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NONPARAMETRIC METHODS FOR IMPERFECT REPAIR MODELS

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In the age-dependent minimal repair model of Block, Borges and Savits (BBS), a system failing at age t undergoes one of two types of repair. With probability $p(t)$, a perfect repair is performed and the system is returned to the "good-as-new" state, while with probability $1 - p(t)$, a minimal repair is performed and the system is repaired, but is only as good as a working system of age t. Whitaker and Samaniego propose an estimator for the system life distribution F when data are collected under this model.

In the present article, an appropriate probability model for the BBS process is developed and a counting process approach is used to extend the large sample theorems of Whitaker and Samaniego to the whole line. Applications of these results to confidence bands and an extension of the Wilcoxon two-sample test are examined.

1. Introduction. Procedures for inference in reliability often assume an i.i.d. model for (inter-) failure times. The i.i.d. assumption may be inadequate, however, for modeling situations in which a failed item may be repaired, and the imperfect repair models of Brown and Proschan (1983), and Block, Borges and Savits (1985) attempt to overcome this difficulty in a mathematically tractable way.

In the Brown–Proschan (BP) model, a device with continuous life distribution F is put on test at time zero. Upon failure, one of the two types of repair is performed. With probability p, the device is returned to the "good-as-new" state (perfect repair), and we consider that its age is returned to zero. With probability $q = 1 - p$, the device is returned to the working state, but is only as good as a working item of age equal to the age of the device at failure (minimal repair). Thus, if a minimal repair is performed on an item failing at age t, the repaired item has survival function $\overline{F}(s|t)$ given by

$$\overline{F}(s|t) = \frac{\overline{F}(s+t)}{\overline{F}(t)}, \qquad s \geq 0,$$

where we use the notation $\overline{G}$ to indicate the survival function $1 - G$ of a life distribution G. The process is continued after repair, with each subsequent failure being followed by a perfect repair with probability p, or a minimal repair with probability q. Under this model, Brown and Proschan showed that

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the time between perfect repairs has distribution H, where $\overline{H}(t) = \overline{F}^{p}(t)$, and the transformation from F to H was shown to preserve certain aging properties, such as the IFR, IFRA, NBU and DMRL properties and their duals, as defined in Barlow and Proschan (1981).

Block, Borges and Savits (BBS) extended these results by proposing a more general model, where the probability of a perfect repair, $p(t)$, depends on the age of the failed device; that is, $p(\cdot)$ is a measurable function p: $[0, \infty) \to [0, 1]$. Under the condition

$$\int_{(0,\infty)} \frac{p(t)}{\overline{F}(t)}\, dF(t) = +\infty, \tag{1.1}$$

Block, Borges and Savits showed that for continuous F, the waiting time between perfect repairs is almost-surely finite with distribution H given by

$$H(t) = 1 - \exp\left\{ -\int_{(0,t]} \frac{p(s)}{\overline{F}(s)}\, dF(s) \right\}, \qquad t \geq 0, \tag{1.2}$$

and the preservation results of Brown and Proschan were shown to hold under suitable hypotheses on $p(\cdot)$.

Whitaker and Samaniego (1989) proposed an estimator for the life distribution when this model is observed until the time of the mth perfect repair. This estimator was motivated by a nonparametric maximum likelihood approach, and was shown to be a "neighborhood MLE." Whitaker and Samaniego derived large-sample results for this estimator by following the methods of Breslow and Crowley (1974).

In this paper we take the more modern approach of using counting process and martingale theory to analyze these models. These methods yield extensions of Whitaker and Samaniego's results to the whole line, and provide a useful framework for further work on the minimal repair model.

In Section 2, we develop an appropriate probability model for the BBS process, and we provide a straightforward extension of (1.1) and (1.2) which allows for discontinuous F. Some basic counting processes and an appropriate filtration are then defined that permit us to study the model through the use of martingale techniques. Following Gill (1983), we rederive and extend the large sample results of Whitaker and Samaniego to the whole line in Section 3. In Section 4, application of these results to obtain confidence bands for F similar to those of Hall and Wellner (1980) and to extend the Mann–Whitney–Wilcoxon two-sample test to the BBS model is explored.

2. The model and the basic martingale.

Let h be a cadlag (right continuous with left limits) function. Throughout this paper, we will use the following standard notation for right continuous functions with left limits: $h(t-)$ represents the left limit of h at the point t, $\Delta h(t) = h(t) - h(t-)$ is the jump in h at t and, if h is of bounded variation, $h^c(t) = h(t) - \sum_{s<t} \Delta h(s)$ is the continuous part of h at t. We also let $s \wedge t$ represent the minimum of s

and t. Uniform random variables are to be taken as uniformly distributed on the unit interval.

A life distribution F is a distribution function with $F(0) = 0$, and we denote the corresponding survival function $1 - F$ by $\bar{F}$. We call F a *sub*distribution function when we wish to allow $\lim_{t\to\infty} F(t) < 1$; in this case, we still define $\bar{F} = 1 - F$, but we refrain from calling $\bar{F}$ a subsurvival function since this contradicts common usage of this term in the survival analysis literature. By convention, we will take $F(\infty) = 1$ for all subdistributions, and we define $\tau_F = \inf\{t \in [0, \infty]: F(t) = 1\}$.

The cumulative hazard function Λ of a life (sub) distribution F is defined by

$$\Lambda(t) = \int_{(0,t]} \frac{dF(s)}{\bar{F}(s-)}, \tag{2.1}$$

and it is well known that

$$\bar{F}(t) = \prod_{(0,t]} (1 - d\Lambda) = \exp(-\Lambda^c(t)) \prod_{s \le t} (1 - \Delta\Lambda(s)), \tag{2.2}$$

where the notation $\prod_{(0,t]}(1 - d\Lambda)$ represents a product integral. A review of the theory of product integration and its applications in statistics is given by Gill and Johansen (1990). In the one-dimensional case considered here, the right-hand side of (2.2) may be taken as the definition of the product integral.

A sequence of failure ages obtained under a model of perpetual minimal repair may be defined as follows. Let F be a life distribution and let $\{X_0 \equiv 0, X_1, X_2, \dots\}$ be a record-value sequence based on F; that is, $(X_k)_{k=1}^{\infty}$ is a Markov process with the conditional distribution of X_k given $X_0, \dots, X_{k-1}$ being given by $\bar{F}(t|X_{k-1}) = \bar{F}(t)/\bar{F}(X_{k-1})$, for $t \ge X_{k-1}$ and $k \ge 1$. In the case that $\Delta F(\tau_F) > 0$, we take $X_j = \infty$ for all j larger than the first k for which $X_k = \tau_F$. Thus, in any case, $X_k < X_{k+1}$ whenever $X_k < \infty$, and we define $X_\infty = \lim_{k\to\infty} X_k$.

It is advantageous at this point to introduce perfect repair into this structure through the use of independent uniform random variables. Let p: $[0, \infty] \to [0, 1]$ be measurable, with $p(t) = 1$ for $t \ge \tau_F$, and suppose that $\{U_1, U_2, \dots\}$ are independent uniform random variables, independent of $(X_k)_{k=1}^{\infty}$. Defining $\delta_k = I(U_k < p(X_k))$ and $\nu = \inf\{k: \delta_k = 1\}$, where we take $\inf \varnothing = \infty$, we see that $P(\delta_k = 1|X_1, \dots, X_k, \delta_1, \dots, \delta_{k-1}) = p(X_k)$. For continuous F, with $\tau_F = \infty$, observing $\{(X_1, \delta_1), \dots, (X_\nu, \delta_\nu)\}$ is equivalent to observing the minimal repair process of BBS until the time of the first perfect repair. Note that ν may be infinite in the case that $F(\tau_F -) = 1$.

Let H be the subdistribution function defined by

$$H(t) = P(X_\nu \le t, \nu < \infty).$$

Assuming that F is continuous and that $F(t) < 1$ for all $t < \infty$, Theorem A.5

of BBS states that

$$\bar{H}(t) = \exp\left(- \int_{(0,t]} p(s) \bar{F}^{-1}(s)\, dF(s) \right).$$

The following proposition provides a useful generalization of this result and shows that in the more general setting the cumulative hazard function of H is given by

$$\Lambda_H(t) = \int_{(0,t]} p(s) \frac{dF(s)}{\bar{F}(s-)}. \tag{2.3}$$

PROPOSITION 2.1. *Under the model described above, H is given by*

$$\begin{aligned} H(t) &= 1 - \prod_{(0,t]} (1 - d\Lambda_H) \\ &= 1 - \exp\left(- \int_{(0,t]} p(s) \frac{dF^c(s)}{\bar{F}(s-)} \right) \prod_{s \le t} \left(1 - p(s) \frac{\Delta F(s)}{\bar{F}(s-)} \right). \end{aligned} \tag{2.4}$$

Moreover, if either

$$\Delta F(\tau_F) > 0 \quad [\text{and } p(\tau_F) = 1] \tag{2.5(i)}$$

or

$$F(\tau_F -) = 1 \quad \text{and} \quad \int_{(0,\tau_F)} p(s) \frac{dF(s)}{\bar{F}(s-)} = +\infty, \tag{2.5(ii)}$$

then H is a proper distribution function and hence ν is almost surely finite. Conversely, if H is a proper distribution, then either (i) *or* (ii) *must hold.*

A direct proof of this result is given in Hollander, Presnell and Sethuraman (1989), where a conditioning argument is used to show that for $t < \tau_F$,

$$\frac{\bar{H}(t)}{\bar{F}(t)} = 1 + \sum_{j=1}^{\infty} \int \cdots \int_{0 < t_1 < \cdots < t_j \le t} d\alpha(t_1) \cdots d\alpha(t_j),$$

where

$$\alpha(t) = \int_{(0,t]} (1 - p(s)) \frac{dF(s)}{\bar{F}(s)}.$$

The result is then obtained from Theorem 4 of Gill and Johansen (1990). As pointed out by a referee, the result can also be obtained through a thinning argument applied to the nonhomogeneous Poisson process obtained from the sequence of minimal repair times.

We will now describe the basic martingale structure needed in the sequel. Define the counting process N^* by

$$N^*(t) = \#\{k: X_k \le t\}$$

and the corresponding filtration $(\mathscr{F}_t^*)_{t \ge 0}$ by

$$\mathscr{F}_t^* = \sigma(\{N^*(s): s \le t\}).$$

Here we assume that our basic probability space $(\Omega, \mathscr{F}, P)$ is complete, and we take each $\mathscr{F}_t^*$ to contain $\mathscr{F}_0$, the σ field consisting of the P-null sets of $\mathscr{F}$ and their complements. Applying Theorem 18.2 of Liptser and Shiryayev (1978), we see that the counting process N^* has $\mathscr{F}_t^*$ compensator A^* given by

$$A^*(t) = \sum_{k=1}^{\infty} \int_{(0, t \wedge X_k]} \frac{dF(s|X_{k-1})}{\bar{F}(s-|X_{k-1})} = \sum_{k=1}^{\infty} \int_{(X_{k-1}, t \wedge X_k]} \frac{dF(s)}{\bar{F}(s-)} = \Lambda(t).$$

By Lemma 18.12 of Liptser and Shiryayev (1978), $M^* \equiv N^* - \Lambda$ is a τ_F-locally square-integrable martingale with respect to $\mathscr{F}_t^*$, with predictable variation process $\langle M^* \rangle$ given by

$$\langle M^* \rangle(t) = \int_{(0,t]} (1 - \Delta\Lambda(s))\, d\Lambda(s).$$

REMARK 2.1. If $\Delta F(\tau_F) > 0$, then M^* is actually a square-integrable martingale, which can be seen by applying Theorem 18.8(a) of Liptser and Shiryayev (1978), with the integrand taken to be identically 1. More generally, for any $\tau < \tau_F$, applying the same result with $I(t \le \tau)$ as the integrand shows that the stopped process $M^{*\tau} = M^*(t \wedge \tau)$ is a square-integrable martingale. Consequently, the localizing sequence for M^* may be taken to be any sequence of constants increasing to (but strictly less than) τ_F.

Now let $\{U_1, U_2, \dots\}$, δ_k and ν be as before. Since the U_k's are independent of $\mathscr{F}_t^*$ for all t, M^* is a τ_F-locally square-integrable martingale with respect to $\mathscr{F}_t$, where

$$\mathscr{F}_t = \mathscr{F}_t^* \vee \sigma(\{U_1, U_2, \dots\}).$$

Note here that X_ν is an $\mathscr{F}_t$-stopping time, so that $Y(t) = I(X_\nu \ge t)$ is predictable. Let $N(t) = N^*(t \wedge X_\nu)$ and

$$M(t) = \int_{(0,t]} Y(s)\, dM^*(s) = N(t) - \int_{(0,t]} Y(s)\, d\Lambda(s).$$

By Theorem 18.8(b) of Liptser and Shiryayev (1978), M is a τ_F-locally square-integrable martingale with respect to $\mathscr{F}_t$, with predictable variation process $\langle M \rangle$ given by

$$\langle M \rangle(t) = \int_{(0,t]} Y^2(s) d\langle M^* \rangle(s) = \int_{(0,t]} Y(s)(1 - \Delta\Lambda(s))\, d\Lambda(s).$$

REMARK 2.2. Even if H is a proper distribution, so that X_ν is almost surely finite, we cannot, in general, claim that M is a square-integrable martingale. As an example, suppose that F is exponential and let $p(t) = I(t \geq 1)t^{-1}$. Then $X_\nu < \infty$ a.s., since $\overline{H}(t) = 1 + (t^{-1} - 1)I(t \geq 1)$, but

$$E\left(\int_0^\infty Y^2(s)\, d\langle M^*\rangle(s)\right) = \int_0^\infty \overline{H}(s-)\, d\Lambda(s) = 1 + \int_1^\infty s^{-1}\, ds = \infty,$$

so that M is not a square-integrable martingale [Theorem 18.8(c), Liptser and Shiryayev (1978)].

These results for the case of observing a single BBS process are now used to derive the basic martingale structure for our sampling scheme of observing n such processes. Let $(X_{jk})_{k=1}^\infty$, $1 \leq j \leq n$, be independent record value sequences from F and let $\{U_{jk}: k \geq 1, 1 \leq j \leq n\}$ be independent uniform random variables, independent of the X_{jk}. We define N_j^* and ν_j in the obvious way and we let $Y_j(t) = I(X_{j\nu_j} \geq t)$, $N_j(t) = N_j^*(t \wedge X_{j\nu_j})$ and $M_j(t) = N_j(t) - \int_{(0,t]} Y_j(s)\, d\Lambda(s)$. We also define $N(t) = \sum_{j=1}^n N_j(t)$, $Y(t) = \sum_{j=1}^n Y_j(t)$ and

$$M(t) = \sum_{j=1}^n M_j(t) = N(t) - \int_{(0,t]} Y(s)\, d\Lambda(s). \tag{2.6}$$

Let

$$\mathscr{F}_t = \sigma\left(\{N_j^*(s): s \leq t, 1 \leq j \leq n\} \cup \{U_{jk}: k \geq 1, 1 \leq j \leq n\}\right).$$

From the previous results and the independence of the processes involved, we see that each M_j is a τ_F-locally square-integrable martingale with respect to $\mathscr{F}_t$ and hence M is a τ_F-local martingale. As before, the localizing sequence may be taken to be any sequence of constants increasing to τ_F. In fact, since we may take the localizing sequence for each M_j to be the same sequence of constants, and since the M_j are independent, Lemma A.2 of Doss and Chiang (1992) yields

$$\langle M\rangle(t) = \sum_{j=1}^n \langle M_j\rangle(t) = \int_{(0,t]} Y(s)(1 - \Delta\Lambda(s))\, d\Lambda(s). \tag{2.7}$$

3. The Whitaker–Samaniego estimator. Henceforth we assume that F is a continuous distribution and that the pair (F, p) satisfies (2.5). The statistical model consists of observing n independent copies of the BBS process, each until the time of its first perfect repair, $X_{j\nu_j}$. Since the $X_{j\nu_j}$ are almost surely finite, this sampling scheme is well defined and the continuity assumption guarantees that there will be no ties in the data.

Let T be the first failure age at which only one item is at risk; that is, $T = \min\{X_{(k)}: Y(X_{(k)}) = 1\}$, where the $X_{(k)}$ are the ordered values of the observed failure ages of the n BBS processes. It is not difficult to see that T is a stopping time. A natural estimator of Λ is the Aalen estimator $\hat{\Lambda}$, given by

$$\hat{\Lambda}(t) = \int_{(0,t]} \frac{J(s)}{Y(s)}\, dN(s), \tag{3.1}$$

where $J(s) = I(s \le T)$ and where we again take $0/0 = 0$. Note that $Y(X_{(k)})$ is the number of processes still being observed, or "at risk," just prior to $X_{(k)}$ and

$$\hat{\Lambda}(t) = \sum_{X_{(k)} \le t \wedge T} \frac{1}{Y(X_{(k)})}.$$

With this notation, the WSE can be simply written as

$$\hat{\hat{F}}(t) = \prod_{X_{(k)} \le t \wedge T} \left(1 - \frac{1}{Y(X_{(k)})}\right) = \prod_{(0,t]} (1 - d\hat{\Lambda}). \tag{3.2}$$

REMARK 3.1. As noted by Whitaker and Samaniego, $\hat{F}$ puts no mass at those failure ages at which only one item was at risk, excepting the first, or smallest, of such ages; that is, $\hat{\hat{F}}(t) = 0$ for all $t \ge T$. Some insight may be had into the difficulty of assigning mass to these points by considering the problem of nonparametric estimation of F when $n = 1$. See also the small example and discussion of nonparametric maximum likelihood estimation in Section 2 of Whitaker and Samaniego. We also note that our choice of the cumulative hazard estimator differs slightly from that of Whitaker and Samaniego, in that we have chosen to allow only one jump of size 1. This is mostly for convenience, although it is natural to do so, since any actual cumulative hazard function, as defined by (2.1), has as most one jump of size 1.

Referring to (2.1) and (2.2), we see that

$$\Lambda^T(t) = \int_{(0,t]} \frac{dF^T(s)}{\bar{F}^T(s-)} \quad \text{and} \quad \bar{F}^T(t) = \prod_{(0,t]} (1 - d\Lambda^T), \tag{3.3}$$

where we use the notation $g^T(t) = g(t \wedge T)$. The product integral representations (3.2) and (3.3), together with Duhamel's equation [Gill and Johansen (1990)] yields the identity

$$\frac{\hat{F}(t) - F^T(t)}{\bar{F}^T(t)} = \int_{(0,t]} \frac{\hat{\hat{F}}(s-)}{\bar{F}^T(s)} d\big[\hat{\Lambda}(s) - \Lambda^T(s)\big]. \tag{3.4}$$

The utility of (3.4) lies in the fact that it expresses $(\hat{F} - F)/\bar{F}$ as a martingale. To see this, note that from (2.6), (3.1) and (3.3) we have

$$\hat{\Lambda}(t) - \Lambda^T(t) = \int_{(0,t]} \frac{J(s)}{Y(s)}\, dM(s),$$

and since $\hat{F}(t) = 1$ and $J(t) = 0$ for all $t > T$, we see that (3.4) may be written as

$$\frac{\hat{F}(t) - F(t)}{\bar{F}(t)} = \int_{(0,t]} \frac{\hat{\hat{F}}(s-)}{\bar{F}(s) Y(s)}\, dM(s). \tag{3.5}$$

For any fixed $\tau < \tau_F$, the integrand on the right-hand side is predictable and bounded by $\bar{F}^{-1}(\tau) < \infty$, for all $t \le \tau$. Hence, $(\hat{F} - F)/\bar{F}$ is a τ_F-locally square-integrable martingale, where we may again take the localizing sequence to be any sequence of constants increasing to τ_F. Using (2.7), we see that

$$\left\langle \frac{\hat{F} - F}{\bar{F}} \right\rangle(t) = \int_{(0,t]} \left(\frac{\hat{\bar{F}}(s-)}{\bar{F}(s)Y(s)} \right)^2 d\langle M \rangle(s) \tag{3.6}$$

$$= \int_{(0,t]} \left(\frac{\hat{\bar{F}}(s-)}{\bar{F}(s-)} \right)^2 \frac{dF(s)}{\bar{F}(s)Y(s)}.$$

These results are of course very similar to those derived by Gill (1983) for the Kaplan–Meier estimator. From here, arguments parallel to those used for the KME yield the expected weak convergence results. The martingale representation of (3.5) is the key identity used to prove these results, the first of which corresponds to Whitaker and Samaniego's Theorem 3.3.

We first need a consistency result for the WSE. Theorem 3.1 of Whitaker and Samaniego establishes the almost-sure convergence of $\sup_{0 \le t \le \tau_F} |\hat{F}(t) - F(t)|$ to zero. Alternatively, following Gill (1983), the martingale representation provides us with a simple proof of convergence in probability, which is sufficient for our needs.

LEMMA 3.1. $\sup_{0 \le t \le \infty} |\hat{F}(t) - F(t)| \to 0$ *in probability, as* $n \to \infty$.

PROOF. Let $\tau < \tau_F$ and note that by (3.6), $\langle (\hat{F} - F)/\bar{F} \rangle(\tau) \le (n\bar{F}^3(\tau)[Y(\tau)/n])^{-1}$. This last expression converges almost surely to zero, since $n^{-1}Y(\tau) \to \bar{H}(\tau -) > 0$, and it follows from a corollary to Lenglart's inequality [see Example B.4.1 of Shorack and Wellner (1986)] that

$$\sup_{0 \le t \le \tau} |\hat{F}(t) - F(t)| \le \sup_{0 \le t \le \tau} \left| (\hat{F}(t) - F(t))/\bar{F}(t) \right|$$

$$\to 0 \quad \text{in probability, as } n \to \infty.$$

The extension to $[0, \infty]$ is straightforward. □

Define $Z = \sqrt{n}(\hat{F} - F)/\bar{F}$. The following weak convergence result can also be found as Theorem 3.3 in Whitaker and Samaniego (1989):

THEOREM 3.1. *Let* $\tau < \tau_F$. *Then* $Z \to_{\mathscr{D}} B(C)$ *in* $D[0, \tau]$, *as* $n \to \infty$, *where* B *is standard Brownian motion on* $[0, \infty)$ *and* $B(C)$ *is the process* $\{B(C(t))\colon t \in [0, \tau]\}$, *where*

$$C(t) = \int_{(0,t]} \frac{dF(s)}{\bar{H}(s-)\bar{F}(s)}.$$

PROOF. It is immediate from (3.6), that Z is a square-integrable martingale on $[0, \tau]$, with

$$\begin{aligned}\langle Z\rangle(t) &= n\int_{(0,t]}\left(\frac{\hat{\bar{F}}(s-)}{\bar{F}(s-)}\right)^2 \frac{dF(s)}{\bar{F}(s)Y(s)}\\ &= \int_{(0,t]}\frac{J(s)}{\bar{F}(s)[Y(s)/n]}\,dF(s)\\ &\quad+\int_{(0,t]}\left(\frac{\hat{\bar{F}}^2(s-)-\bar{F}^2(s-)}{\bar{F}^2(s-)}\right)\frac{J(s)\,dF(s)}{\bar{F}(s)[Y(s)/n]}.\end{aligned}$$

Using Lemma 3.1 and the fact that $n^{-1}Y(\tau) \to \bar{H}(\tau-) > 0$ almost surely, we see that the second term on the right-hand side of this expression converges uniformly to zero in probability as $n \to \infty$. By the Glivenko–Cantelli theorem, $n^{-1}Y(s)$ converges uniformly to $\bar{H}(s-)$ almost surely as $n \to \infty$, and so, for n sufficiently large, $n^{-1}Y(\tau) > \bar{H}(\tau-)/2 > 0$. Since Y is nonincreasing, we have $[Y(s)/n]^{-1} \le 2[\bar{H}(\tau-)]^{-1}$ for all $s \le \tau$, and the bounded convergence theorem thus applies to give, for all $t \le \tau$,

$$\int_{(0,t]}\frac{J(s)}{\bar{F}(s)[Y(s)/n]}\,dF(s) \to C(t), \quad \text{a.s., as } n \to \infty.$$

It follows that $\langle Z\rangle(t) \to C(t)$ in probability, for all $t \le \tau$.

In order to apply Rebolledo's martingale central limit theorem [Theorem B.5.2 of Shorack and Wellner (1986)], it remains only to show that Z satisfies the strong ARJ(2) condition. Since $\Delta M = \Delta N$ and the jumps of N are of size 1, we have

$$\begin{aligned}\sigma^\varepsilon[Z](t) &\equiv \sum_{s\le t}|\Delta Z(s)|^2 I(|\Delta Z(s)|>\varepsilon)\\ &= \int_{(0,t]}\left(\frac{\sqrt{n}\,\hat{\bar{F}}(s-)}{\bar{F}(s)Y(s)}\right)^2 I\left(\frac{\hat{\bar{F}}(s-)}{\bar{F}(s)[Y(s)/n]} > \varepsilon\sqrt{n}\right)dN(s).\end{aligned}$$

Thus, the compensator $\tilde{\sigma}^\varepsilon[Z]$ of $\sigma^\varepsilon[Z]$ on $[0, \tau]$ is given by

$$\begin{aligned}\tilde{\sigma}^\varepsilon[Z](t) &= \int_{(0,t]}\left(\frac{\sqrt{n}\,\hat{\bar{F}}(s-)}{\bar{F}(s)Y(s)}\right)^2 I\left(\frac{\hat{\bar{F}}(s-)}{\bar{F}(s)[Y(s)/n]} > \varepsilon\sqrt{n}\right)Y(S)\,d\Lambda(s)\\ &= \int_{(0,t]}\left(\frac{\hat{\bar{F}}(s-)}{\bar{F}(s)}\right)^2\frac{1}{[Y(s)/n]}I\left(\frac{\hat{\bar{F}}(s-)}{\bar{F}(s)[Y(s)/n]} > \varepsilon\sqrt{n}\right)d\Lambda(s).\end{aligned}$$

Now, we again use the fact that $n^{-1}Y(\tau) \to \bar{H}(\tau-) > 0$ to get that, almost surely, the indicator in the above expression is identically zero for n sufficiently large, and hence that $\tilde{\sigma}^{\varepsilon}[Z](t) \to 0$ almost surely, as $n \to \infty$, for all $t \in [0, \tau]$. Thus Rebolledo's theorem applies to Z, and the theorem follows. □

The weak convergence results of Theorem 3.1 can be extended to $D[0, \tau_F]$ following the methods of Gill (1983). The main result needed is analogous to Gill's Theorem 2.1 and can be proven in the same way. Note that $\int h\, dZ$ and $\int Z\, dh$ are to be interpreted as the processes whose value at t is defined as the integral over the interval $(0, t]$.

THEOREM 3.2. *Let h be a nonnegative, continuous, nonincreasing function on $[0, \tau_F]$ such that*

$$\int_{(0, \tau_F)} h(t)^2\, dC(t) < \infty.$$

Then the processes hZ, $\int h\, dZ$ and $\int Z\, dh$ converge jointly in distribution in $D[0, \tau_F]$ to processes hZ_∞, $\int h\, dZ_\infty$ and $\int Z_\infty\, dh$, where $Z_\infty = B(C)$ and

$$hZ_\infty = \int h\, dZ_\infty + \int Z_\infty\, dh.$$

COROLLARY 3.1.

$$\sqrt{n}\,(\hat{F} - F) \to_{\mathscr{D}} \bar{F} \cdot B(C) \quad \text{in } D[0, \tau_F], \text{ as } n \to \infty.$$

PROOF. This result follows immediately from Theorem 3.2 since

$$\int_0^{\tau_F} \bar{F}^2(t)\, dC(t) = \int_0^{\tau_F} \frac{\bar{F}(t)}{\bar{H}(t-)}\, dF(t) \le \int_0^{\tau_F} dF(t) = 1. \qquad \square$$

REMARK 3.2. Note that no additional assumption is needed to obtain Corollary 3.1 from Theorem 3.2. To obtain an analogous result for the KME, some assumption such as (1.1) of Gill (1983) appears to be required.

COROLLARY 3.2. *Let $K = C/(1 + C)$. Then*

$$\sqrt{n}\,\frac{\bar{K}}{\bar{F}}(\hat{F} - F) \to_{\mathscr{D}} B^0(K) \quad \text{in } D[0, \tau_F], \text{ as } n \to \infty,$$

where B^0 is a Brownian bridge on $[0, 1]$.

PROOF. Noting that

$$\int_0^{\tau_F} \bar{K}^2(t)\, dC(t) = \int_0^{\tau_F} \frac{dC(t)}{(1 + C(t))^2} = 1,$$

Theorem 3.2 implies convergence to $\bar{K} \cdot B(C)$, and a check of covariances yields the corollary. □

These corollaries are used in the next section to develop asymptotic confidence bands for F and to determine the large sample theory for an extension of the Mann–Whitney–Wilcoxon two-sample test to the current model.

4. Applications. The result of Corollary 3.2 suggests an asymptotic $100(1-\alpha)\%$ confidence band for F of the form

$$\hat{F} \pm n^{-1/2}\lambda_\alpha \hat{\bar{F}}/\hat{\bar{K}}, \tag{4.1}$$

where λ_α is the upper αth percentile of the distribution of $\sup|B^0(t)|$. Here we let $\hat{\bar{H}}(s-) = Y(s)/n$, so that $\hat{\bar{H}}$ is the empirical survival function of the perfect repair ages, $X_{1\nu_1}, \dots, X_{n\nu_n}$, and we take $\hat{\bar{K}} = 1/(1+\hat{C})$, where

$$\hat{C}(t) = \int_{(0,t]} \frac{d\hat{F}(s)}{\hat{\bar{H}}(s-)\hat{\bar{F}}(s)} = \sum_{X_{(k)} \le t} \frac{n}{Y(X_{(k)})\big(Y(X_{(k)})-1\big)}. \tag{4.2}$$

This is of course analogous, when multiplied by $\hat{\bar{F}}(t)$, to the usual Greenwood formula for the estimated variance of the KME. Since $\hat{C}(T) = \infty$, we take $\hat{\bar{F}}(t)/\hat{\bar{K}}(t) = \hat{\bar{F}}(T-)/\hat{\bar{K}}(T-)$ for all $t \ge T$. In general one could also define $\hat{\bar{H}}$ to be any estimate of $\bar{H}$ with cumulative hazard given by

$$\hat{\Lambda}_H(t) = \int_{(0,t]} \hat{p}(s)\frac{d\hat{F}(s)}{\hat{\bar{F}}(s-)}, \tag{4.3}$$

where $\hat{p}$ is some model-dependent, consistent estimator of p.

Remark 4.1. In applications, there may be ties in the data. To construct the bands in this case, T should be taken as the first age at which the number of units failing is equal to the number at risk; that is, the first t for which $\Delta N(t) = Y(t)$. Also, $\hat{F}$ should be given by

$$\hat{\bar{F}}(t) = \prod_{s \le t}\left(1 - \frac{\Delta N(s)}{Y(s)}\right)$$

and $\hat{\bar{K}} = 1/(1+\hat{C})$, where

$$\hat{C}(t) = \sum_{s \le t} \frac{n\,\Delta N(s)}{Y(s)(Y(s)-\Delta N(s))}.$$

Use of these bands on $[0,\tau]$, $\tau < \tau_F$, is justified by straightforward arguments to show that $\sup_{0 \le t \le \tau}|\hat{\bar{K}}(t)/\hat{\bar{F}}(t) - \bar{K}(t)/\bar{F}(t)| \to 0$ in probability, so that

$$\sqrt{n}\,\frac{\hat{\bar{K}}}{\hat{\bar{F}}}(\hat{F}-F) = \sqrt{n}\,\frac{\bar{K}}{\bar{F}}(\hat{F}-F) + \left(\frac{\hat{\bar{K}}}{\hat{\bar{F}}} - \frac{\bar{K}}{\bar{F}}\right)\sqrt{n}\,(\hat{F}-F)$$

$$\to_{\mathscr{D}} B^0(K) \quad \text{in } D[0,\tau], \text{ as } n \to \infty.$$

This justifies asymptotic confidence bands on the interval $[0, \tau]$ of the form

$$\hat{F} \pm n^{-1/2}\lambda_\alpha(K(\tau))\hat{\bar{F}}/\hat{\bar{K}}, \tag{4.4}$$

where $\lambda_\alpha(\beta)$ is the upper αth quantile of the distribution of $\sup_{0 \le t \le \beta} |B^0(t)|$. Of course, in particular, one would approximate $K(\tau)$ by $\hat{K}(\tau)$, or, conservatively, use $\lambda_\alpha(1)$. Partial tables for the values of $\lambda_\alpha(\beta)$ are provided by Koziol and Byar (1975) and by Hall and Wellner (1980).

In order to rigorously justify the use of these bands on the whole line, it is necessary to show that

$$\sqrt{n}\,\frac{\hat{\bar{K}}}{\hat{\bar{F}}}(\hat{F} - F) \to_{\mathscr{D}} B^0(K) \quad \text{in } D[0, \tau_F], \text{ as } n \to \infty. \tag{4.5}$$

However, our early simulation studies with the BP model indicated that (4.5) does not hold without additional conditions on (F, p). Unlike the analogous result for the KME as given by Theorem 1.2(ii) of Gill (1983), (4.5) is not an immediate consequence of Corollary 3.1, since in our case $\hat{\bar{K}}/\hat{\bar{F}}$ is a nondecreasing function (as is $\bar{K}/\bar{F}$) and cannot, in general, be uniformly bounded in t and n by a fixed constant. The corresponding function in the censored data model, on the other hand, is nonincreasing and bounded by 1.

The monotonicity of $\hat{\bar{K}}/\hat{\bar{F}}$ is implied by Proposition 4.1, which should be contrasted with (1.2) of Gill (1983) and the accompanying remarks. Note that application of the proposition to $\hat{\bar{K}}/\hat{\bar{F}}$ is justified by the fact that $\hat{\bar{H}}$ is equivalent, on $[0, T)$, to the survival function specified by $\hat{\Lambda}_H$ of (4.3), with $\hat{p}$ taken as 1 at T and at all $X_{i\nu_i}$, and 0 elsewhere. The proposition also applies if $\hat{\bar{H}}$ is specified by (4.3) with $\hat{p}$ taken as some model specific estimator of p and $\hat{\bar{K}} = 1/(1 + \hat{C})$, where $\hat{C}$ is defined by (4.2).

PROPOSITION 4.1. *Under the BBS model, as described in Section* 2, $\bar{F}/\bar{K}$ *is a nonincreasing function, with*

$$\frac{\bar{F}_-}{\bar{H}_-} \le \frac{\bar{F}}{\bar{K}} \le 1. \tag{4.6}$$

PROOF. Using the integration by parts formula given in Lemma 18.7 of Liptser and Shiryayev (1978), we have for $t < \tau_F$,

$$\begin{aligned} d\left(\frac{\bar{F}}{\bar{K}}\right) &= \left(\frac{\bar{F}_-}{\bar{F}\bar{H}_-} - \left(1 + \int_{(0,\cdot]} \frac{dF}{\bar{F}\bar{H}_-}\right)\right) dF \\ &= \left(\int_{(0,\cdot)} \left(\frac{dH}{\bar{H}\bar{H}_-} - \frac{dF}{\bar{F}\bar{H}_-}\right)\right) dF. \end{aligned} \tag{4.7}$$

But since $d\Lambda_H = p\,d\Lambda$,

$$\frac{dH}{\bar{H}\bar{H}_-} - \frac{dF}{\bar{F}\bar{H}_-} = \left(\frac{p-1}{(1-p\,\Delta\Lambda)(1-\Delta\Lambda)}\right)\frac{d\Lambda}{\bar{H}_-}, \tag{4.8}$$

and since $p - 1 \le 0$ and $\Delta\Lambda \le 1$, it follows that $d(\bar{F}/\bar{K}) \le 0$ and thus $\bar{F}/\bar{K}$ is nonincreasing.

Since $\bar{F}/\bar{K}$ is nonincreasing, the upper bound in (4.6) is obvious. For the lower bound, (4.8) implies that the measure defined by $(\bar{H}\bar{H}_-)^{-1}\,dH - (\bar{F}\bar{H}_-)^{-1}\,dF$ is a negative measure, and thus, by (4.7),

$$\frac{\bar{F}(t)}{\bar{K}(t)} = 1 + \int_{(0,t)}(\bar{F}(u) - \bar{F}(t))\left(\frac{dH(u)}{\bar{H}(u)\bar{H}(u-)} - \frac{dF(u)}{\bar{F}(u)\bar{H}(u-)}\right)$$

$$\ge 1 + \int_{(0,t)}\bar{F}(u)\left(\frac{dH(u)}{\bar{H}(u)\bar{H}(u-)} - \frac{dF(u)}{\bar{F}(u)\bar{H}(u-)}\right) = \frac{\bar{F}(t-)}{\bar{H}(t-)}. \qquad \square$$

Note that the fact that $\hat{\bar{K}}/\hat{\bar{F}}$ is nondecreasing implies that the suggested confidence bands will decrease in width as t increases. This is in contrast to the similarly constructed bands based on the Kaplan–Meier estimator [Hall and Wellner (1980)], which increase in width, and to the usual Kolmogorov–Smirnov bands for i.i.d. sampling, which have constant width. Intuitively, this may be seen as a consequence of the fact that relative to i.i.d. sampling, the minimal repair sampling scheme yields additional information about the tail of the distribution while right-censored samples yield less.

REMARK 4.2. We conjecture that

$$\int_0^t (1 - p(s))\,d\Lambda(s) < \infty \tag{4.9}$$

is a sufficient condition for (4.5), but we have not yet been able to prove this. This condition is similar to Gill's (1983) condition (1.1), which as noted earlier in Remark 3.2, was needed to prove an analogue of Corollary 3.1 for the KME. In fact, while Gill's condition may be seen as a limitation on the amount of censoring, (4.9) limits the amount of imperfect repair by forcing $p(t)$ to be *close* to 1, and in light of the last proposition seems a natural condition since it implies that $\bar{K}/\bar{F}$ is bounded.

As an example, we have computed and plotted in Figure 1 the WSE and the corresponding 95% confidence bands for the oft analyzed Boeing air conditioner data, originally presented by Proschan (1963). The original data are reproduced in Table 1. For this analysis, we have treated the intervals between failures as interfailure times between minimal repairs. Proschan omits any failure interval immediately following a major overhaul (indicated by $**$ in Table 1), and we have omitted the intervals following such an interval from our analysis, since it is impossible to determine the age of the unit after the

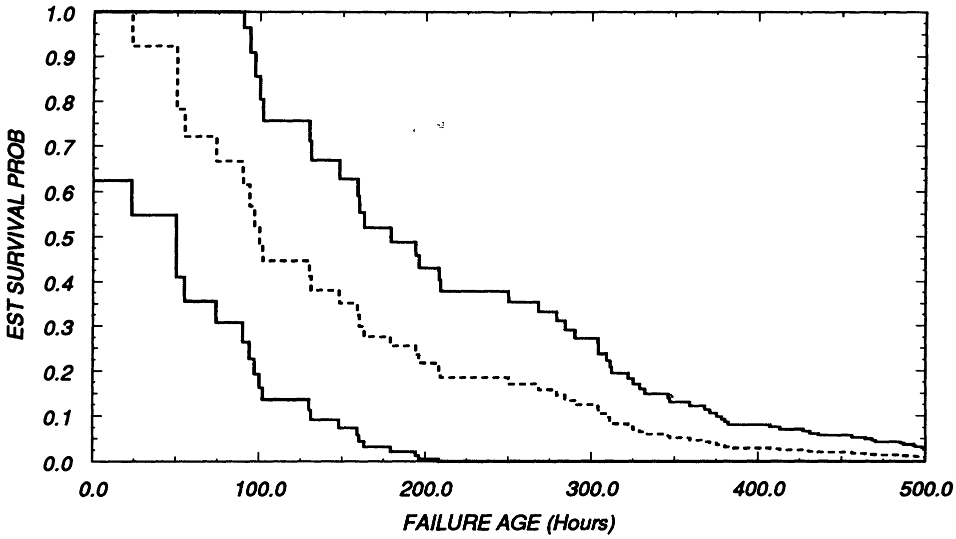

FIG. 1. *Confidence bands for Boeing data*

overhaul. Thus the values below the $**$'s in Table 1 are not used. We have treated the age at which a major overhaul occurs as the time of the first perfect repair for that airplane. This affects planes 7908, 7909, 7910 and 7911. For purposes of this example, we treat the last observed failure ages of the remaining planes as the times of their first perfect repair. We have arbitrarily chosen to compute bands for the interval from 0 to 500 h. Thus we apply the formula of (4.7) with $n = 13$ and $\tau = 500$. Our program for computing the confidence bands uses the tables of Koziol and Byar (1975) to look up $\lambda_\alpha(\hat{K}(\tau))$, after rounding $\hat{K}(\tau)$ to the nearest tenth. In this case $\hat{K}(500)$ is 0.867. The plot shows clearly the decreasing width of the confidence bands.

REMARK 4.3. Justification of the use of the minimal repair model in cases such as this is not immediate of course. For a critical discussion of minimal repair models, see, for example, Arjas and Norros (1989) and Natvig (1990).

In Hollander, Presnell and Sethuraman (1989), we provide results of simulation studies of the coverage probabilities of these bands on finite intervals under the BP model, with various p and F. The tables of Koziol and Byar (1975) were used to look up the appropriate value of $\lambda_\alpha(\hat{K}(\tau))$. In all the cases examined, a sample size of 100 was adequate to insure a true coverage probability at worst 1 or 2% less than the nominal coverage probability.

In the two-sample problem we assume that for $i = 1, 2$ we observe n_i BBS processes based on (p_i, F_i), each until its first perfect repair. In general, we wish to test the null hypothesis H_0: $F_1 = F_2$, with typical one-sided

TABLE 1
Intervals between failures of Boeing air conditioner systems

Plane number												
7907	**7908**	**7909**	**7910**	**7911**	**7912**	**7913**	**7914**	**7915**	**7916**	**7917**	**8044**	**8045**
194	413	90	74	55	23	97	50	359	50	130	487	102
15	14	10	57	320	261	51	44	9	254	493	18	209
41	58	60	48	56	87	11	102	12	5		100	14
29	37	186	29	104	7	4	72	270	283		7	57
33	100	61	502	220	120	141	22	603	35		98	54
181	65	49	12	239	14	18	39	3	12		5	32
	9	14	70	47	62	142	3	104			85	67
	169	24	21	246	47	68	15	2			91	59
	447	56	29	176	225	77	197	438			43	134
	184	20	386	182	71	80	188				230	152
	36	79	59	33	246	1	79				3	27
	201	84	27	**	21	16	88				130	14
	118	44	**	15	42	106	46					230
	**	59	153	104	20	206	5					66
	34	29	26	35	5	82	5					61
	31	118	326		12	54	36					34
	18	25			120	31	22					
	18	156			11	216	139					
	67	310			3	46	210					
	57	76			14	111	97					
	62	26			71	39	30					
	7	44			11	63	23					
	22	23			14	18	13					
	34	62			11	191	14					
		**			16	18						
		130			90	163						
		208			1	24						
		70			16							
		101			52							
		208			95							

** Omission by Proschan (1963) of any failure interval immediately following a major overhaul.

alternatives specifying $\int_0^\infty F_1\,dF_2 > 1/2$ and two-sided alternatives specifying $\int_0^\infty F_1\,dF_2 \neq 1/2$.

A statistic which generalizes the Mann–Whitney form of the Wilcoxon two-sample test statistic W to the current situation is

$$W = \int_0^\infty \hat{F}_1\,d\hat{F}_2 = \sum_{\Delta N_2(s)>0} \hat{F}_1(s)\hat{F}_2(s-)\frac{\Delta N_2(s)}{Y_2(s)},$$

where $\hat{F}_i$ is the WSE, $\Delta N_i(s)$ is the number of failures at age s and $Y(s)$ is the number of items at risk at age s in the ith sample. This statistic is a natural estimator of $\int_0^\infty F_1\,dF_2$, which is equal to $P(X_1 \le X_2)$, where X_1 and X_2 are independent random variables, with $X_i \sim F_i$. Assuming continuous distributions, $P(X_1 \le X_2) = 1/2$ under H_0 and, in the one-sided case, significantly large values of W provide evidence against H_0 in the direction of $\int_0^\infty F_1\, dF_2 > 1/2$.

Although W is intuitively appealing, useful exact distribution or exact moment results are difficult to obtain. If we assume that $p_1 \equiv p_2$, then inference could be based on the use of the reference distribution obtained by computing W for each of the $\binom{n_1+n_2}{n_1}$ possible arrangements of the observed BBS processes into two groups of sizes n_1 and n_2. In the general situation, we can use the results of Section 3 to obtain the asymptotic result of Proposition 4.2. A complete proof of this theorem and detailed verifications of the subsequent remarks are provided in Hollander, Presnell and Sethuraman (1989). The theorem could also be obtained as a consequence of the Wilcoxon example in Gill (1989).

PROPOSITION 4.2. *If F_1 and F_2 are continuous and the pairs (F_1, p_1) and (F_2, p_2) describe regular repair schemes and if $n_1, n_2 \to \infty$ in such a way that $n_1/(n_1+n_2) \to \lambda$, $0 < \lambda < 1$, then*

$$\sqrt{n_1+n_2}\left[W - \int_0^\infty F_1\, dF_2\right] \to_{\mathscr{D}} N\left(0, \frac{1}{\lambda}\sigma_1^2 + \frac{1}{1-\lambda}\sigma_2^2\right), \tag{4.10}$$

where

$$\begin{aligned} \sigma_1^2 &= 2\int_0^\infty \int_t^\infty \bar{F}_1(s)\bar{F}_1(t)C_1(t)\, dF_2(s)\, dF_2(t), \\ \sigma_2^2 &= 2\int_0^\infty \int_t^\infty \bar{F}_2(s)\bar{F}_2(t)C_2(t)\, dF_1(s)\, dF_1(t). \end{aligned} \tag{4.11}$$

Under the null hypothesis, H_0: $F_1 = F = F_2$,

$$\sigma_i^2 = 2\int_0^\infty \bar{F}(t)C_i(t)\left(\int_t^\infty \bar{F}(s)\, dF(s)\right) dF(t) = \frac{1}{4}\int_0^\infty \frac{\bar{F}^3(s)}{\bar{H}_i(s-)}\, dF(s),$$

which is consistently estimated by

$$\hat{\sigma}_i^2 = \frac{1}{4}\int_0^\infty \frac{\hat{\bar{F}}_i^3(s)}{\hat{\bar{H}}_i(s-)}\, d\hat{F}_i(s) = \frac{1}{4}\sum_{\Delta N_i(s)>0} \frac{n\hat{\bar{F}}_i^3(s)\hat{\bar{F}}_i(s-)\,\Delta N_i(s)}{Y_i^2(s)}.$$

For purposes of testing the null hypothesis in the large sample case, we thus propose referring the test statistic

$$Z = \left(W - \frac{1}{2}\right)\Big/\left(\frac{\hat{\sigma}_1^2}{n_1} + \frac{\hat{\sigma}_2^2}{n_2}\right)^{1/2}$$

to a standard normal distribution. Here H_i is the empirical distribution of the perfect repair ages in the ith sample.

Under the Brown–Proschan model, the preceding expressions simplify greatly if H_0 is assumed. If $F_1 = F_2 = F$, then $\overline{H}_i = \overline{F}^{p_i}$ and the asymptotic variance in (4.10) reduces to

$$\frac{1}{\lambda}\sigma_1^2 + \frac{1}{1-\lambda}\sigma_2^2 = \frac{1}{\lambda}\left(\frac{1}{4(4-p_1)}\right) + \frac{1}{1-\lambda}\left(\frac{1}{4(4-p_2)}\right).$$

The p_i's are of course consistently estimated by their MLE's, $\hat{p}_i$, the ratio of n_i to the total number of failures in the ith sample, and for large samples, the statistic Z', given by

$$Z' = \left(W - \frac{1}{2}\right) \Big/ \left[\frac{1}{4n_1(4-\hat{p}_1)} + \frac{1}{4n_2(4-\hat{p}_2)}\right]^{1/2},$$

can be referred to a standard normal distribution in order to test the null hypothesis. Note also that if $p_1 = p_2 = 1$, then we are in the usual i.i.d. two-sample model, the WSE's reduce to the empirical c.d.f.'s, W is just a multiple of the Mann–Whitney form of the Wilcoxon rank-sum statistic and the preceding convergence results reduce to the usual results for the Mann–Whitney–Wilcoxon test.

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