|C)\|^2$, for all x in R^k. With a cone C is associated the so-called polar cone C^p, $C^p = \{y \in R^k | \langle x, y \rangle \leq 0,\ \forall\ x \in C\}$. If $C = L$ is a linear subspace of R^k, then $C^p = L^{\perp}$ is the orthogonal complement of L. When C is a closed convex cone, $x - p(x|C) = p(x|C^p)$ and then $p(x|C^p)$ is orthogonal to $p(x|C)$. The closure of a cone C, that is the smallest closed and convex set containing C, will be denoted by $\mathrm{cl}(C)$. In applications, C is often a polyhedral cone, defined by a finite number of linear inequalities, $C = \hat{A} = \{a'_i x \geq 0,\ i = 1, \ldots, r\}$. Some results such as Lemmas 2.2 and 3.2 in Raubertas, Nordheim and Lee (1986), used frequently throughout this paper, are usually presented for polyhedral cones, but they can be generalized to closed convex cones.

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Throughout this paper, we consider a k-dimensional random normal vector $Y \sim N_k(\theta, \Gamma)$, with mean θ restricted to belong to a closed convex cone C of R^k, and address the problem of testing H_0: $\theta \in C \cap L$, where L is a linear subspace, against $H_a - H_0$, with H_a: $\theta \in C$.

The corresponding likelihood ratio test (LRT) is defined by the statistic

$$T(Y) = \|Y - p(Y|C \cap L)\|^2 - \|Y - p(Y|C)\|^2, \tag{1.1}$$

where $\|y\|^2 = y'\Gamma^{-1}y$. The projections $p(Y|C \cap L)$ and $p(Y|C)$ define the MLE's for θ under H_0 and H_a, respectively.

When testing H_0 against $H_a - H_0$ one needs to find the null hypothesis distribution of T in order to determine a critical value for the test. However, in this situation H_0 is a composite hypothesis, so that one needs to find $\sup_{\theta \in C \cap L} P_\theta(T(Y) \geq t)$ and determine the distribution of $T(Y)$ at the value of θ where the supremum occurs.

Two cases may arise. In the first case the null hypothesis is a linear subspace, $L \subset C$. The most characteristic example of this situation is the problem of testing homogeneity against monotonicity [Bartholomew (1961)]. In such cases the LRT provides a satisfactory method. The supremum previously mentioned typically occurs at $\theta = 0$ (homogeneity) and the null distribution of $T(Y)$ is a chi-bar-squared distribution. This condition is sometimes referred as "homogeneity is the least favourable configuration under H_0." Some works along this line are Raubertas, Nordheim and Lee (1986), Shapiro (1988) and Robertson, Wright and Dykstra (1988).

A property of interest, present when $L \subset C$, is

$$p(p(x|C)|C \cap L) = p(x|C \cap L), \qquad \forall\, x \in R^k, \tag{1.2}$$

and it is said that H_0 and H_a are not oblique. The term "oblique" was introduced by Warrack and Robertson (1984).

Property (1.2) may also hold in the second case, namely, when the null hypothesis H_0: $\theta \in C \cap L$ is not a linear subspace. When H_0 and H_a are not oblique, the null distribution of $T(Y)$ is also a chi-bar-squared distribution and the supremum of the power function over H_0 is attained at $\theta = 0$ [Menéndez, Rueda and Salvador (1991)]. A particular instance of this situation occurs when H_0 is a face of a right polyhedral cone C, with C the cone specified in H_a.

Difficulties arise when H_0 and H_a are oblique. Then the preceding supremum is often not attained at $\theta = 0$ and $\sup_{\theta \in C \cap L} P_\theta(T(Y) \geq t) = P_0(T^*(Y) \geq t)$, where $T^*(Y)$ is the likelihood ratio statistic for testing a reduced problem H_0^* against $H_a^* - H_0^*$, with $H_0 \subset H_0^*$ and $H_a \subset H_a^*$.

The first reference to a problem of this type is Warrack and Robertson (1984). See also Menéndez and Salvador (1990), where the domination of the LRT for testing a face of an acute cone against the cone is proved. A different example is given by Robertson (1986). For testing the symmetry and unimodality of k means against unimodality, the LRT was shown to be dominated by another "reduced test."

This paper provides a general framework in which the results in the preceding three papers are viewed as particular cases. In Section 2, for the case Γ known, a general condition on C and L is given under which the LRT is dominated by some other test, in the sense that this test has the same size and at least the same power on the alternative hypothesis. This dominating test becomes the LRT for testing a reduced problem, which is specified. Also, a necessary and sufficient condition is given for the equivalence of the LRT to a simpler test.

The same questions are considered in Section 3 in the case where variances are known up to a constant σ^2. In Section 4 several applications are given that illustrate the previous results.

Two related references are Berger (1989) and Tang (1991) where the significance level of LRT's, for other restricted problems, is attained at an infinite point of the null hypothesis and LRT's also become dominated.

It is of interest to note that projections are preserved by linear transformations of the entire statistical problem. Hence without loss of generality we assume $\Gamma = I$ and use the unit metric on R^k.

2. Dominance and equivalence of the LRT.

Lemma 2.1 provides three equivalent conditions which are important in determining when the LRT is dominated by, or equivalent to, another test.

LEMMA 2.1. *Let C and L, respectively, be a closed convex cone and a linear subspace. Then the following conditions are equivalent*:

(i) $p(p(x|L)|C) \in L$, $\forall\, x \in R^k$.
(ii) $p(p(x|C)|L) \in C$, $\forall\, x \in R^k$.
(iii) $p(x|L) = p(x|C \cap L)$, $\forall\, x \in C$.

PROOF. (i) $\Rightarrow$ (ii) First note that $\|p(y|C)\|^2 = \langle y, p(y|C)\rangle$ for any closed convex cone C [see (8.2.6) in Robertson, Wright and Dykstra (1988)]. For any point x of R^k, let $z = p(p(x|C)|L)$. Then,

$$\begin{aligned}\|z - p(z|C)\|^2 &= \|p(z|C^p)\|^2 = \langle z, p(z|C^p)\rangle \\ &= \langle z - p(x|C), p(z|C^p)\rangle + \langle p(x|C), p(z|C^p)\rangle.\end{aligned}$$

But $z - p(x|C) = -p(p(x|C)|L^{\perp}) \in L^{\perp}$ and $p(z|C^p) = z - p(z|C) \in L$ because $z \in L$ and $p(z|C) = p(p(p(x|C)|L)|C) \in L$ by assumption. Hence $\|z - p(z|C)\|^2 = 0$, implying $z = p(z|C) \in C$. We are in debt to a referee for this part of the proof.

(ii) $\Rightarrow$ (i) Let us denote $u = p(p(x|L)|C)$ for any $x \in R^k$. Since $p(x|L) - p(u|L) \in L$ and $u - p(u|L) \in L^{\perp}$, we can decompose $\|u - p(x|L)\|^2 = \|u - p(u|L)\|^2 + \|p(u|L) - p(x|L)\|^2$. But u is the projection of $p(x|L)$ onto C and $p(u|L) \in C$ by assumption, hence $\|u - p(x|L)\|^2 \le \|p(u|L) - p(x|L)\|^2$, so that $\|u - p(u|L)\|^2 = 0$. Therefore $u = p(U|L) \in L$.

The equivalence between (ii) and (iii) is proved by observing that they are both equivalent to the equality $p(p(x|C)|L) = p(p(x|C)|C \cap L)$, $\forall\, x \in R^k$. $\square$

Theorem 2.1 establishes the dominance of the LRT when any of the conditions in Lemma 2.1 is met. Although the case $T = T^*$ a.s. for every $\theta \in H_a$ is included in Theorem 2.1, Theorem 2.2 provides a necessary and sufficient condition for such a situation, where the dominance of the LRT becomes an equivalence to a simpler test.

It is necessary to note that, in general, $C \cap L$ is not defined by only a linear subspace L. Throughout this paper L is taken to be the subspace of smallest dimension amongst those that contain $C \cap L$. Note that $\dim(L) = \dim(C \cap L)$.

THEOREM 2.1. *Assume* $Y \sim N_k(\theta, I)$ *and* L *and* C *satisfy any of the conditions in Lemma* 2.1. *Then the* LRT *for testing* H_0: $\theta \in C \cap L$ *against* $H_a - H_0$, *with* H_a: $\theta \in C$, *is dominated by the* LRT *for testing* H_0^*: $\theta \in L$ *against* $H_a^* - H_0^*$, *with* H_a^*: $\theta \in C^*$, $C^* = \mathrm{cl}(C + L)$.

PROOF. Similarly to T in (1.1), we denote by $T^*(Y) = \|Y - p(Y|L)\|^2 - \|Y - p(Y|C^*)\|^2$ the LR statistic for testing H_0^* against $H_a^* - H_0^*$.

From (8.2.6) in Robertson, Wright and Dykstra (1988) we can write

$$T(y) = \|p(y|C)\|^2 - \|p(y|C \cap L)\|^2 \qquad \forall\, y \in R^k, \tag{2.1}$$

$$T^*(y) = \|p(y|C^*)\|^2 - \|p(y|L)\|^2 \qquad \forall\, y \in R^k. \tag{2.2}$$

On the other hand, from Lemma 2.2 in Raubertas, Nordheim and Lee (1986), we have, since $L \subset C^*$

$$p(p(y|C^*)|L) = p(y|C), \qquad \forall\, y, \tag{2.3}$$

and also

$$\begin{aligned} p(y|C \cap L) &= p(p(y|L)|C \cap L) = p(p(p(y|C^*)|L)|C \cap L) \\ &= p(p(y|C^*)|C \cap L), \qquad \forall\, y. \end{aligned} \tag{2.4}$$

Now in a first step, we will prove that,

$$T(y) \le T(p(y|C^*)) \le T^*(p(y|C^*)) = T^*(y), \qquad \forall\, y \in R^k. \tag{2.5}$$

We have

$$\|p(y|C)\|^2 = \|y - p(y|C^p)\|^2 \le \|p(p(y|C^*)|C)\|^2, \tag{2.6}$$

since

$$\begin{aligned} \|p(p(y|C^*)|C)\|^2 &= \|p(y|C^*) - p(p(y|C^*)|C^p)\|^2 \\ &= \|y - (y - p(y|C^*) + p(p(y|C^*)|C^p))\|^2 \end{aligned}$$

and $y - p(y|C^*) + p(p(y|C^*)|C^p) \in C^p$ because $y - p(y|C^*) \in C^{*p} \subset C^p$.

From (2.1), (2.4) and (2.6),

$$T(y) \le \|p(p(y|C^*)|C)\|^2 - \|p(p(y|C^*)|C \cap L)\|^2 = T(p(y|C^*))$$

and the first inequality in (2.5) holds.

In order to prove the second inequality in (2.5), consider the decomposition,

$$T(y) = \|y - p(y|L)\|^2 + 2\langle y - p(y|L), p(y|L) - p(y|C \cap L)\rangle + \|p(y|L) - p(y|C \cap L)\|^2 - \|y - p(y|C)\|^2, \tag{2.7}$$

where the inner product is zero because $y - p(Y|L) \in L^{\perp}$.

From (2.19), $p(y|C \cap L)$ can be considered as the point in C closest to $p(y|L)$. Therefore by condition (ii) in Lemma 2.1 we have

$$\|p(y|L) - p(y|C \cap L)\|^2 \leq \|p(y|L) - p(p(y|C)|L)\|^2 \leq \|y - p(y|C)\|^2, \tag{2.8}$$

the last inequality by the contractivity of a projection onto a subspace. This inequality and the preceding decomposition prove

$$T(y) \leq \|y - p(y|L)\|^2, \qquad \forall\, y \in R^k. \tag{2.9}$$

Since, for any $y \in C^*$, $T^*(y) = \|y - p(y|L)\|^2$, the second inequality in (2.5) follows from (2.9).

The last equality in (2.5) is obvious from (2.2) and (2.3).

The second step is devoted to proving that the tests with critical regions $\{T \geq c\}$ and $\{T^* \geq c\}$ have the same significance level.

From (2.5), by any c, the test with critical region $\{T^*(Y) \geq c\}$ is more powerful than that defined by $\{T(Y) \geq c\}$,

$$P_\theta(T(Y) \geq c) \leq P_\theta(T^*(Y) \geq c), \qquad \forall\, \theta, \forall\, c, \tag{2.10}$$

so that in order to show that both tests reach the same significance level in H_0 by applying Lemma 3.2 in Raubertas, Nordheim and Lee (1986), we only need to prove that

$$\sup_{\theta \in C \cap L} P_\theta(T(Y) \geq c) = P_0(T^*(Y) \geq c) = \alpha. \tag{2.11}$$

Assume $\dim(C \cap L) = r$ and let $z_1, \dots, z_r$ be linearly independent vectors in $C \cap L$. Since $\dim(L) = \dim(C \cap L)$ implies that $\{z_1, \dots, z_r\}$ is a basis for L, then for any fixed $z \in L$ we may decompose $z = \lambda_1 z_1 + \cdots + \lambda_r z_r$. Then for $\theta = z_1 + \cdots + z_r$, taking $\delta = \max_{i=1,\dots,r}|\lambda_i|$, we have $z + \delta\theta \in C$. Now if $y \in C + L$, then $y = x + z$ for some $x \in C$ and $z \in L$ so that we have

$$\text{for each } y \in C + L \text{ there is } \delta > 0 \text{ for which } y + \delta\theta \in C. \tag{2.12}$$

Let E be a sphere centered at the origin with $P_0(Y \in E) \geq 1 - \xi$ and $P_0(\{T^*(Y) \geq c\} \cap \{Y \in E\}) \geq \alpha - \xi$ for some fixed $\xi > 0$.

From (2.12), $p(y|L) = p(y|C \cap L)\ \forall\ y \in E + \delta\theta$ for some δ. As a consequence, from (2.1) and (2.2) we have

$$T^*(y) - T(y) = \|p(y|C^*)\|^2 - \|p(y|C)\|^2, \qquad \forall\, y \in E + \delta\theta. \tag{2.13}$$

We will first consider the case $C + L$ closed. In this case $C^* = C + L$ and, given $\delta_n \uparrow \infty$, from (2.12) and the compactness of $E \cap C^*$, there is N such that for any $n \geq N$, $(E + \delta_n \theta) \cap C^* = (E + \delta_n \theta) \cap C$. Therefore, from (2.13), $T^*(y) = T(y)$ for any y in $E + \delta_n \theta$, $n \geq N$.

Now taking some $\delta > \delta_N$, we have

$$\begin{aligned} P_{\delta\theta}(T(Y) \geq c) &\geq P_{\delta\theta}(\{T(Y) \geq c\} \cap \{E + \delta\theta\}) \\ &= P_{\delta\theta}(\{T^*(Y) \geq c\} \cap \{E + \delta\theta\}) \\ &= P_0(\{T^*(Y) \geq c\} \cap E) \geq \alpha - \xi \end{aligned}$$

and (2.11) follows.

Consider now the case in which $C + L$ is not closed. Since C^* is the closure of $C + L$ and from (2.12) we have, for $\delta_n \uparrow \infty$,

$$\lim_{n \to \infty} (E + \delta_n \theta) \cap C = \lim_{n \to \infty} (E + \delta_n \theta) \cap C^* = \lim_{n \to \infty} \left[(E \cap C^*) + \delta_n \theta \right].$$

Let us consider $C_n = [(E + \delta_n \theta) \cap C] - \delta_n \theta$, a collection of bounded, closed and convex sets, verifying $C_n \uparrow E \cap C^*$, so that $p(y|C_n) \to_{n \to \infty} p(y|C^*)$ for all $y \in E$ uniformly, which implies that

$$\forall\, \varepsilon > 0\ \exists\, M \text{ such that } \forall\, n \geq M,$$

$$\|p(y|C^*)\|^2 - \|p(y|C_n)\|^2 \leq \varepsilon, \qquad \forall\, y \in E. \tag{2.14}$$

But $p(y|C^*) = p(y - \delta_n \theta | C^*) + \delta_n \theta$ and $p(y|C) = p(y - \delta_n \theta | C_n) + \delta_n \theta$, $\forall\ y \in E + \delta_n \theta$, so that, for any $\varepsilon > 0$, $n \geq M$ and $y \in E + \delta_n \theta$, $T^*(y) - T(y) \leq \varepsilon$ follows from (2.13) and (2.14). By taking $\delta > \delta_M$,

$$\begin{aligned} P_{\delta\theta}(T(Y) \geq c) &\geq P_{\delta\theta}(\{T(Y) \geq c\} \cap \{E + \delta\theta\}) \\ &\geq P_{\delta\theta}(\{T^*(Y) \geq c + \varepsilon\} \cap \{E + \delta\theta\}) \\ &= P_0(\{T^*(Y) \geq c + \varepsilon\} \cap E) \geq \alpha - \eta - \xi, \end{aligned}$$

where $\eta \downarrow 0$ as $\varepsilon \downarrow 0$. Then, taking $\varepsilon \downarrow 0$, (2.11) follows. □

The preceding proof shows that the size of the LRT with critical region $\{T \geq c\}$ is attained at values of the parameter of the form $\delta\theta$ for $\theta \in C \cap L$ and $\delta \to \infty$. Also note that, since $C \cap L \subset L \subset C^*$, the null distribution of T^* is a chi-bar-squared, as we pointed out in the introduction.

Lemma 2.2 relates each condition in Lemma 2.1 to the nonoblique condition (1.2).

LEMMA 2.2. *Let C and L be a closed convex cone and a linear subspace, respectively. For every $x \in R^k$, the relation*

$$p(p(x|C)|L) = p(p(x|L)|C) \tag{2.15}$$

holds if and only if x satisfies the three conditions

$$p(p(x|L)|C) \in L, \tag{2.16}$$

$$p(p(x|C)|L) \in C, \tag{2.17}$$

$$p(p(x|C)|C \cap L) = p(x|C \cap L). \tag{2.18}$$

PROOF. (2.16) and (2.17) are obviously implied by (2.15). The rest of the proof is straightforward after taking the following two statements into account:

(2.17) implies that $p(p(x|C)|L) = p(p(x|C)|C \cap L)$.

(2.16) and Lemma 2.2 in Raubertas, Nordheim and Lee (1986) imply

$$p(p(x|L)|C) = p(p(x|L)|C \cap L) = p(x|C \cap L). \tag{2.19}$$

If L satisfies the equivalent conditions in Lemma 2.1, then, from Lemma 2.2, either (2.15) is satisfied $\forall\ x \in R^k$ and $C \cap L$ and C are not oblique, or (2.18) fails for some x; that is, $C \cap L$ and C are oblique. □

Lemma 2.3 will be used in the proof of Theorem 2.2.

LEMMA 2.3. *Let L and C verifying* (2.15). *Then*:

(a) $p(p(x|L^{\perp})|C) = p(p(x|C)|L^{\perp}) = p(x|L^{\perp} \cap C)$, $\forall\ x \in R^k$;

(b) $\mathrm{cl}(C + L) = L + (L^{\perp} \cap C)$.

PROOF. (a) From Lemma 5.12 in Zarantonello (1971) it follows that $p(p(x|C)|L^{\perp}) = p(x|L^{\perp} \cap C)$. Applying Lemma 2.1, $p(p(x|L^{\perp})|C) \in L^{\perp}$ and then $p(p(x|L^{\perp})|C) = p(p(x|L^{\perp})|L^{\perp} \cap C)$. Now Lemma 2.2 in Raubertas, Nordheim and Lee (1986) implies that $p(p(x|L^{\perp})|L^{\perp} \cap C) = p(x|L^{\perp} \cap C)$ and the results follows.

(b) Obviously $L + (L^{\perp} \cap C) \subset \mathrm{cl}(C + L)$. In order to prove the converse, let $x = y + z \in C + L$, $y \in C$, $z \in L$; x can be decomposed as

$$\begin{aligned} x &= p(x|L) + p(x|L^{\perp}) = z + p(y|L) + p(y|L^{\perp}) \\ &= z + p(y|L) + p\big(p(y|C)|L^{\perp}\big). \end{aligned}$$

Then, from (a), $x \in L + (L^{\perp} \cap C)$ and $C + L \subset L + (L^{\perp} \cap C)$. Because $L + (L^{\perp} \cap C)$ is closed, equality in (b) is proven. □

THEOREM 2.2. *Consider $Y \sim N_k(\theta, I)$; let C and L, respectively, be a closed convex cone and a linear subspace. Let T be the* LR *statistic for testing H_0: $\theta \in C \cap L$ against $H_a - H_0$, with H_a: $\theta \in C$ and let T^* be the* LR *statistic for testing H_0^*: $\theta \in L$ against $H_a^* - H_0^*$, with H_a^*: $\theta \in C^*$, $C^* = \mathrm{cl}(C + L)$. Then $T(y) = T^*(y)$, $\forall\ y \in R^k$, if and only if L and C satisfy the condition* (2.15) $\forall\ y \in R^k$.

PROOF. Sufficiency: From (2.1) and (2.2), since $C^* = L + (L^{\perp} \cap C)$, $T^*(y)$ can be reduced to a simpler expression:

$$T^*(y) = \|p(y|L)\|^2 + \|p(y|L^{\perp} \cap C)\|^2 - \|p(y|L)\|^2 = \|p(y|L^{\perp} \cap C)\|^2.$$

In a similar way, from Lemmas 2.2 and 2.3,

$$\begin{aligned} T(y) &= \|p(y|L^{\perp} \cap C)\|^2 + \|p(y|L \cap C)\|^2 - \|p(y|L \cap C)\|^2 \\ &= \|p(y|L^{\perp} \cap C)\|^2. \end{aligned}$$

Necessity: $T(y) = T^*(y)$, $\forall\ y \in R^k$, implies all of the inequalities in (2.5) become equalities since $T^*(y) = T^*(p(y|C^*))$ from (2.2) and (2.3). Also (2.6) is now an equality and the argument proving (2.6) is useful to state that $p(y|C^p) = y - p(y|C^*) + p(p(y|C^*)|C^p)$, $\forall\ y$, since the closest point to y in the convex cone C^p is unique. Equivalently,

$$p(y|C) = p(p(y|C^*)|C), \qquad \forall\, y \in R^k. \tag{2.20}$$

For any $y \in C^*$, $T(y) = T^*(y) = \|y - p(y|L)\|^2$ and from decomposition (2.7),

$$\|p(y|L) - p(y|C \cap L)\|^2 = \|y - p(y|C)\|^2, \qquad \forall\, y \in C^*, \tag{2.21}$$

which implies

$$p(p(y|C)|L) \in C, \qquad \forall\, y \in C^*, \tag{2.22}$$

because if not, for $z \in C^*$ with $p(p(z|C)|L) \notin C$ (2.21) fails applied to $p(z|C^*)$.

Using (2.21) and (2.22) the first inequality in (2.8) becomes an equality for any $y \in C^*$, which proves that the condition (2.15) is satisfied for any $y \in C^*$. The following chain of equalities proves the result:

$$\begin{aligned} p(p(y|L)|C) &= p(p(p(y|C^*)|L)|C) = p(p(p(y|C^*)|C)|L) \\ &= p(p(y|C)|L), \qquad \forall\, y \in R^k; \end{aligned}$$

the first equality from (2.3), the second one because $p(y|C^*) \in C^*$ satisfy the condition (2.15) and the last equality from (2.20). □

Theorem 2.2 deals with a situation where $\mathrm{cl}(C + L) = L + (L^{\perp} \cap C)$ and the LRT's are equivalent. Also some other particular cases are of special interest.

CASE A. Consider the same situation as in Theorem 2.1, but further assume that $L \cap \mathrm{ri}(C) \neq \varnothing$, where $\mathrm{ri}(C)$ is the relative interior of C. Let S_C be the subspace of R^k of smallest dimension containing C.

Note that, for any fixed point $z \in \mathrm{ri}(C)$, for each $x \in S_C$ there is some $\delta > 0$ such that $x + \delta z \in C$. This can be seen by considering an orthogonal transformation of S_C onto the subspace $L = \{x | x_{d+1} = \cdots = x_k = 0\}$, where $d = \dim(S_C)$. Without loss of generality we can therefore suppose that $S_C = R^k$. Since $\mathrm{ri}(C)$ is the interior of C, we apply Corollary 6.4.1 in Rockafellar (1970) with $\delta = 1/\varepsilon > 0$, which yields $x + \delta z \in C$.

Then, for each, $x \in S_C$ taking $z \in L \cap \mathrm{ri}(C)$ we get $x \in L + C$ and $\mathrm{cl}(C + L)$ turns out to be S_c.

An example of this particular situation is a problem studied in Robertson (1986) as we will show in Section 4, Example B.

CASE B. Now consider the problem of testing a face of a polyhedral cone $\hat{A}_B$ against the cone $\hat{A}$. We have for this situation the expressions

$$\hat{A} = \{a'_i x \geq 0,\, i = 1, \dots, r\},$$

$$\hat{A}^B = \{a'_i x \geq 0,\, i \in B\},$$

$$L_B = \{a'_i x = 0,\, i \in B\},$$

$$\hat{A}_B = \{a'_i x = 0,\, i \in B;\, a'_i x \geq 0,\, i \notin B\}, \quad \text{for any } B \subset \{1, \dots, r\}.$$

It is straightforward to show that $\hat{A}_B = \hat{A} \cap L_B$ and $\mathrm{cl}(\hat{A} + L_B) = \hat{A}^B$.

Further assume that $\hat{A}$ is an acute cone. As defined by Martin and Salvador (1988), a cone $\hat{A}$ is said to be acute whether $a'_i p(x|\hat{A}) = 0$ for any x with $a'_i x \leq 0$. From the definition it is obvious that

$$(2.23) \quad p\big(p(x|L_B)|\hat{A}\big) \in L_B \quad \text{for any } x \in R^k \quad \text{and} \quad B \subset \{1, \dots, r\}.$$

Then Theorem 2.2 in Menéndez and Salvador (1991) is a consequence of Theorem 2.1.

In a similar form, when $\hat{A}$ is a right cone, that is, if $a'_i p(x|\hat{A}) = 0$ is only verified by x with $a'_i x \leq 0$, Theorem 2.2 shows that the corresponding LR tests are equivalent.

Note that (2.23) becomes not only a particular property of acute cones but also a sufficient condition for a cone to be acute. In order to prove that, let x verifying $a'_i x \leq 0$ for some $i \in \{1, \dots, r\}$; obviously $a'_i p(x|\hat{A}) \geq 0$, hence there exists $z \in L_i$ with $z = \delta x + (1 - \delta)p(x|\hat{A})$ for some $\delta \in [0, 1]$. Such a z verifies $p(x|\hat{A}) = p(z|\hat{A}) = p(p(z|L_i)|\hat{A})$, which in turn belongs to L_i by (2.23). Therefore $a'_i p(x|\hat{A}) = 0$ and $\hat{A}$ is acute.

3. Dominance in case of unknown variances.

In this section we deal with the case in which the covariance is given by $\sigma^2\Gamma$, where Γ is completely known but the scalar σ^2 is unknown.

Consider $Y_1, \dots, Y_n$ to be n independent vector observations from a $N_k(\theta, \sigma^2\Gamma)$ population and let Y be the sample mean vector. Without loss of generality $\Gamma = I$ can be taken. In this framework, with H_0 and H_a as in Theorem 2.1, the LRT for testing H_0 against $H_a - H_0$ has $\{S^2(Y) \geq c\}$ as a critical region, where

$$S^2(Y) = T(Y)/\big(u^2 + \|Y - p(Y|C)\|^2\big),$$

$T(Y)$ defined by (1.1) and $u^2 = (1/n)\Sigma\|Y_i - Y\|^2$.

In a similar way,

$$S^{*2}(Y) = T^*(Y)/\big(u^2 + \|Y - p(Y|C^*)\|^2\big)$$

allows testing the hypothesis H_0^* against $H_a^* - H_0^*$, with H_0^* and H_a^* as in Theorem 2.1.

The statistic $S^{*2}(Y)$ and related distributions under H_0^* were considered in Raubertas, Nordheim and Lee (1986), Robertson, Wright and Dykstra (1988) and Shapiro (1988).

Now, we get a result similar to Theorem 2.1.

THEOREM 3.1. *Consider L and C satisfying any of conditions in Lemma 2.1. The* LRT *for testing H_0: $\theta \in C \cap L$ against $H_a - H_0$, with H_a: $\theta \in C$, is dominated by the* LRT *for testing H_0^*: $\theta \in L$ against $H_a^* - H_0^*$, with H_a^*: $\theta \in C^*$, $C^* = \mathrm{cl}(C + L)$.*

PROOF. Since $C \subset C^*$ implies $\|y - p(y|C^*)\|^2 \le \|y - p(y|C)\|^2$ from (2.5) we have that

$$S^2(y) \le S^{*2}(y), \qquad \forall\, y \in R^k. \tag{3.1}$$

On the other hand, following the second part of the proof of Theorem 2.1, a large sphere $E + \delta\theta$ can be selected in such a way that for every y in the sphere above $S^2(y)$ is as close to $S^{*2}(y)$ as we want. This implies that for some $\theta \in C \cap L$,

$$\lim_{\delta \to \infty} P_{\delta\theta}\big(S^2(Y) \ge c\big) = P_0\big(S^{*2}(Y) \ge c\big); \tag{3.2}$$

hence the theorem follows from (3.1) and (3.2). □

It is of interest to note that a result similar to Theorem 2.2 does not hold in this case of unknown σ^2, because if $M = \{y|p(y|C^*) \notin C\}$, then $P_\theta(M) > 0$, $\forall\ \theta$, and $S^2(y) < S^{*2}(y)$, $\forall\ y \in M$, although $T(y)$ may be the same as $T^*(y)$. A result similar to Theorem 2.2 only holds asymptotically. In fact, $k(n-1)S^2(Y)$ and $T(Y)$ are asymptotically equivalent, since $\|y - p(y|C)\|^2 \to 0$ a.e. and $(k(n-1))^{-1}\sigma^2 u^2 \to 1$ a.e. because $\sigma^2 u^2$ has a $\chi^2_{k(n-1)}$ distribution [cf. Robertson, Wright and Dykstra (1988)]. Similarly, $k(n-1)S^{*2}(Y)$ and $T^*(Y)$ are asymptotically equivalent. Therefore, S^2 and S^{*2} define asymptotically equivalent tests under condition (2.15), because T and T^* coincide.

4. Applications. This section is devoted to illustrating and clarifying, through several examples, the applicability of the preceding results. Examples A, B, C and D are particular cases of problems involving polyhedral cones that have appeared in the literature. Example E is included as a case of a nonpolyhedral cone for which $C + L$ is not closed.

EXAMPLE A. Assume Y to be a $N_k(\theta, \Gamma)$, where the components of θ are totally ordered; that is, $\theta \in \hat{A} = \{\theta \in R^k | \theta_1 \le \cdots \le \theta_k\}$. Let us consider Γ to be a diagonal matrix so that $\hat{A}$ becomes an acute cone [see Martín and Salvador (1988)]. Then, from (2.23), $\hat{A}$ and the subspace L_B associated to any face $\hat{A}_B$ of $\hat{A}$ are oblique, and verify any conditions (i)–(iii) in Lemma 2.1. Theorem 2.1 shows the LRT based on T for testing $\theta \in \hat{A}_B$ against $\theta \in \hat{A} - \hat{A}_B$

to be dominated by T. In this example $T \neq T^*$ unless $L_B \subset \hat{A}$, as is the case with the null hypothesis of homogeneity.

Note that $\hat{A}$ is not always an acute cone. There exist nondiagonal metrics, defined by Γ^{-1}, under which Theorem 2.1 fails to show the domination above.

EXAMPLE B. Let Y be an $N_k(\theta, W^{-1})$, where W is a diagonal matrix with positive diagonal elements $w_1, \dots, w_k$, and consider the problem of testing the symmetry S and unimodality U, in the components of θ, against the unimodality U.

Robertson (1986) deals with this problem and shows how the LRT is dominated when $w_i = w_{k-i+1}$, $i = 1, \dots, (k-1)/2$. Theorem 2.1 shows that the LRT is dominated whenever $w_i w_{k-i} = w_{i+1} w_{k-i+1}$, $i = 1, \dots, (k-3)/2$, since this condition on the diagonal metric is valid if and only if $p(p(y|U)|S) \in U$, $\forall\, y \in R^k$.

EXAMPLE C. An example is now presented in which a face of a partial order is tested against the partial order. Let Y be a $N_4(\theta, W^{-1})$, where W is a definite positive diagonal matrix, and consider the cone $C = \{x_1 \leq x_2 \leq x_3 \geq x_4\}$ and the subspace $L = \{x_1 = x_2\}$, so that $C \cap L = \{x_1 = x_2 \leq x_3 \geq x_4\}$ and $C + L = \{x_1 \leq x_2\}$.

Conditions in Lemma 2.1 are satisfied by C and L since

$$p(y|L) = \left(\frac{y_1 w_1 + y_2 w_2}{w_1 + w_2}, \frac{y_1 w_1 + y_2 w_2}{w_1 + w_2}, y_3, y_4 \right)$$

$$\text{for any } y = (y_1, y_2, y_3, y_4)' \in C.$$

Then, Theorem 2.1 proves that the LRT for testing H_0: $\theta \in C \cap L$ against $H_a - H_0$, with H_a: $\theta \in C$, is dominated by the LRT for testing H_0^*: $\theta \in L$ against H_a^*: $\theta \in C + L$.

This example shows that, in order to get the dominance of the LRT for testing a face of a cone against the cone, the acuteness of the cone is not necessary. We only need this acuteness between, it might be said, the face and the rest of the cone.

EXAMPLE D. An example illustrating Theorem 2.2 is provided by the so-called star-shaped restrictions on the means of a normal model. Shaked (1979) and Dykstra and Robertson (1983) studied the MLE under these restrictions.

These star-shaped restrictions define a right cone C in the parameter set. If L is the subspace associated to any face of C, then L and C satisfy the condition (2.15), for every x in R^k. Therefore, Theorem 2.2 can be used for proving the equivalence between the LRT for testing $C \cap L$ against C and the LRT for testing L against $C + L$.

EXAMPLE E. We deal in this example with a situation in which the parameter is constrained to verify restrictions defined by a nonpolyhedral cone.

Let Y be a $N_3(\theta, I)$, where $\theta \in C_\lambda = \{\theta \in R^3 | \theta_2 \geq 0, \theta_1^2 + \theta_3^2 - \lambda\theta_2\theta_3 \leq 0\}$, $0 < \lambda \leq 1$, and consider the linear subspace $L = \{\theta_1 = \theta_3 = 0\}$. C_λ and L satisfy any condition in Lemma 2.1; therefore, Theorem 2.1 proves that the LRT for testing $\theta \in C_\lambda \cap L = \{\theta_2 \geq 0, \theta_1 = \theta_3 = 0\}$ against $\theta \in C_\lambda$ is dominated by the LRT for testing $\theta \in L$ against $\theta \in C^*$, where $C^* = \text{cl}(C_\lambda + L) = \{\theta_3 \geq 0\}$.

Note that in this case $C_\lambda + L$ is not closed, so that $C^* \neq C_\lambda + L$. Also note $C_\lambda \cap L$ and $C_1 \cap L$ are the same, for every λ, $0 < \lambda \leq 1$.

On the other hand, $C_\lambda \downarrow C_1 \cap L$ as $\lambda \downarrow 0$, which could be interpreted as C_λ giving a more and more constrained information about θ, but the dominance above makes this information absolutely useless.

The answer to this apparent contradiction is that the "information" about θ contained in every cone C_λ is similar as we take θ sufficiently far on $C_1 \cap L$. It is only a matter of scale.

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