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LOCAL ADMISSIBILITY AND LOCAL UNBIASEDNESS IN HYPOTHESIS TESTING PROBLEMS

BY LAWRENCE D. BROWN[1] AND JOHN I. MARDEN[2]

Cornell University and University of Illinois, Urbana–Champaign

In this paper we give necessary conditions and sufficient conditions for a test to be locally unbiased, we define local admissibility and we characterize local admissibility in hypothesis testing problems with simple null hypotheses. Applications are presented involving same-sign alternatives, ordered alternatives and independence testing of several variables.

1. Introduction. In most hypothesis testing problems, there is no uniformly most powerful test, even when restricting to unbiased tests or invariant tests. In such cases the class of admissible tests one has to choose from can be unwieldy, with many of the tests difficult to implement. One way to solve the problem is always to use the generalized likelihood ratio test. In many applications, the likelihood ratio test has good properties. However, in other applications it can be difficult to calculate, fail to have an easily approximated null distribution, or be inadmissible. These difficulties are especially evident when the parameter space is restricted. An alternative approach evaluates testing procedures on their power for alternatives very close to the null. A locally most powerful test maximizes the local power among all tests of its level. Such tests might exist, possibly after restriction to unbiased or invariant tests, when there is no uniformly most powerful test. Examples of tests that are locally most powerful among unbiased, or among locally unbiased, tests can be found in Cohen, Sackrowitz and Strawderman (1985). It is more common, however, that there are no uniformly best tests locally.

The search for locally optimal hypothesis testing procedures goes back to Neyman and Pearson (1936, 1938). When the parameter space is one-dimensional, they define Type A tests, which are the locally most powerful unbiased tests. In two dimensions, they propose Type C tests, which maximize the local power among unbiased tests whose local power is constant on ellipses with given principal axes and ratio of lengths of principal axes. Issaacson (1951) defined Type D tests which are applicable for arbitrary dimensions. A Type D test is unbiased and maximizes the local curvature of the power function among unbiased tests. That is, it maximizes the determinant of the second derivative matrix of the power function evaluated at the null. Sen Gupta and Vermeire (1986) defined locally most mean power unbiased tests as those

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which maximize the local power averaged over a sphere or, more generally, over an ellipsoid of fixed orientation, among locally unbiased tests. A related result is in Giri (1968), where he proved that Hotelling's T^2 test is locally minimax.

Our purpose in this paper is to unify some of the above results in problems with a simple null hypothesis. We give conditions for local unbiasedness, present a definition of local admissibility, and obtain a characterization of local admissibility. A similar approach to the problem of local optimality can be found in Kudô (1961). He presented a definition of locally complete classes and a characterization of the tests in such classes. Our definitions differ a bit from Kudô's, and our results are somewhat more explicit.

As it should be, a locally most powerful locally unbiased test is locally admissible according to our definition. A locally most powerful unbiased test need not be since it is possible that it can be dominated locally by a test that is locally unbiased but not globally unbiased. In some cases, all tests have a singular matrix of second derivatives, hence all unbiased tests are trivially Type D [see Isaacson (1951)]. In such cases, Type D tests need not be locally admissible. It is still open exactly when Type D, or Type C, tests are locally admissible. In Section 4 we show that the locally most mean power unbiased tests are locally admissible, and that if we extend the definition of such tests slightly, the class of these tests coincides with the class of locally admissible locally unbiased tests.

Our setup starts with a sample space $\mathbf{X}$ and a family of densities $\{f_\theta(x): \theta \in \Theta\}$ with respect to the sigma-finite measure ν on $\mathbf{X}$, where Θ is a subset of $\mathbf{R}^p$ that contains 0, and ν is absolutely continuous with respect to f_0. We test

$$H_0: \theta = 0 \quad \text{versus} \quad H_A: \theta \in \Theta_A \equiv \Theta - \{0\}. \tag{1.1}$$

A test is a measurable function

$$\phi: \mathbf{X} \to [0, 1]$$

with the interpretation that if $x \in \mathbf{X}$ is observed, then $\phi(x)$ is the probability of rejecting H_0. A test is evaluated via its risk function:

$$r_\theta(\phi) = \begin{cases} E_0(\phi), & \text{if } \theta = 0, \\ 1 - E_\theta(\phi), & \text{if } \theta \in \Theta_A. \end{cases} \tag{1.2}$$

We next define terms associated with local admissibility. Let

$$\Theta_\varepsilon = \{\theta \in \Theta \mid \|\theta\| \leq \varepsilon\}.$$

We assume that $\Theta_\varepsilon - \{0\}$ is nonempty for every $\varepsilon > 0$.

DEFINITIONS 1.1. A test ϕ is *locally dominated* if for each sufficiently small $\varepsilon > 0$ there exists a test ψ_ε, depending on ϕ, such that

$$r_\theta(\phi) \geq r_\theta(\psi_\varepsilon) \quad \text{for all } \theta \in \Theta_\varepsilon \tag{1.3}$$

and

$$r_\theta(\phi) > r_\theta(\psi_\varepsilon) \quad \text{for some } \theta \in \Theta_\varepsilon.$$

A test is *locally inadmissible* if it is locally dominated. Otherwise it is *locally admissible*.

The notion of local dominance that we use is slightly different than that used by Kudô. He demanded that a locally complete class have the property that for any test ϕ not in the class, for each compact subset C of Θ there exists a test ψ_C that dominates ϕ on C. Our definition only requires dominance on sufficiently small neighborhoods of zero. Note that when Θ is compact, Kudô's approach requires a locally admissible test to be admissible in the usual sense. This is not necessary with our definition. For example, suppose $\Theta = [0, 1]$, the distribution when $\theta < 1$ obtains is the Normal$(\theta, 1)$, and the distribution when $\theta = 1$ is point mass 1 at $x = 1$. Then the test that rejects when $x > 1.645$ is *locally* admissible, but not admissible in the usual sense, or Kudô's sense, since it is dominated by the test that rejects when $x > 1.645$ or $x = 1$.

An alternative, and weaker, notion of local admissibility is to declare the test ϕ locally inadmissible if there exists a test ψ which dominates ϕ as in (1.3) for all sufficiently small $\varepsilon > 0$. That is, (1.3) holds with ψ_ε not depending on ε. However, since our stronger definition leaves us with a nice class of tests, we feel satisfied to use Definitions 1.1.

Now we define local unbiasedness.

DEFINITION 1.2. A test ϕ is *locally unbiased* if, for some $\varepsilon > 0$,

$$E_\theta(\phi) \geq E_0(\phi) \quad \text{for } \theta \in \Theta_\varepsilon.$$

A test is *locally inadmissible among locally unbiased* tests if it is locally unbiased and for each sufficiently small $\varepsilon > 0$ there exists a locally unbiased test ψ_ε for which (1.3) holds. Otherwise, a locally unbiased test is *locally admissible among locally unbiased* tests. In Theorem 3.1 we show that a test is locally admissible among locally unbiased tests if and only if it is locally admissible and locally unbiased.

The exact assumptions that we use are given in Section 2. Here we summarize some of the results. The local properties of tests rely heavily on the behavior of f_θ for θ near zero. The derivatives needed are given now. Let

$$R_\theta(x) = \frac{f_\theta(x)}{f_0(x)}, \tag{1.4}$$

$$l(x) = \{l_i(x)\}_{i=1}^p, \qquad l_i(x) = \frac{\partial}{\partial \theta_i} R_\theta(x)\bigg|_{\theta=0} \tag{1.5}$$

and

$$V(x) = \{V_{ij}(x)\}_{i,j=1}^p, \qquad V_{ij}(x) = \frac{\partial^2}{\partial \theta_i \, \partial \theta_j} R_\theta(x)\bigg|_{\theta=0}. \tag{1.6}$$

The assumptions are sufficient to prove that for any test ϕ, as $\|\theta\| \to 0$, the

power function can be written

$$E_\theta(\phi) = \alpha_\phi + \theta' l_\phi + \tfrac{1}{2}\theta' V_\phi \theta + o(\|\theta\|^2), \tag{1.7}$$

where

$$\alpha_\phi = E_0(\phi), \qquad l_\phi = E_0(\phi(X)l(X))$$

and

$$V_\phi = E_0(\phi(X)V(X)).$$

If one considers a Bayes test with respect to H_ε that has mass confined to an ε-neighborhood of zero, we see from (1.7) and (1.2) that the test is very close to the test obtained by finding ϕ to maximize

$$\alpha_\phi H_\varepsilon(\{0\}) + H_\varepsilon(\Theta - \{0\})(1 - \alpha_\phi) + \left[E_{H_\varepsilon}\theta\right]' l_\phi + \tfrac{1}{2}\operatorname{tr}\left(\left[E_{H_\varepsilon}\theta\theta'\right]V_\phi\right). \tag{1.8}$$

The possible limiting values for the pair of expectations in (1.8), suitably normalized, as ε approaches zero are given by a set Λ, a subset of $\mathbf{R}^p \times \mathbf{S}_p$, where $\mathbf{S}_p$ is the set of $p \times p$ symmetric nonnegative definite matrices. See Section 2. This set depends on the local structure of Θ. The Neyman–Pearson lemma can be used to find the test maximizing (1.8). The limits of such tests as $\varepsilon \to 0$ are the ones of interest. To whit, we consider tests that essentially reject H_0 when

$$\lambda' l(x) + \tfrac{1}{2}\operatorname{tr}(MV(x)) > c, \tag{1.9}$$

where $(\lambda, M) \in \Lambda$ and c is a constant. Under these conditions, the class of all such tests is exactly the class of all locally admissible tests. See Theorems 2.1 and 2.2.

Next consider local unbiasedness. When 0 is in the interior of Θ, Sen Gupta and Vermeire (1986) show that a necessary (sufficient) condition for a test to be locally unbiased is that

$$l_\phi = 0 \quad \text{and} \quad V_\phi \text{ is nonnegative (positive) definite.}$$

In Theorem 3.2 we show that, in general, a necessary (sufficient) condition is that

$$\lambda' l_\phi + \tfrac{1}{2}\operatorname{tr}(MV_\phi) \geq (>)\, 0 \quad \text{for all } (\lambda, M) \in \Lambda - \{(0,0)\}. \tag{1.10}$$

In Section 4 we show that when 0 is in the interior of Θ, the class of locally admissible among locally unbiased tests is equivalent to the locally most mean power unbiased tests of Sen Gupta and Vermeire if we extend their definition a little.

Section 5 contains several examples. We exhibit some particular sets Λ and make applications to problems with same-sign alternatives, ordered alternatives and independence testing of several variables.

REMARK. One is rarely concerned with the power of tests at alternatives arbitrarily close to the null, hence it might seem that it is not worthwhile paying too much attention to these locally optimally tests. However, in a few

examples, it appears that if the sample size is not too small, locally optimal tests compare quite favorably to other tests such as the likelihood ratio test. See Schatzoff (1966) for the multivariate analysis of variance problem, Marden (1983) for the general multivariate analysis of variance problem and Marden (1981) for the problem of testing that a bivariate normal correlation is zero when the variances are known. These results are in contrast to asymptotically (as $\theta \to \infty$) optimal tests, which can perform very poorly over much of the parameter space.

We note that the local tests often are easy to calculate, and, since they are like sums or sums of squares, it is often easy to find an approximation to their null distributions. See Sections 5 and 6.

2. Necessary and sufficient conditions for local admissibility. We begin with the first two assumptions, parallel to the first two in Section 2 of Brown and Marden (1989).

ASSUMPTION 2.1. For some $\varepsilon_0 > 0$, the function $R_\theta(x)$ in (1.4) can be extended to a function on $\overline{\Theta}_{\varepsilon_0}$, the closure of Θ_{ε_0} in $\mathbf{R}^p$, such that for each $x \in \mathbf{X}$, $R_\theta(x)$ is continuous on $\overline{\Theta}_{\varepsilon_0}$, and

$$0 < R_\theta(x) < \infty \quad \text{for all } \theta \in \overline{\Theta}_{\varepsilon_0}. \tag{2.1}$$

ASSUMPTION 2.2. For each $x \in \mathbf{X}$, $R_\theta(x)$ has all first and second derivatives with respect to θ at $\theta = 0$ given by (1.5) and (1.6). Also, $E_0(\|l(X)\|) < \infty$, $E_0(\|V(X)\|) < \infty$,

$$R_\theta^{(2)}(x) \equiv R_\theta(x) - 1 - \theta' l(x) - \tfrac{1}{2}\theta' V(x)\theta = o(\|\theta\|^2) \tag{2.2}$$

for each x, and for some $\varepsilon_0 > 0$,

$$\int_{\mathbf{X}} \sup_{\theta \in \Theta_{\varepsilon_0}} \frac{|R_\theta^{(2)}(x)|}{\|\theta\|^2} f_0(x)\nu(dx) < \infty. \tag{2.3}$$

We can take the same ε_0 in the two assumptions. Now for (l, V) and (λ, M) in $\mathbf{R}^p \times \mathbf{S}_p$, define

$$\rho(\lambda, M; l, V) = \lambda' l + \tfrac{1}{2} \operatorname{tr} MV. \tag{2.4}$$

Let $\mathbf{G}_1$ denote the set of nonnegative finite measures G on R^p such that

$$\left\| \int \theta G(d\theta) \right\| + \int \|\theta\|^2 G(d\theta) = 1.$$

Let

$$\Delta_\varepsilon = \left\{ (\lambda, M) \in \mathbf{R}^p \times \mathbf{S}_p \middle| \text{there exists } G \in \mathbf{G}_1, \text{ support}(G) \subset \Theta_\varepsilon \right.$$
$$\left. \text{and } (\lambda, M) = \left(\int \theta G(d\theta), \int \theta\theta' G(d\theta) \right) \right\}, \tag{2.5}$$

and

$$\Delta = \bigcap_{\varepsilon > 0} \overline{\Delta}_\varepsilon.$$

Note that $\|\lambda\| + \operatorname{tr} M = 1$ for all $(\lambda, M) \in \Delta_\varepsilon$, hence Δ is compact, and $0 \notin \Delta$. Note also that since each Θ_ε is nonempty, so is each Δ_ε, and since Δ is an intersection of nested, closed, bounded, nonempty sets, it too is nonempty. For any test function ϕ and $(\lambda, M) \in \mathbf{R}^p \times \mathbf{S}_p$ define

$$\rho(\lambda, M; \phi) = E_0(\rho(\lambda, M; l(X), V(X))\phi(X)) \equiv \rho(\lambda, M; l_\phi, V_\phi), \tag{2.6}$$

where $l(\cdot)$, $V(\cdot)$, l_ϕ and V_ϕ are defined in (1.5), (1.6) and (1.7).

The main theorem follows. Its proof echoes some of the elements of Brown and Marden (1989).

THEOREM 2.1. *Under Assumptions* 2.1 *and* 2.2, *a test* ϕ *of size* α, $0 \le \alpha \le 1$, *is locally admissible level* α *only if there is a* $(\lambda_0, M_0) \in \Delta$ *and a constant* c, $-\infty \le c \le \infty$, *such that*

$$\phi(x) = \begin{cases} 1, & \text{if } \rho(\lambda_0, M_0; l(x), V(x)) > c, \\ 0, & \text{if } \rho(\lambda_0, M_0; l(x), V(x)) < c, \end{cases} \quad a.e.\,[\nu]. \tag{2.7}$$

PROOF. By Definitions 1.1, there exists a sequence of positive numbers $\varepsilon_i \downarrow 0$ such that ϕ is admissible on Θ_{ε_i}, $i = 1, 2, \ldots$. A standard decision-theoretic result [e.g., Brown (1986), Theorem 4A.10] yields the existence of a test ϕ_i with $r_\theta(\phi_i) = r_\theta(\phi)$ for $\theta \in \Theta_{\varepsilon_i}$, and a sequence of priors G_{ij} supported on Θ_{ε_i} having Bayes procedures ϕ_{ij} such that $\phi_{ij} \to \phi_i$ in the weak topology on $\mathbf{L}_\infty$. [This topology is defined so that $\phi_{ij} \to \phi_i$ if and only if

$$\int \big(\phi_{ij}(x) - \phi_i(x)\big) g(x) \nu(dx) \to 0$$

for all g satisfying $\int |g(x)| \nu(dx) < \infty$.] A minor corollary to this result shows that the sequence G_{ij} can also be chosen so that $E_0(\phi_{ij}(X)) = E_0(\phi_i(X)) = E_0(\phi(X)) = \alpha$ for all i, j.

Because $\phi_{ij} \to \phi_i$, and because $E_0(\|l(X)\|) < \infty$ and $E_0(\|V(X)\|) < \infty$, there is a $J(i) < \infty$ such that $j > J(i)$ implies

$$\left\| E_0\big[\big(\phi_{ij}(X) - \phi_i(X)\big) l(X)\big] \right\| < \frac{1}{i} \tag{2.8}$$

and

$$\left\| E_0\big[\big(\phi_{ij}(X) - \phi_i(X)\big) V(X)\big] \right\| < \frac{1}{i}.$$

Choose a subsequence, if necessary, so that $\phi_i \to \phi'$, say. Then a routine diagonalization argument yields a sequence $\{\phi_{i, j(i)}\}$ with $j(i) > J(i)$ such that $\phi_{i, j(i)} \to \phi'$.

If $\alpha = 0$ or 1, then (2.7) holds with $c = \infty$ or $-\infty$, respectively. Now assume $0 < \alpha < 1$. Then $0 < G_{i, j(i)}(\{0\}) < 1$ for all i sufficiently large; say for all i

with no loss of generality. Define

$$H_i = \frac{G_{i,j(i)}}{\left\| \int \theta G_{i,j(i)}(d\theta) \right\| + \int \|\theta\|^2 G_{i,j(i)}(d\theta)}.$$

The denominator of this expression is positive since $G_{i,j(i)}(\{0\}) < 1$, so that H_i is well defined, $H_i \in G_1$.

Let

$$m_i = \frac{G_{i,j(i)}(\{0\}) - G_{i,j(i)}(\Theta_{\varepsilon_i} - \{0\})}{\left\| \int \theta G_{i,j(i)}(d\theta) \right\| + \int \|\theta\|^2 G_{i,j(i)}(d\theta)}.$$

Choose a further subsequence, if necessary, so that

$$\lambda_i = \int \theta H_i(d\theta) \to \lambda_0, \qquad M_i = \int \theta\theta' H_i(d\theta) \to M_0 \tag{2.9}$$
$$\text{and } m_i \to c, \ -\infty \le c \le \infty.$$

By (2.5), $(\lambda_0, M_0) \in \Delta$ since $H_i \in G_1$, and each H_i is supported on (a subset of) Θ_{ε_i}. Note that

$$\phi_{i,j(i)}(x) = \begin{cases} 1, & \text{if } -m_i + \rho(\lambda_i, M_i; l(x), V(x)) \\ & \qquad + \int R_\theta^{(2)}(x) H_i(d\theta) > 0, \\ 0, & \text{if } -m_i + \rho(\lambda_i, M_i; l(x), V(x)) \\ & \qquad + \int R_\theta^{(2)}(x) H_i(d\theta) < 0, \end{cases} \tag{2.10}$$

a.e. $[\nu]$. Now, $\int R_\theta^{(2)}(x) H_i(d\theta) \to 0$ by (2.2). Hence (2.9), (2.10) and $\phi_{i,j(i)} \to \phi'$ imply that ϕ' satisfies (2.7) and that $-\infty < c < \infty$ since $0 < \alpha < 1$.

It remains to show that ϕ also satisfies (2.7). Note that

$$\begin{aligned} \int \big(r_\theta(\phi) - r_\theta(\phi_{i,j(i)})\big) H_i(d\theta) &= \int \big(r_\theta(\phi_i) - r_\theta(\phi_{i,j(i)})\big) H_i(d\theta) \\ &= \int\int \big[\phi_i(x) - \phi_{i,j(i)}(x)\big] \\ &\qquad \times \big[-m + \rho(\theta, \theta\theta'; l(x), V(x)) + R_\theta^{(2)}(x)\big] \\ &\qquad \times f_0(x) \nu(dx) H_i(d\theta) \\ &\to 0 \end{aligned} \tag{2.11}$$

since $\int (\phi_{i,j(i)}(x) - \phi_i(x)) f_0(x) \nu(dx) = 0$,

$$\begin{aligned} &\int E_0\big[\big(\phi_{i,j(i)}(X) - \phi_i(X)\big)\rho(\theta, \theta\theta'; l(X), V(X))\big] H_i(d\theta) \\ &\quad = E_0\big[\big(\phi_{i,j(i)}(X) - \phi_i(X)\big)\rho(\lambda_i, M_i; l(X), V(X))\big] \to 0 \end{aligned}$$

by (2.8) and the fact that $\|\lambda_i\| + \operatorname{tr}(M_i) = 1 < \infty$, and $\int E_0(|R_\theta^{(2)}(X)|)H_i(d\theta) \to 0$ by (2.3), (2.2) and the dominated convergence theorem.

Since $\phi_{i,j(i)} \to \phi'$, it follows as above that

$$\int \bigl(r_\theta(\phi) - r_\theta(\phi_{i,j(i)})\bigr)H_i(d\theta)$$

$$= \int \bigl[\phi_{i,j(i)}(x) - \phi_i(x)\bigr]\bigl[-m_i + \rho(\lambda_i, M_i; l(x), V(x))\bigr] f_0(x)\nu(dx) \tag{2.12}$$

$$\to \int \bigl[\phi'(x) - \phi(x)\bigr]\bigl[-c + \rho(\lambda_0, M_0; l(x), V(x))\bigr] f_0(x)\nu(dx).$$

The integrand in the right-hand expression above is nonnegative, a.e. $[\nu]$, since ϕ' satisfies (2.7). Furthermore, it is positive with positive ν-probability unless ϕ also satisfies (2.7), a.e. $[\nu]$. Combining the result of (2.11) with (2.12) thus yields that ϕ satisfies (2.7), a.e. $[\nu]$, which is the desired result. □

Now we give an extra condition that is sufficient for local admissibility.

THEOREM 2.2. *Let ϕ be defined by* (2.7) *for some* $(\lambda_0, M_0) \in \Delta$. *If*

$$\nu\bigl(\{x|\rho(\lambda_0, M_0; l(x), V(x)) = c\}\bigr) = 0, \tag{2.13}$$

then ϕ is locally admissible.

PROOF. By virtue of (2.13) and the Neyman–Pearson lemma, we have that there does not exist a size-α test ϕ' essentially different than ϕ such that $\rho(\lambda, M; \phi') \le \rho(\lambda, M; \phi)$ for all $(\lambda, M) \in \Delta \subset \overline{\Delta}_\varepsilon$. Thus for any $\varepsilon > 0$ there is no size-α test $\phi' \ne \phi$ such that $\rho(\lambda, M; \phi') \le \rho(\lambda, M; \phi)$ for all $\theta \in \Theta_\varepsilon$. This implies that ϕ is locally admissible. □

A question that arises is whether (2.7) alone, without (2.13), implies that ϕ is locally admissible. It is possible to construct examples that show that such an implication is logically false.

In some cases, $l(x) \equiv 0$ [e.g., in testing for sphericity of a normal covariance matrix as in Cohen and Marden (1988) or in our Examples 5.5 and 5.7]. In such cases, (2.7) provides no information since any test can be so represented by taking $M_0 = 0$. We handle this situation as follows. Let $\mathbf{G}_1^*$ denote the set of nonnegative finite measures G on $\mathbf{G}^p$ such that $\int \|\theta\|^2 G(d\theta) = 1$. Also, let

$$\Delta_\varepsilon^* = \Bigl\{M \in \mathbf{S}_p \bigm| \text{there exists } G \in \mathbf{G}_1^*, \text{ support}(G) \subset \Theta_\varepsilon \text{ and } M = \int \theta\theta' G(d\theta)\Bigr\}, \tag{2.14}$$

$$\Delta^* = \bigcap_{\varepsilon > 0} \overline{\Delta}_\varepsilon^*. \tag{2.15}$$

The following theorem can be proven by dropping the "λ"-terms in the proofs of Theorems 2.1 and 2.2.

THEOREM 2.3. *Suppose* $l(x) \neq 0$, *a.e.* $[\nu]$. *Under Assumptions* 2.1 *and* 2.2, *a test* ϕ *of size* α, $0 \leq \alpha \leq 1$ *is locally admissible only if there is a* $M_0 \in \Delta^*$ *and constant* c, $0 \leq c \leq \infty$, *such that*

$$\phi(x) = \begin{cases} 1, & \text{if } \operatorname{tr} M_0 V(x) > c, \\ 0, & \text{if } \operatorname{tr} M_0 V(x) < c, \end{cases} \quad a.e.\,[\nu]. \tag{2.16}$$

In addition, if

$$\nu(\{x | \operatorname{tr} M_0 V(x) = c\}) = 0, \tag{2.17}$$

then ϕ *is locally admissible.*

REMARK. The statements of the theorems in this section and in Section 3 remain true if we replace Δ or Δ^* with $\Lambda - \{(0,0)\}$ or $\Lambda^* - \{0\}$, respectively, where

$$\Lambda = \operatorname{Cone}(\Delta) \equiv \{\beta\Delta | \beta \geq 0\} \quad \text{and} \quad \Lambda^* = \operatorname{Cone}(\Delta^*). \tag{2.18}$$

In Sections 4 and 5, we find it easier to present Λ and Λ^*, which also conforms to the notation in Brown and Marden (1989).

3. Local unbiasedness. We first prove the following.

THEOREM 3.1. *Suppose Assumptions* 2.1, 2.2, *and* (2.13) [(2.17)] *hold. Then the class of tests that are locally admissible among locally unbiased tests consists exactly of those tests of the form* (2.7) [(2.16)] *that are locally unbiased.*

PROOF. Clearly, if ϕ is locally admissible and locally unbiased, it is locally admissible among locally unbiased tests. On the other hand, suppose ϕ is locally unbiased, but locally inadmissible among all tests. Then for each sufficiently small ε (1.3) holds for some ψ_ε. Now by (1.3) and Definition 1.2, it must be that ψ_ε is also locally unbiased. Hence ϕ is locally inadmissible among locally unbiased tests. □

Now we look more closely at local unbiasedness. We have the following.

THEOREM 3.2. *Suppose* (1.7) *holds for all tests* ϕ. *A necessary condition for a test* ϕ *to be locally unbiased is that*

$$\rho(\lambda, M; \phi) \geq 0 \quad \text{for all } (\lambda, M) \in \Delta, \tag{3.1}$$

and a sufficient condition is that

$$\rho(\lambda, M; \phi) > 0 \quad \text{for all } (\lambda, M) \in \Delta. \tag{3.2}$$

PROOF. Suppose (3.1) is violated, so that

$$\rho(\lambda, M; \phi) < -\eta \quad \text{for some } (\lambda, M) \in \Delta \text{ and } \eta > 0. \tag{3.3}$$

Hence, by (2.3), for each $\varepsilon > 0$ there is a $(\lambda, M) \in \Delta_\varepsilon$ such that (3.3) holds. This implies that there exists $\theta \in \Theta_\varepsilon$ for which

$$\rho(\theta, \theta\theta'; \phi) < -\|\theta\|^2 \eta/2. \tag{3.4}$$

Hence (1.7) implies that for all ε sufficiently small there is a $\theta \in \Theta_\varepsilon$ such that $r_\theta(\phi) < \alpha$. Thus ϕ is locally biased.

Conversely, suppose that (3.2) is satisfied. Then $\rho(\lambda, M; \phi) \geq \eta > 0$ for all $(\lambda, M) \in \Delta$ since Δ is compact. Consequently there is an $\varepsilon > 0$ such that $\rho(\lambda, M; \phi) > \eta/2$ for all $(\lambda, M) \in \Delta_\varepsilon$. It then follows that there is an $\varepsilon' \leq \varepsilon$ such that $r_\theta(\phi) > \alpha$ for all $\theta \in \Delta_{\varepsilon'}$. Thus ϕ is locally unbiased. □

The necessary condition (3.1) cannot be improved upon in that the test $\phi(x) \equiv \alpha$ is locally unbiased and achieves equality in (3.1) for all (λ, M). The sufficient condition (3.2), however, can be improved. For example, if $\Lambda = \mathbf{R}^p \times \mathbf{S}_p$ [see (2.18)], then, by taking $(\lambda, 0) \in \mathbf{R}^p \times \mathbf{S}_p$ for various values of λ, we see that we need $l_\phi = 0$. Assuming $l_\phi = 0$, then, for local unbiasedness we need only that (3.2) holds for all $(\lambda, M) \in \mathbf{R}^p \times \mathbf{S}_p$ with $M \neq 0$. For another example, take the problem of testing H_0 based on a p-variate $N(\theta, I)$ for $p \geq 2$. If $\Theta = \mathbf{R}^p$, then the test that rejects H_0 when $|X_1| > 1.96$ is unbiased, hence locally unbiased, but violates (3.2) for any M with zero as the upper left element.

4. Local admissibility and locally most mean power unbiased tests. In this section we assume that Assumptions 2.1 and 2.2 and (2.13) or (2.17) hold, and that 0 is in the interior of Θ. From Brown and Marden (1989), Example 4.4, we have that $\Lambda = \mathbf{R}^p \times \mathbf{S}_p$ and $\Lambda^* = \mathbf{S}_p$. Sen Gupta and Vermeire (1986) define a test ϕ to be locally most mean power unbiased (LMMPU) of level α if it is locally unbiased and, for any other essentially different level α locally unbiased test ψ, there exists an $\varepsilon_0 > 0$ such that for $M = I_p$

$$\int_{\Psi_\varepsilon(M)} E_\theta(\phi)\, d\theta > \int_{\Psi_\varepsilon(M)} E_\theta(\psi)\, d\theta \quad \text{for all } \varepsilon < \varepsilon_0, \tag{4.1}$$

where, for $M \in \mathbf{S}_p$, $\Psi_\varepsilon(M)$ is the ellipsoid

$$\Psi_\varepsilon(M) = \{\theta | \theta' M \theta \leq \varepsilon\}.$$

As in Sen Gupta and Vermeire, we can use (1.7) to show that a sufficient condition for a test to be LMMPU level α is that it be locally unbiased and be the essentially unique test that maximizes $\operatorname{tr}(V_\phi)$ among locally unbiased level α tests. In their Theorem 2 they use the Neyman–Pearson lemma to show that a LMMPU test satisfies (2.7) with $M_0 = I_p$, where $\lambda \in \mathbf{R}^p$ and c are chosen to make the level be α and $l_\phi = 0$. Clearly this test is locally admissible by either Theorem 2.2 or 2.3. In their Remark 2.f, Sen Gupta and Vermeire

note that one could change parametrizations using a smooth transformation with invertible Jacobian J at $\theta = 0$, and then find the LMMPU test under the new parametrization. In general, this test will be different than the original unless $JJ' = I$. In fact, the new test will maximize $\operatorname{tr}((JJ')^{-1}V_\phi)$ among the level α locally unbiased tests, and will be as in (2.7) or (2.16) with $M = (JJ')^{-1}$. Equivalently, this new test is the one for which (4.1) holds with $M = (JJ')^{-1}$. If when M in (4.1) is of rank q, $q < p$, we interpret $d\theta$ to be q-dimensional Lebesgue measure on the q-dimensional space in which $\Psi_\varepsilon(M)$ lies, then we obtain the following.

THEOREM 4.1. *The class of tests in Theorem* 3.1 *consists of those tests that are LMMPU* (4.1) *for some* $M \in \mathbf{S}_p$.

5. Examples. Section 4 of Brown and Marden (1989) contains a discussion of the set Λ in (2.18), from which it is easy to find Δ, for different parameter spaces Θ. As mentioned in the previous section, if 0 is in the interior of Θ, then $\Lambda = \mathbf{R}^p \times \mathbf{S}_p$. Other results are referred to below. Unless otherwise stated, we assume that Assumptions 2.1 and 2.2 hold, as well as (2.13) or (2.17) as appropriate.

EXAMPLE 5.1 (One-parameter families). When Θ is a subset of the real line, $l(x) = [\partial f_\theta(x)/\partial\theta]_{\theta=0}/f_0(x)$ and $V(x) = [\partial^2 f_\theta(x)/\partial\theta^2]_{\theta=0}/f_0(x)$. In one-sided cases, for example, $\Theta \subset [0, \infty)$, it can be shown that $\Delta = \{(1, 0)\}$, hence the only locally admissible level α test is the locally most powerful (LMP) test that has rejection region of the form $\{x|l(x) > c\}$. If $\Theta \subset (-\infty, 0]$, then the LMP test has rejection region $\{x|l(x) < c\}$. Interestingly, it can be the case that the test for the two-sided situation [$0 \in \text{Interior}(\Theta)$] created by combining the two one-sided LMP tests into a rejection region of the form $R \equiv \{x|l(x) < c_-\} \cup \{x|l(x) > c_+\}$ is locally inadmissible. One can imagine this eventuality by noting that in the two-sided case, any locally admissible test rejects H_0 when

$$al(x) + bV(x) > c, \tag{5.1}$$

for some a, b, c with $b \geq 0$, $(a, b) \neq (0, 0)$, and realizing that the set R cannot in general be put in the form (5.1). Thus, choosing c_- and c_+ so that R is locally unbiased does not necessarily yield a LMP locally unbiased test. One example of this phenomenon can be found in Section 6.1 of Brown and Marden (1989). As another example, consider the curved exponential family with two-dimensional statistic

$$f_\theta(x) = \exp\{\theta x_1 + \kappa(\theta^2/2)x_2 - \psi(\theta, \kappa\theta^2/2)\}.$$

If the family is normalized so that $E_0(X) = 0$ and $\operatorname{Var}_0(X) = 1$, then κ is the statistical curvature. Now $l(x) = x_1$ and $V(x) = x_1^2 + \kappa x_2 - 1$. We have that R^c is an interval depending only on x_1, while the test in (5.1) has a rejection region with parabolic boundary in $\mathbf{X}$ unless $\kappa = 0$ or $b = 0$. If $b = 0$, then we revert to a one-sided test, hence in particular when $\kappa \neq 0$ no unbiased test of the form R can be LMP among locally unbiased tests (unless the sample space

and measure ν are taken in a very special way). If $\kappa = 0$, then the tests of the form R will be locally admissible.

EXAMPLE 5.2 (Exponential families). Suppose we have an exponential family of densities with respect to Lebesgue measure on $\mathbf{R}^p$ with natural parameter θ and with 0 in the interior of the natural parameter space. We then have

$$f_\theta(x) = a(x)e^{\theta' x - \psi(\theta)}.$$

It is not difficult to verify the assumptions. Also,

$$l(x) = x - \mu_0 \quad \text{and} \quad V(x) = (x - \mu_0)(x - \mu_0)' + \Sigma_0,$$

where μ_0 and Σ_0 are the mean and covariance, respectively, of X when $\theta = 0$. Thus the statistics in the tests in Theorem 2.1 are (possibly degenerate) ellipsoids in x. Which ellipsoids yield locally admissible tests then depends on the set Λ.

EXAMPLE 5.3 (Locally pointed alternatives). Suppose that Θ is locally a pointed closed convex cone, that is, there exists a closed convex cone C in $\mathbf{R}^p$ with vertex 0 such that for some $\gamma \in \mathbf{R}^p$ and $\beta < 0$,

$$\frac{\gamma'\theta}{\|\theta\|} < \beta \quad \text{for all } \theta \in C,$$

and

$$C \cap \{\theta \in \mathbf{R}^p | \|\theta\| \leq \varepsilon\} = \Theta_\varepsilon.$$

Examples of such spaces include the nonnegative orthant, and the ordered parameter space in Example 5.6. In this case

$$\Lambda = \{(\lambda, 0) | \lambda \in C\}.$$

Then Theorem 2.1 shows that a necessary condition for a test to be locally admissible is that it essentially rejects H_0 if

$$\lambda' l(x) > c \tag{5.2}$$

for some $(\lambda, c) \in C \times \mathbf{R}$. Test (5.2) is locally most powerful along the ray $\{r\lambda | r > 0\}$. Theorem 3.2 shows that a necessary (sufficient) condition for a test to be locally unbiased is that

$$\lambda' l_\phi \geq (>) 0 \quad \text{for all } \lambda \in C.$$

EXAMPLE 5.4 (Half-, quarter-, etc., alternatives). For a given t, $1 < t < p$, define

$$C_t = \{\theta \in \mathbf{R}^p | \theta_i \geq 0 \text{ for } i = 1, \ldots, t\}. \tag{5.3}$$

Thus when $p = 2$ and $t = 1$, C_t is the right half-plane. If, for some $\varepsilon > 0$,

$$C_t \cap \{\theta \in \mathbf{R}^p | \|\theta\| \leq \varepsilon\} = \Theta_\varepsilon, \tag{5.4}$$

then, by Example 4.6 in Brown and Marden (1989),

$$\Lambda = \{(\lambda, M) \in \mathbf{R}^p \times \mathbf{S}_p | \lambda \in C_t \text{ and } M^{(1)} = 0\}, \tag{5.5}$$

where $M^{(1)}$ is the upper left $t \times t$ submatrix of M. Let $M^{(2)}$ be the lower $(p-t) \times (p-t)$ submatrix, let $\lambda^{(1)}$ ($\lambda^{(2)}$) be the upper $t \times 1$ [lower $(p-t) \times 1$] subvector of λ and partition l_ϕ and V_ϕ similarly. Then by Theorem 3.2, a necessary (sufficient) condition for a test to be locally unbiased is that $l_\phi^{(2)} = 0$ and

$$\lambda^{(1)\prime} l_\phi^{(1)} + \tfrac{1}{2} \operatorname{tr}\left(M^{(2)} V_\phi^{(2)}\right) \geq (>)\, 0$$

for all

$$(\lambda^{(1)}, M^{(2)}) \in \mathbf{R}^t \times \mathbf{S}^{p-t} - \{(0,0)\}.$$

EXAMPLE 5.5 (Same-sign alternatives). In problem (1.1), one might have a priori information that all the parameters have the same sign, so that

$$\Theta = C_{ss} \equiv \{\theta | \theta_i \geq 0 \text{ for all } i, \text{ or } \theta_i \leq 0 \text{ for all } i\}. \tag{5.6}$$

It can be shown that

$$\Lambda = \{(\lambda, M) \in \mathbf{R}^p \times \mathbf{S}_p | \text{all elements of } M \text{ are nonnegative}\}.$$

As in the case $\Lambda = \mathbf{R}^p \times \mathbf{S}_p$, a necessary condition for a test to be locally unbiased is that $l_\phi = 0$. However, because of the restriction on the M's it is not in general necessary for V_ϕ to be nonnegative definite. It is sufficient, for example, if $l_\phi = 0$, the diagonal elements of V_ϕ are positive and the off-diagonal elements are nonnegative.

We apply this result to a normal linear model, that is,

$$Y = Z\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + r, \tag{5.7}$$

where Y and r are $n \times 1$ vectors, Z is a known $n \times (p+q)$ matrix of full rank, $p > 0$ and $p + q < n$, β_1 and β_2 are $p \times 1$ and $q \times 1$ vectors, respectively, and

$$r \sim N_n(0, \sigma^2 I_n)$$

for $\sigma^2 > 0$. We want to test

$$H_0: \beta_1 = 0 \quad \text{versus} \quad H_A: \beta_1 \in C_{ss} - \{0\},$$

where β_2 and σ^2 are unspecified. For example, β_1 might represent comparisons of p treatments to the same control, where all treatment effects are known to go in the same direction as compared to the control.

The null is not simple, so we first reduce the problem by invariance and sufficiency. A sufficient statistic consists of the least-squares estimates of β_1 and β_2 and the sum of squared residuals,

$$\left(\hat{\beta}_1, \hat{\beta}_2, S\right) \tag{5.8}$$

so that

$$\begin{pmatrix}\hat\beta_1\\ \hat\beta_2\end{pmatrix} \sim N\left(\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}, \sigma^2(X'X)^{-1}\right), \quad \text{independent of } S \sim \sigma^2\chi^2_{n-p-q}.$$

The problem is invariant under the affine group $G \equiv [\mathbf{R} - \{0\}] \times \mathbf{R}^q$, which acts via

$$(a, b) \cdot \left(\hat\beta_1, \hat\beta_2, S\right) \to \left(a\hat\beta_1, a\hat\beta_2 + b, a^2S\right).$$

If we restrict consideration to the subgroup G^* of G that requires that $a > 0$, we have that the maximal invariant statistic and parameter are, respectively,

$$X \equiv \hat\beta_1/\sqrt{S} \quad \text{and} \quad \theta \equiv \beta_1/\sigma.$$

The problem is then (1.1) with Θ as in (5.6). The density of X can be easily derived by first conditioning on S and then integrating:

$$\begin{aligned} f^*_\theta(x) &= (\text{constant})\exp\left(-\tfrac{1}{2}\theta'H^{-1}\theta\right)\\ &\quad \times \int_0^\infty s^{[(n-q)/2]-1}\exp\left(-\tfrac{1}{2}s[1 + x'H^{-1}x]\right)\exp\left(\sqrt{s}\,\theta'H^{-1}x\right) ds \qquad (5.9)\\ &= (\text{constant})(1 + x'H^{-1}x)^{(n-q)/2}\exp\left(-\tfrac{1}{2}\theta'H^{-1}\theta\right)\sum_{k=1}^{\infty}\frac{c_k}{k!}\left(\sqrt{2}\,\theta'u\right)^k, \end{aligned}$$

where

$$u = \frac{H^{-1}x}{\sqrt{1 + x'H^{-1}x}}, \qquad c_k = \frac{\Gamma((n - q + k)/2)}{\Gamma((n - q)/2)}, \qquad (5.10)$$

and H is the upper left $p \times p$ submatrix of $(Z'Z)^{-1}$. The density required when considering invariance under the full group G is then

$$\begin{aligned} f_\theta(x) &= \frac{f^*_\theta(x) + f^*_\theta(-x)}{2}\\ &= (\text{constant})(1 + x'H^{-1}x)^{(n-q)/2}\exp\left(-\tfrac{1}{2}\theta'H^{-1}\theta\right)\sum_{i=0}^{\infty}\frac{c_{2i}}{(2i)!}\left(2(\theta'u)^2\right)^i. \end{aligned}$$

Assumptions 2.1 and 2.2 can be verified in a straightforward manner, and it is easy to see that $l(x) \equiv 0$ and

$$V(x) \equiv \tfrac{1}{2}\left[(n - q)uu' - H^{-1}\right]. \qquad (5.11)$$

Equation (2.17) holds since when $M \neq 0$, the function $\operatorname{tr} MV(x)$ is convex in u and strictly convex in at least one u_i, and the distribution of u is absolutely continuous with respect to Lebesgue measure on $\mathbf{R}^p$. Thus (5.11) and Theorem 2.3 show that the locally admissible tests are those that essentially reject H_0 when

$$u'Mu > c \qquad (5.12)$$

for $(M, c) \in \Lambda^* \times \mathbf{R}$, where

$$\Lambda^* = \{M \in \mathbf{S}_p | \text{all elements of } M \text{ are nonnegative}\}.$$

In Section 6 we consider the null distribution of the statistic $u'Mu$. In general, the distribution can be given as a linear combination of nonindependent beta variables. We derive the first two moments. When $M = H$, the distribution is a simple beta.

If the matrix $(Z'Z)^{-1}$ happens to be permutation-invariant, then we can perform further reductions. Let P be the group of $p \times p$ permutation matrices, and consider the group $G \times P$ which acts on (5.8) via

$$(a, b, \pi) \cdot \left(\hat{\beta}_1, \hat{\beta}_2, S\right) \to \left(a\pi\hat{\beta}_1, a\hat{\beta}_2 + b, a^2 S\right).$$

The class of locally admissible invariant tests under $G \times P$ are those in (2.17) but with V replaced by

$$V^I(x) \equiv \frac{1}{p!} \sum V(\pi x). \tag{5.13}$$

Thus $V^I(x)$ has all diagonal elements equal [to $v_0(x)$, say] and all off-diagonal elements equal [to $v_1(x)$, say]. Then, for $M \in \Lambda^*$,

$$\operatorname{tr}(MV^I(x)) = v_0(x) \sum_{i=1}^{p} M_{ii} + v_1(x) \sum_{i \neq j} M_{ij} \equiv m_0 v_0(x) + m_1 v_1(x). \tag{5.14}$$

Thus we can use (5.11), (5.13), and (5.14) to show that the locally admissible tests for the $G \times P$-reduced problem have statistics

$$m_0 \|u\|^2 + m_1 \sum_{i \neq j} u_i u_j, \tag{5.15}$$

where $m_0 \geq m_1 \geq 0$.

EXAMPLE 5.6 (Ordered alternatives). Suppose we have $p + 1$ means, and we wish to test the null hypothesis that they are all equal versus the alternative that they are in nonincreasing order with respect to their indices. See Barlow, Bartholomew, Bremner and Brunk (1972). For the ith mean μ_i, we have a set of independent $N(\mu_i, \sigma^2)$ observations, with the $p + 1$ sets of observations independent. We assume $\sigma^2 > 0$ is unknown. This is a special case of the model (5.7). Let Y denote the $(p + 1) \times 1$ vector of sample means from the sets, and let S denote the pooled sum of squares about the means. Thus Y and S are independent with

$$Y \sim N(\mu, \sigma^2 D) \quad \text{and} \quad S \sim \chi^2_{n-p-1},$$

where μ is the vector of means, D is a diagonal matrix with diagonal elements inverses of the individual sample sizes and n is the total sample size. We test

$$H_0: \mu_1 = \mu_2 = \cdots = \mu_{p+1} \qquad \text{versus}$$

$$H_A: \mu_1 \geq \mu_2 \geq \cdots \geq \mu_{p+1}, \qquad \text{not all equal.}$$

We again use invariance, noting that the problem is invariant under the affine group $G \equiv \{(a, b) | a > 0 \text{ and } b \in \mathbf{R}\}$ which acts via

$$(a, b) \cdot (y, s) \to (ay + b\mathbf{1}, a^2 s),$$

where $\mathbf{1}$ is the vector of 1's. The maximal invariant statistic and parameter are, respectively,

$$x \equiv \frac{1}{\sqrt{s}} \begin{pmatrix} y_1 - y_{p+1} \\ y_2 - y_{p+1} \\ \vdots \\ y_p - y_{p+1} \end{pmatrix} \quad \text{and} \quad \theta \equiv \frac{1}{\sigma} \begin{pmatrix} \mu_1 - \mu_{p+1} \\ \mu_2 - \mu_{p+1} \\ \vdots \\ \mu_p - \mu_{p+1} \end{pmatrix}.$$

The density of X can be found to be f_θ^* in the previous example, where H is now the covariance matrix of $\sqrt{S}X$, and $q = 1$. Also,

$$\Theta = \{\theta \in \mathbf{R}^p | \theta_1 \geq \theta_2 \geq \cdots \geq \theta_p \geq 0\}.$$

Since Θ is a pointed closed convex cone as in Example 5.3, we know that $\Lambda = \{(\lambda, 0) \in \mathbf{R}^p \times \mathbf{S}_p | \lambda \in \Theta\}$. We only need $l(x)$, which can be found from (5.9) to be

$$l(x) = c_1 \sqrt{2}\, u.$$

Thus by Theorems 2.1 and 2.2, the locally admissible tests are those that essentially reject H_0 if and only if

$$\lambda' u > c \tag{5.16}$$

for some $(\lambda, c) \in \Theta \times \mathbf{R}$.

In Section 6 we derive the null distribution of $\lambda' U$. Without loss of generality take $\lambda' H^{-1} \lambda = 1$. Then, when $\theta = 0$,

$$\frac{\sqrt{n-2}\,\lambda' U}{\sqrt{1 - (\lambda' U)^2}} \sim t_{n-2}. \tag{5.17}$$

We also show that for test (5.16),

$$\lambda_0 l_\phi \text{ has the same sign as } \lambda_0' H^{-1} \lambda. \tag{5.18}$$

Thus by Theorem 3.2 and (5.6), a necessary (sufficient) condition for the test to be locally unbiased is that

$$\lambda_0' H^{-1} \lambda \geq (>) 0 \quad \text{for all } \lambda_0 \in \Theta - \{0\}.$$

For example, if $\lambda' = (1, 0, \ldots, 0)$, then, since H is positive definite and $\lambda_{01} > 0$, the test is locally unbiased.

EXAMPLE 5.7 (Testing independence of p variables). We are given n independent observations from a p-variate normal distribution with mean zero and nonsingular covariance matrix Σ, and wish to test the independence of the p variables. That is, we test

$$H_0: \Sigma > 0 \text{ is diagonal} \quad \text{versus } H_A: \Sigma > 0, \text{ arbitrary}. \tag{5.19}$$

By sufficiency, we can base the tests on the sample sum of squares and cross-products S, so that S has a p-variate Wishart distribution with n degrees of freedom and mean $n\Sigma$. The first step is again to invoke invariance. Let G be the group of $p \times p$ diagonal matrices with nonzero diagonal elements which acts on S via

$$A \cdot S = ASA.$$

The maximal invariant statistic is R, the sample correlation matrix arising from S. We take the maximal invariant parameter to be

$$\{\theta_{ij}\}_{1 \le i < j \le p},$$

the off-diagonal elements of Ω $(\equiv \Omega_\theta)$ that has elements

$$\omega_{ij} = \frac{\sigma^{ij}}{\sqrt{\sigma^{ii}\sigma^{jj}}} \qquad 1 \le i, j \le p,$$

where the σ^{ij}'s are the elements of Σ^{-1}. A straightforward application of Wijsman's theorem (1967) yields the function R_θ of (1.4) for the reduced problem to be

$$R_\theta(R) = K|\Omega_\theta|^{n/2} \int_G (\text{abs}|A|)^{n-1} \exp\left[-\tfrac{1}{2}\operatorname{tr}(\Omega_\theta ARA)\right] dA,$$

where abs$|A|$ is the absolute value of the determinant of A, and

$$K = \left[2^{n/2}\Gamma\left(\frac{n}{2}\right)\right]^{-p}.$$

Since both R and Ω_θ have all diagonal elements 1 and A is diagonal, we can write

$$R_\theta(R) = K|\Omega_\theta|^{n/2} \int_G \left(\prod_{i=1}^{p} (a_i^2)\right)^{(n-1)/2} \exp\left(-\frac{1}{2}\sum_{i=1}^{p} a_i^2\right) \times \exp\left(-\sum_{i>j} a_i a_j \theta_{ij} r_{ij}\right) da_1 \cdots da_p.$$

The second exponential in the integral above equals

$$1 - \sum_{i>j} a_i a_j \theta_{ij} r_{ij} + \frac{1}{2}\left[\sum_{i>j} a_i a_j \theta_{ij} r_{ij}\right]^2 + o(\|\theta\|^2).$$

Since

$$\int_0^\infty a_i (a_i^2)^{(n-1)/2} \exp\left(-\tfrac{1}{2}a_i^2\right) da_i = 0$$

and

$$\int_{-\infty}^\infty a_i^2 (a_i^2)^{(n-1)/2} \exp\left(-\tfrac{1}{2}a_i^2\right) da_i = 2^{n/2+1}\Gamma\left(\frac{n}{2} + 1\right),$$

we have that

$$R_\theta(R) = |\Omega_\theta|^{n/2}\left[1 + \frac{n^2}{2}\sum_{i>j}\theta_{ij}^2 r_{ij}^2 + o(\|\theta\|^2)\right]. \tag{5.20}$$

The first derivatives of $|\Omega_\theta|$ at $\theta = 0$ are all 0, and the second derivative matrix at $\theta = 0$ is $2I_{p(1-p)/2}$. Thus we obtain from (5.20) that

$$l(R) = 0 \quad \text{and} \quad V(R) = \left[n^2 \operatorname{diag}\left(\{r_{ij}^2\}_{1\le i<j\le p}\right) + nI_{p(p-1)/2}\right].$$

We apply Theorem 2.3. The requisite assumptions can be verified as in previous examples. The only task left is to find Λ^*. The restriction on $\theta \in \Theta$ is that Ω_θ be positive definite. Since for any θ with $\|\theta\|^2$ small enough, Ω_θ is positive definite, Θ_ε for small enough $\varepsilon > 0$ is the ε-ball about 0. Hence $\Lambda^* = \mathbf{S}_{p(1-p)/2}$. The theorem then gives that a test is locally admissible if and only if it essentially rejects when

$$\sum_{i>j} m_{ij} r_{ij}^2 > c,$$

where $m_{ij} \ge 0$ for all i, j.

Although the exact null distribution of the above statistic is difficult to find, the first two moments can be calculated easily since the R_{ij}^2's are pairwise independent Beta$(1/2, (n-1)/2)$ variables. Thus

$$E_0\left[\sum_{i>j} m_{ij} R_{ij}^2\right] = \frac{1}{n}\sum_{i>j} m_{ij} \tag{5.21}$$

and

$$\operatorname{Var}\left[\sum_{i>j} m_{ij} R_{ij}^2\right] = \frac{2(n-1)}{n^2(n+2)}\sum_{i>j} m_{ij}^2.$$

We note that the reduced problem is invariant under the group of $p \times p$ permutation matrices P which acts via $\pi \cdot R \to \pi R \pi'$. As in (5.13), we can find the V^I for the permutation invariant problem to be

$$V^I(R) = \frac{1}{p!}\sum_{\pi\in P} V(\pi R \pi') = n\left[\frac{4}{p(p-1)}\left(\sum_{i>j} r_{ij}^2\right) + 1\right] I_{p(1-p)/2}.$$

Therefore, the only locally admissible tests are those based on $\sum_{i>j} r_{ij}^2$. It is interesting that this statistic is analogous to the one minimized in factor analysis when using the MINRES criterion [see Harman (1976)].

APPENDIX

Consider the statistic U defined in (5.10). Working back to (5.8), we see that under H_0, U has the same distribution as

$$W \equiv \frac{H^{-1/2}Y}{\sqrt{S + Y'Y}}, \tag{6.1}$$

where

$$Y \sim N(0, I_p) \quad \text{independent of } S \sim \chi^2_m, \tag{6.2}$$

and $m = n - p - q$. Now take $M \in \Lambda^* - \{0\}$ and look at $W'MW$. By taking an appropriate orthogonal transformation of Y to Y^*, we can obtain

$$W'MW = \sum_{i=1}^{p} \frac{d_i Y_i^{*2}}{S + Y^{*\prime} Y^*} \equiv \sum_{i=1}^{p} d_i B_i, \tag{6.3}$$

where Y^* has the same properties as Y in (6.2), and $d_1, \ldots, d_p$ are the eigenvalues of $H^{-1/2}MH^{-1/2}$. Each B_i is a Beta$(1/2, (m + p - 1)/2)$ variable, but the B_i's are clearly not independent. When $H^{-1/2}MH^{-1/2}$ is the identity, then (6.3) is a Beta$(p/2, m/2)$ variable.

The first two moments of (6.3) are now derived. We know that

$$E_0[B_i] = \frac{1}{m + p} \quad \text{and} \quad \operatorname{Var}_0[B_i] = \frac{2(m + p - 1)}{(m + p)^2(m + p + 2)}. \tag{6.4}$$

We will show below that

$$\operatorname{Cov}_0[B_i, B_j] = \frac{2\pi}{(m + p)(m + p - 2)} - \frac{1}{(m + p)^2}. \tag{6.5}$$

From (6.4) and (6.5) we obtain that

$$E_0[W'MW] = \frac{1}{m + p} \sum_{i=1}^{p} d_i$$

and

$$\operatorname{Var}_0[W'MW] = \left[\sum_{i=1}^{p} d_i^2\right] \frac{2(m + p - 1)}{(m + p)^2(m + p + 2)} + \left[2 \sum_{i>j} d_i d_j\right] \left(\frac{2\pi}{(m + p)(m + p - 2)} - \frac{1}{(m + p)^2} \right).$$

We can reexpress the variance in terms of H and M by noting that

$$\sum_{i=1}^{p} d_i^2 = \operatorname{tr}\left([MH^{-1}]^2\right) \quad \text{and} \quad 2 \sum_{i>j} d_i d_j = \left[\operatorname{tr}(MH^{-1})\right]^2 - \operatorname{tr}\left([MH^{-1}]^2\right).$$

Now to (6.5). For $z > 0$,

$$E\left[Y_1^{*2} Y_2^{*2} \middle| Y_1^{*2} + Y_2^{*2} = z\right] = z^2 \int_0^{2\pi} \sin^2(\omega) \cos^2(\omega)\, d\omega = \frac{\pi}{4} z^2$$

since, conditionally, (Y_1^*, Y_2^*) is uniformly distributed on the circle of radius $\sqrt{z}$.

Thus

$$E\left[\frac{Y_1^{*2}Y_2^{*2}}{(S+Y^{*\prime}Y^*)^2}\right] = E\left\{E\left[\frac{Y_1^{*2}Y_2^{*2}}{(S+Y^{*\prime}Y^*)^2}\middle| Y_1^{*2}+Y_2^{*2}, Y_3^*, \ldots, Y_p^*, \text{and } S\right]\right\}$$

$$= \frac{\pi}{4}E\left[\left(\frac{Y_1^{*2}+Y_2^{*2}}{S+Y_1^{*2}+Y_2^{*2}+\sum_{i=3}^p Y_i^{*2}}\right)^2\right]$$

$$= \frac{\pi}{4}E\left[\text{Beta}\left(1, \frac{m+p-2}{2}\right)^2\right]$$

$$= \frac{2\pi}{(m+p)(m+p-2)},$$

which with (6.4) verifies (6.5).

Go back to W in (6.1). We want to find the distribution of the statistic in (5.16), which is the same as that for $\lambda' W$. Suppose $\lambda' H^{-1}\lambda = 1$, and let $\gamma = H^{-1/2}\lambda$, so that $\|\gamma\| = 1$ and

$$\lambda' W = \frac{\gamma' Y}{\sqrt{S+Y'Y}}.$$

Again, with an appropriate orthogonal transformation, we obtain Y^* as above so that

$$\lambda' W = \frac{Y_1^*}{\sqrt{S+Y^{*\prime}Y^*}}.$$

From this equation it is a short calculation to show that

$$\frac{\sqrt{m+p-1}\,\lambda' W}{\sqrt{1-(\lambda' W)^2}} \sim t_{m+p-1},$$

proving (5.17).

Finally, we show that for λ and λ_0 in $\mathbf{R}^p$, (5.18) holds. For the test ϕ as in (5.16) with λ_0, we have that

$$\lambda_0' l_\phi = c_1\sqrt{2}\,\lambda_0' \int_{\{\lambda' u > c\}} u f_0(u)\, du,$$

where f_0 is given in (5.9). Set $v = H^{-1/2}u$, $\gamma = H^{-1}\lambda$, and $\gamma_0 = H^{-1}\lambda_0$, and suppose without loss of generality that $\|\gamma\| = \|\gamma_0\| = 1$. Then

$$\lambda_0' l_\phi = (\text{constant}) \int_{\{\gamma' v > c\}} \frac{\gamma_0' v}{(1+\|v\|^2)^{(n-q)/2}}\, dv.$$

Next, let Γ be any $p \times p$ orthogonal matrix whose first row is γ_0', and put

$v^* = \Gamma v$ to obtain

$$\lambda_0' l_\phi = (\text{constant}) \int_{\{\gamma'\Gamma' v^* > c\}} \frac{v_1^*}{\left(1 + \|v^*\|^2\right)^{(n-q)/2}} \, dv^*.$$

Since the integral over v_1^* with $v_2^*, \ldots, v_p^*$ fixed is symmetric about $v_1^* = 0$, it has the same sign as the first component of $\Gamma\gamma$. Thus the integral over all of v^* has the same sign as

$$\Gamma\gamma \equiv \gamma_0'\gamma \equiv \lambda_0' H^{-1}\lambda,$$

which completes the proof of (5.18).

REFERENCES

BARLOW, R., BARTHOLOMEW, D., BREMNER, J. M. and BRUNK, H. D. (1972). *Statistical Inference Under Order Restrictions*. Wiley, New York.

BROWN, L. D. (1986). *Fundamentals of Statistical Exponential Families*. IMS, Hayward, Calif.

BROWN, L. D. and MARDEN, J. I. (1989). Complete class results for hypothesis testing problems with simple null hypothesis. *Ann. Statist.* **17** 209–235.

COHEN, A. and MARDEN, J. I. (1988). Minimal complete classes of invariant tests for equality of normal covariance matrices and sphericity. *J. Multivariate Anal.* **27** 139–150.

COHEN, A., SACKROWITZ, H. B. and STRAWDERMAN, W. E. (1985). Multivariate locally most powerful unbiased tests. In *Multivariate Analysis VI* (P. R. Krishnaiah, ed.) 121–144. North-Holland, Amsterdam.

GIRI, N. (1968). Locally and asymptotically minimax tests of a multivariate problem. *Ann. Math. Statist.* **39** 171–178.

HARMAN, H. H. (1976). *Modern Factor Analysis*. Univ. Chicago Press.

ISAACSON, S. L. (1951). On the theory of unbiased tests of simple statistical hypotheses specifying the values of two or more parameters. *Ann. Math. Statist.* **22** 217–234.

KUDÔ, H. (1961). Locally complete classes of tests. *Bull. Inst. Internat. Statist.* **38** 173–180.

MARDEN, J. I. (1981). One-sided testing problems on the correlation coefficient when the variances are known: Small and large sample properties. Unpublished manuscript.

MARDEN, J. I. (1983). Admissibility of invariant tests in the general multivariate analysis of variance problem. *Ann. Statist.* **11** 1086–1099.

NEYMAN, J. and PEARSON, E. S. (1936). Contributions to the theory of testing statistical hypotheses: Part I. *Statistical Research Memoirs* **1** 113–137.

NEYMAN, J. and PEARSON, E. S. (1938). Contributions to the theory of testing statistical hypotheses: Parts II and III. *Statistical Research Memoirs* **2** 25–57.

SCHATZOFF, M. (1966). Sensitivity comparisons among tests of the general linear hypothesis. *J. Amer. Statist. Assoc.* **61** 415–435.

SEN GUPTA, A. and VERMEIRE, L. (1986). Locally optimal tests for multiparameter hypotheses. *J. Amer. Statist. Assoc.* **81** 819–825.

WALD, A. (1950). *Statistical Decision Functions*. Wiley, New York.

DEPARTMENT OF MATHEMATICS
WHITE HALL
CORNELL UNIVERSITY
ITHACA, NEW YORK 14853-7901

DEPARTMENT OF STATISTICS
101 ILLINI HALL
UNIVERSITY OF ILLINOIS
725 SOUTH WRIGHT STREET
CHAMPAIGN, ILLINOIS 61820