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A PURE-TAIL ORDERING BASED ON THE RATIO OF THE QUANTILE FUNCTIONS[1]

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In the intuitive approach, a distribution function F is said to be not more heavily tailed than G if $\limsup_{x\to\infty} \bar{F}/\bar{G} < \infty$. An alternative is to consider the behavior of the ratio $F^{-1}(u)/G^{-1}(u)$, in a neighborhood of one. The present paper examines the relationship between these two criteria and concludes that the intuitive approach gives a more thorough comparison of distribution functions than the ratio of the quantile functions approach in the case F or G have tails that decrease faster than, or at, an exponential rate. If F or G have slowly varying tails, the intuitive approach gives a less thorough comparison of distributions. When F or G have polynomial tails, the approaches agree.

1. Introduction. The concept of tail-heaviness of a distribution function F permeates both the theory and practice of statistics. Among others, the following examples illustrate the importance of the concept. In the problem of estimating the location parameter of a symmetric distribution, tail behavior of the underlying probability distribution has a direct effect on the efficiency of the estimators. In extreme value theory, the tail behavior of F determines the limiting distributions of the extreme value statistics. In nonparametric density estimation, certain methods of selecting the smoothing parameter work well for short-tailed distributions [Schuster and Gregory (1981)], but the presence of a moderate outlier causes the methods to choose too large a smoothing parameter.

Until recently, the literature on tail ordering [e.g., van Zwet (1964), Barlow and Proschan (1975), Loh (1984), Capéraà (1988)] has been concerned with orderings of the whole distribution, with the resulting ordering strongly affected by the behavior of the distributions in the center.

The most common approach to a pure tail ordering defines tail weight in terms of the rate at which the density tends to 0 at infinity. More precisely, a density f is said to have a lighter tail than g if

$$f(x)/g(x) \to 0 \quad \text{as } x \to \infty. \tag{1.1}$$

With this intuitive definition, a normal density has a lighter tail than a logistic or double exponential density, which in turn are lighter tailed than a t distribution. An alternative approach is suggested in Parzen (1979) and Lehmann (1988) in terms of density quantile functions. The latter author

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defines the distribution F to be lighter tailed than the distribution G if

$$(F^{-1}(u) - F^{-1}(v))/(G^{-1}(u) - G^{-1}(v)) \leq M \tag{1.2}$$

for $0 < v < u < 1$ and some $M > 0$. When both F^{-1} and G^{-1} are differentiable, (1.2) reduces to gG^{-1}/fF^{-1} being bounded on $(0, 1)$. Then, as an alternative to (1.1), the following criterion which compares the densities at the same quantile, rather than at the same x, could replace (1.1),

$$gG^{-1}(u)/fF^{-1}(u) \to 0 \quad \text{as } u \to 1. \tag{1.3}$$

When f or g do not exist, the ratios given in (1.1) and (1.3) may be replaced by their integrated versions,

$$\overline{F}(x)/\overline{G}(x) \tag{1.4}$$

and

$$F^{-1}(u)/G^{-1}(u), \tag{1.5}$$

where $\overline{F} = 1 - F$ and similarly for $\overline{G}$. The definitions to be adopted in Section 2 will be in terms of (1.4) and (1.5), and it will turn out that, in some cases, the tail orderings defined in terms of (1.4) and (1.5) agree. In other cases, one ordering provides a more detailed comparison than the other. The purpose of the present paper is to give an account of this relationship. Section 2 provides the definitions and considers some of their possible drawbacks. In Section 3, the classes of distributions with swiftly varying tails, polynomial tails and scale-invariant tails are introduced and it is concluded that the definition in terms of (1.4) gives a more thorough comparison of distribution functions than that in terms of (1.5) in the case F or G have tails which decrease faster than, or at, an exponential rate. When F or G have slowly varying tails, the approach based on (1.5) gives a more thorough comparison of distributions than the one based on (1.4). For polynomial tails, the two approaches agree.

2. The definitions. The statement of the definitions to be adopted are now given.

DEFINITION 1. Let F and G be probability distribution functions. Then

$$F \leq_D G \quad \text{if } \limsup_{x \to \infty} \overline{F}(x)/\overline{G}(x) < \infty,$$
$$F <_D G \quad \text{if } F \leq_D G \text{ but } G \nleq_D F,$$
$$F \sim_D G \quad \text{if } F \leq_D G \text{ and } G \leq_D F.$$

When the limit in (1.1) exists, the limit of (1.4) as $x \to \infty$ exists and they are equal. However, the limit of (1.4) may exist, for example, when F and G are the Poisson and geometric distributions, without the ratio in (1.1) even being defined.

Although Definition 1 is satisfactory for some purposes, we mention two possible drawbacks: (i) If F and G have finite support, the ratios in (1.1) and

(1.4) are undefined for large x, and the definitions therefore do not apply, (ii) the definition is not location nor scale invariant; this is illustrated by the following example.

EXAMPLE 1. Let f and g represent normal densities with means (μ, θ) and variances (σ^2, τ^2), respectively. Then, the behavior of f/g is as follows:

$$\lim_{x \to \infty} f(x)/g(x) = \begin{cases} 0, & \text{if either } \tau^2 > \sigma^2 \text{ or } \tau^2 = \sigma^2 \text{ and } \theta > \mu, \\ \infty, & \text{if either } \tau^2 < \sigma^2 \text{ or } \tau^2 = \sigma^2 \text{ and } \theta < \mu, \\ 1, & \text{otherwise.} \end{cases}$$

Although for some applications, for example, efficiency of point estimators, a normal distribution $N(\theta, \sigma^2)$ has lighter tails than $N(\theta, 2\sigma^2)$, for other applications, for example, selection of the smoothing parameters in nonparametric density estimation, a definition under which all normal distributions are considered to have equally heavy tails, might be preferred. One possibility for an alternative definition is suggested by the work of Parzen (1979) and Lehmann (1988). The latter author defined F to have a lighter tail than G if (1.2) holds for $0 < v < u < 1$, and some $M > 0$. When both F^{-1} and G^{-1} are differentiable, (1.2) reduces to

$$gG^{-1}(u)/fF^{-1}(u) \le M, \qquad 0 \le u \le 1. \tag{2.1}$$

Parzen calls the function fF^{-1} the density-quantile function and classifies probability distributions according to the limiting behavior of $fF^{-1}(u)$ as $u \to 1$ or 0 [see also Schuster (1984)]. When $M = 1$, the ordering defined by (2.1) has been named dispersive ordering in the literature and was originally proposed, under a different name, by Doksum (1969).

Hereinafter, F^{-1} will denote the left-continuous version of the inverse of the distribution function F. That is,

$$F^{-1}(u) = \inf\{x: F(x) \ge u\}.$$

Thus, in particular, the inequalities $F^{-1}F(x) \le x$ and $FF^{-1}(u) \ge u$ are valid.

DEFINITION 2. Let F and G be distribution functions with inverses F^{-1} and G^{-1}, respectively. Then

$$\begin{aligned} F \le_q G \quad & \text{if } \limsup_{u \to 1} F^{-1}(u)/G^{-1}(u) < \infty, \\ F <_q G \quad & \text{if } F \le_q G \text{ but } G \not\le_q F, \\ F \sim_q G \quad & \text{if } F \le_q G \text{ and } G \le_q F. \end{aligned}$$

When $G^{-1}(u) \to \infty$ as $u \to 1$, and $\lim_{u \to 1} gG^{-1}(u)/fF^{-1}(u)$ exists, then, by L'Hôpital's rule, $\lim_{u \to 1} F^{-1}(u)/G^{-1}(u) = \lim_{u \to 1} gG^{-1}(u)/fF^{-1}(u)$, but the ratio of the quantile functions may be bounded in a neighborhood of one without the density quantile functions even being defined [e.g., $F(x) = 1 - q_1^x$ and $G(x) = 1 - q_2^x$ with $0 < q_1 < q_2 < 1$, $x = 1, 2, 3, \ldots$].

It is easy to see that q-ordering is capable of comparing distribution functions with finite support. Also, it follows easily that under very mild conditions, q-ordering is location and scale invariant. A more detailed comparison of the two definitions is given in Sections 5 and 6 of Rojo (1988). For example, it can be shown that q-ordering remains invariant under general monotone transformations.

This section ends with two examples that suggest a property of the q-ordering which will be studied in more detail in Section 3. Namely, for distributions with tails which decrease at a faster rate than, or at, an exponential rate, the q-ordering may be thought of as a smoothed out version of the D-ordering in the sense that it ignores the low order terms in the tail of the distribution function.

EXAMPLE 2 [Parzen (1979)]. Let $\bar{F}_i(x) = e^{-x-c_i \sin x}$, $x \geq 0$, $|c_i| < 1$, $i = 1, 2$. Then $\bar{F}_1(x)/\bar{F}_2(x) = e^{(c_2-c_1)\sin x}$ and $F_1^{-1}(u)/F_2^{-1}(u) = 1 + (c_2 \sin F_2^{-1}(u) - c_1 \sin F_1^{-1}(u))/F_2^{-1}(u)$. Thus, while $\bar{F}_1/\bar{F}_2$ oscillates between $e^{c_2-c_1}$ and $e^{c_1-c_2}$, F_1^{-1}/F_2^{-1} converges to 1.

EXAMPLE 3. Let $f(x)$ be the standard normal density function and let $g(x) = |x|e^{-(1/2)x^2}/2$, $-\infty < x < \infty$. Then $\bar{F}(x)/\bar{G}(x) < (2/\pi)^{1/2}/x \to 0$ as $x \to \infty$. Thus, $F <_D G$. On the other hand, $F^{-1}(u)/G^{-1}(u) \to 1$. It follows that $F \sim_q G$ and the q-ordering ignored the low order term x present in the density g.

3. Relationships between q and D orderings. Example 3 suggests that D-ordering provides a more precise comparison of tail-heaviness of distribution functions than q-ordering. However, as illustrated by the following example, q-ordering may order F and G strongly while D-ordering cannot distinguish between F and G.

EXAMPLE 4. Define $\bar{F}(x) = 1/\ln(x)$ for $x > e$ and $\bar{G}(x) = 1/2\ln(x)$ for $x > e^{1/2}$. Then $\bar{F}(x)/\bar{G}(x) = 1/2$ and since $F^{-1}(\mu) = \exp(1/(1-\mu))$ and $G^{-1}(\mu) = \exp(1/(2(1-\mu)))$, $G^{-1}(\mu)/F^{-1}(\mu) \to 0$ as $\mu \to 1$. Therefore, while $F \sim_D G$, $G <_q F$.

It is clear from Examples 3 and 4 that general statements regarding relationships between q-ordering and D-ordering are not possible without restricting attention to suitable classes of distribution functions. One obvious difference between the distributions considered in Example 3 and Example 4 is that in the former example, the distributions have tails which decrease rapidly, while the latter example considers distributions whose tails decrease slowly. Somewhat more precisely, since q-ordering is scale-invariant while D-ordering is not, q-ordering will distinguish between distributions F and G only when the tail of the long-tailed distribution cannot be made shorter than that of the short-tailed distribution by a scale transform, while the D-ordering only distinguishes between F and G if the largest of $\bar{F}$ and $\bar{G}$ cannot be made smaller than the smallest of $\bar{F}$ and $\bar{G}$ by multiplication with a small positive constant. These considerations lead, then, into focusing our attention in the following classes of distributions.

DEFINITION 3. A distribution function F is said to have a swiftly varying right tail, ($F \in$ SVT), if there is a $t > 1$ such that

$$\liminf_{x\to\infty} \frac{\bar{F}(x)}{\bar{F}(tx)} > 1.$$

DEFINITION 4. A distribution function F is said to have a scale-invariant right tail ($F \in$ SIT), if

$$\limsup_{x\to\infty} \frac{\bar{F}(x)}{\bar{F}(2x)} < \infty.$$

Examples of distributions with swiftly varying right tails include distributions with regularly varying tails at infinity. Thus, the t, F and Pareto distributions have swiftly varying tails. Other examples include the exponential, normal and extreme value distributions. On the other hand, examples of distributions with tails which are not swiftly varying are given by $\bar{F}(x) = l(x)$, where l is a nonincreasing slowly varying function. In fact, it can be shown, Rojo (1988), that $F \in$ SVT if and only if $\bar{F}$ is not asymptotically equivalent to a slowly varying function. By contrast, if $\bar{F}(x) = l(x)$ for some slowly varying function l, then $F \in$ SIT. Other examples of distributions with scale-invariant right tails are given by distributions with regularly varying tails, while distributions that do not have scale-invariant tails include the normal, exponential and extreme value distributions.

The following theorem establishes the main relationships between the q and D orderings.

THEOREM 1. *Let F and G be distribution functions.* (i) *If either $F \in SIT$ or $G \in SIT$, then $F \leq_q G$ implies $F \leq_D G$.* (ii) *If either $F \in SVT$ or $G \in SVT$, then $F \leq_D G$ implies $F \leq_q G$.*

PROOF. (i) Suppose that $F \leq_q G$ and $F \in$ SIT or $G \in$ SIT. Note that $F \in$ SIT implies $F(x) < 1$ for all x so that $\lim_{\mu\to 1} F^{-1}(\mu) = \infty$, and therefore $F \leq_q G$ yields $\lim_{\mu\to 1} G^{-1}(\mu) = \infty$, or equivalently, $G(x) < 1$ for all x. On the other hand, $G \in$ SIT implies directly that $G(x) < 1$ for all x. Therefore, the assumptions of (i) imply that $G(x) < 1$ for all x. Define $\phi(x) = F^{-1}G(x)$. Then

$$(3.1) \qquad \limsup_{x\to\infty} \frac{\phi(x)}{x} \leq \limsup_{x\to\infty} \frac{F^{-1}G(x)}{G^{-1}G(x)} = \limsup_{\mu\to 1} \frac{F^{-1}(\mu)}{G^{-1}(\mu)} < \infty,$$

where the first inequality above follows from the fact that $G^{-1}G(x) \leq x$ and the last inequality follows from our assumptions. Next note that, since $FF^{-1}(\mu) \geq \mu$,

$$\bar{F}(\phi(x)) \leq \bar{G}(x) \quad \text{for all } x, \tag{3.2}$$

so that (3.1) then implies that there exist $k > 0$ and $x_0 > 0$ such that

$$\bar{F}(kx) \leq \bar{G}(x) \quad \text{for } x > x_0. \tag{3.3}$$

Now choose N so that $k \leq 2^N$. Then, when $F \in$ SIT, there exists A and x_0' such that $\bar{F}(x) \leq A\bar{F}(2x)$ for $x \geq x_0'$. Therefore, (3.3) yields

$$\bar{F}(x) \leq A^N \bar{G}(x) \quad \text{for } x \geq \max(x_0, x_0') \tag{3.4}$$

and, therefore, $F \leq_D G$.

When $G \in$ SIT, a similar argument leads to

$$\bar{F}(x) \leq \bar{G}\left(\frac{x}{k}\right) \quad \text{for } x > x_0'' \text{ and some } k > 0, \text{ some } x_0'' > 0$$

and

$$\bar{G}\left(\frac{x}{k}\right) \leq A^N \bar{G}(x) \quad \text{for } x > \max(x_0, x_0'') \text{ and some } A > 0,$$

and, therefore, $F \leq_D G$ follows.

(ii) If $F \leq_D G$, then $G(x) < 1$ for all x and hence $\lim_{\mu \to 1} G^{-1}(\mu) = \infty$. Also, there exist x_0 and $A > 0$ such that

$$\bar{F}(x) \leq A\bar{G}(x) \quad \text{for } x > x_0.$$

For $F \in$ SVT, there exist $t > 1$, $\alpha > 1$ and x_0' such that

$$\bar{F}(x) \geq \alpha \bar{F}(tx) \quad \text{for } x > x_0'.$$

Now, choose an integer N with $\alpha^N \geq A$. Then

$$\bar{F}(x) \geq \alpha^N \bar{F}(t^N x) \geq A\bar{F}(t^N x) \quad \text{for } x > x_0',$$

so that

$$\bar{F}(t^N x) \leq \bar{G}(x) \quad \text{for } x > x_1 = \max(x_0, x_0'). \tag{3.5}$$

Now, since $GG^{-1}(\mu) \geq \mu$ and $F(x) \geq \mu$ is equivalent to $x \geq F^{-1}(\mu)$, (3.5) implies $F(t^N G^{-1}(\mu)) \geq \mu$ for all μ with $G^{-1}(\mu) \geq x_1$, or equivalently,

$$F^{-1}(\mu) \leq t^N G^{-1}(\mu) \quad \text{for } \mu > \mu_0, \tag{3.6}$$

where $\mu_0 \in (0, 1)$. Therefore, $F \leq_q G$.

If instead $G \in$ SVT, then

$$\bar{G}(t^{-N}x) \geq \alpha^N \bar{G}(x) \geq A\bar{G}(x) \quad \text{for } x \geq x_0'$$

for some t, $\alpha > 1$, and an argument similar to the one for the case $F \in$ SVT yields

$$\bar{F}(x) \leq \bar{G}(t^{-N}x) \quad \text{for } x > x_2 = \max(x_0, x_0'').$$

As N is fixed, (3.6) follows for large μ and therefore $F \leq_q G$. □

The following corollary is immediate from Theorem 1.

COROLLARY 1. (i) *If F or $G \in SVT$, then $F \sim_D G$ implies $F \sim_q G$. Moreover, if $F <_q G$, then either $F <_D G$ or F and G cannot be D-ordered.*

(ii) *If F or $G \in SIT$, then $F \sim_q G$ implies $F \sim_D G$. Moreover, if $F <_D G$, then either $F <_q G$ or F and G cannot be q-ordered.*

PROOF. (i) That $F \sim_D G$ implies $F \sim_q G$ when F or $G \in$ SVT, follows immediately from Theorem 1, part (ii) by interchanging the roles of F and G. Now, if F and G are D-ordered and $F <_q G$, then it follows that $F <_D G$ since $F <_q G$ is inconsistent with $F \sim_D G$ and $G <_D F$.

(ii) The proof is similar to that of (i). □

It is not at all obvious that $F <_q G$ does not imply that F and G are D-ordered when F or $G \in$ SVT. When F or $G \in$ SIT, it is also possible to have $F <_D G$ but F and G not q-ordered. These remarks are illustrated by the following examples.

EXAMPLE 5. Let $\overline{G}(x) = e^{-\{\ln x - (\ln x)^{1/2} h_1(x)\}^2}$ and $\overline{F}(x) = e^{(-(\ln x)^2 + \ln x)}$ for $x > e$, where $h_1(x) = (1 + \sin(\ln \ln x))$ so that $F \in$ SVT. Then, for x_n such that $\ln \ln x_n = (3\pi/2) + 2n\pi$, $n = 0, 1, 2, \dots$, $\overline{G}(x_n) = e^{-(\ln x_n)^2}$ so that $\overline{G}(x_n)/\overline{F}(x_n) \to 0$ as $n \to \infty$. On the other hand, for y_n such that $\ln \ln y_n = (\pi/2) + 2n\pi$, $n = 0, 1, 2, \dots$, $\overline{G}(y_n) = e^{-(\ln y_n - 2(\ln y_n)^{1/2})^2}$ so that

$$\overline{G}(y_n)/\overline{F}(y_n) = e^{4(\ln y_n)^{3/2} - 5 \ln y_n} \to \infty \quad \text{as } n \to \infty,$$

and therefore F and G are not D-ordered. However, note that $F^{-1}(u)$ satisfies the equation

$$-\ln(1 - u) = (\ln F^{-1}(u))^2 - \ln(F^{-1}(u))$$

and therefore $F^{-1}(u) = e^{(1/2) + ((1/4) - \ln(1-u))^{1/2}}$. Also,

$$(-\ln(1 - u))^{1/2} = \ln G^{-1}(u) - (\ln G^{-1}(u))^{1/2} h_1(G^{-1}(u)).$$

Therefore, for all u, $G^{-1}(u) \geq e^{(-\ln(1-u))^{1/2}}$. It follows that

$$\limsup_{u \to 1} F^{-1}(u)/G^{-1}(u) < \infty.$$

On the other hand, for u such that $h_1(G^{-1}(u)) = 2$, $(-\ln(1 - u))^{1/2} = \ln G^{-1}(u) - 2(\ln G^{-1}(u))^{1/2}$ so that $G^{-1}(u) = e^{(1 + [1 + (-\ln(1-u))^{1/2}]^{1/2})^2}$ and hence, $\liminf_{u \to 1} F^{-1}(u)/G^{-1}(u) = 0$ so that $F <_q G$.

EXAMPLE 6. Define

$$\overline{F}(x) = \frac{c}{\ln x}\left(\frac{1}{\ln \ln x} + h_2(x)\right), \qquad x > e^e,$$

where $c = e/2$ and $h_2(x) = 1 + \sin(\ln\ln\ln x)$ and let $\overline{G}(x) = c/\ln x$, $x > e^{e/2}$. Clearly, $G \in$ SIT. Now, $\overline{F}/\overline{G} \sim 1/\ln\ln x$ when $h_2(x) = 0$ so that $\liminf_{x\to\infty} \overline{F}/\overline{G} = 0$. On the other hand, $\overline{F}/\overline{G} < (1/\ln\ln x) + 2$ so that $\limsup_{x\to\infty} \overline{F}/\overline{G} < \infty$ and therefore, $F <_D G$. Now note that $G^{-1}(u) = e^{c/(1-u)}$ while $F^{-1}(u)$ satisfies the equation

$$1 - u = \frac{c}{\ln F^{-1}(u)}\left(\frac{1}{\ln\ln F^{-1}(u)} + h_2(F^{-1}(u))\right).$$

Thus, if $h_2(F^{-1}(u)) = 2$, $(2c/\ln F^{-1}(u)) < 1 - u$ and therefore $F^{-1}(u) > e^{2c/(1-u)}$. It follows that $\limsup_{u\to 1} F^{-1}(u)/G^{-1}(u) = \infty$. Now, if $h_2(F^{-1}(u)) = 0$, then $\ln F^{-1}(u)\ln\ln F^{-1}(u) = c/(1-u)$ so that for $\ln\ln F^{-1}(u) > 2$, $\ln F^{-1}(u) < (c/2(1-u))$. It follows that $F^{-1}(u) < e^{c/2(1-u)}$ and hence $\liminf_{u\to 1} F^{-1}(u)/G^{-1}(u) = 0$. Thus, F and G are not q-ordered.

It is easy to see that the existence of the limit of F^{-1}/G^{-1} as $u \to 1$ is sufficient for $F <_q G$ to imply $F <_D G$ when F or $G \in$ SVT. Similarly, if F or $G \in$ SIT and $\overline{F}/\overline{G}$ converges as $x \to \infty$, then $F <_D G$ implies $F <_q G$. Theorem 1 and Corollary 1, together with the previous remarks, show that q-ordering and D-ordering agree in many cases. However, the nongenerality of this agreement has been illustrated by Examples 3, 4, 5 and 6. Nonetheless, there is a large class of distributions for which q-ordering and D-ordering are in agreement. The class is defined as follows:

DEFINITION 5. A distribution function F is said to have a polynomial tail ($F \in \mathscr{P}$) if $F \in$ SVT and $F \in$ SIT.

It is easy to see that any distribution function F with $\overline{F} = h(x)x^{-\alpha}$, where $\alpha > 0$ and $h(tx)/h(x)$ is bounded away from 0 and ∞, has a polynomial tail. The following corollary relating q-ordering and D-ordering when $F \in \mathscr{P}$ or $G \in \mathscr{P}$ follows immediately from Theorem 1 and Corollary 1.

COROLLARY 2. *If either $F \in \mathscr{P}$ or $G \in \mathscr{P}$, then q-ordering and D-ordering agree.*

Examples of distributions which do not have polynomial tails include the normal and exponential distributions, and more generally, distributions of the form

$$\overline{F}(x) = h(x)e^{-x^p}, \qquad p > 0, \tag{3.7}$$

where

$$h(x)e^{-kx^\varepsilon} \to 0 \quad \text{and} \quad h(x)e^{kx^\varepsilon} \to \infty \quad \text{for all } \varepsilon > 0, \text{all } k > 0. \tag{3.8}$$

Distributions satisfying (3.7) and (3.8) above may be classified as having exponential tails. We now define the class of exponentially-tailed distributions more generally as follows:

DEFINITION 6. A distribution function F is said to have an exponential tail ($F \in \mathscr{E}$) if, for some $t > 1$,

$$\liminf_{x\to\infty} \frac{-\log \bar{F}(tx)}{-\log \bar{F}(x)} > 1 \quad \text{and} \quad \limsup_{x\to\infty} \frac{-\log \bar{F}(tx)}{-\log \bar{F}(x)} < \infty. \tag{3.9}$$

Clearly, if F satisfies (3.7) and (3.8) above, then $F \in \mathscr{E}$. It is easy to see that $\mathscr{E} \in$ SVT. On the other hand, that $\mathscr{E} \notin$ SIT follows easily by taking, for example, $\bar{F}(x) = e^{-x}$. It follows from Theorem 1 and Corollary 1 that if F or $G \in \mathscr{E}$, then $F \leq_D G$ implies $F \leq_q G$, $F <_q G$ implies that $F <_D G$ or F and G are not D-ordered at all and $F \sim_D G$ implies $F \sim_q G$. On the other hand, Example 3 shows that while $F <_D G$, $F \sim_q G$ and hence the q and D orderings do not agree on the family of the exponentially-tailed distributions.

For distributions satisfying (3.7) and (3.8) above, a somewhat more precise statement may be made.

THEOREM 2. *Let* $\bar{G}(x) = h_1(x)e^{-x^{\alpha_1}}$ *and* $\bar{F}(x) = h_2(x)e^{-x^{\alpha_2}}$ *where* h_1, h_2 *satisfy* (3.8) *with* $\alpha_1, \alpha_2 > 0$. *Then* (i) $\alpha_1 < \alpha_2$ *implies that* $F <_q G$ *and* $F <_D G$; (ii) *if* h_1, h_2 *are continuous*, $\alpha_1 = \alpha_2$ *implies* $F \sim_q G$ *but* $F \sim_D G$ *may not hold*.

PROOF. (i) Since $\alpha_2 > \alpha_1$ and h_1, h_2 satisfy (3.8) for sufficiently large x and $\alpha_1 < \alpha_1' < \alpha_2' < \alpha_2$,

$$\bar{F}(x) \geq e^{-x^{\alpha_1'}} \quad \text{and} \quad \bar{G}(x) \leq e^{-x^{\alpha_2'}}$$

so that, for $\mu > \mu_0$, $\mu_0 \in (0,1)$,

$$F^{-1}(\mu) \geq (-\log(1-\mu))^{1/\alpha_1'} \quad \text{and} \quad G^{-1}(\mu) \leq (-\log(1-\mu))^{1/\alpha_2'}.$$

Therefore, $F >_D G$ and $F >_q G$.

(ii) In the case $\alpha_1 = \alpha_2 = \alpha$ with h_1 and h_2 continuous, we have

$$(F^{-1}(\mu))^{\alpha} = -\log(1-\mu) + \log h_1(F^{-1}(\mu)),$$

and since h_1 satisfies (3.8), for every $\varepsilon > 0$,

$$(F^{-1}(\mu))^{\alpha} = -\log(1-\mu) + O\big((F^{-1}(\mu))^{\varepsilon}\big) \quad \text{as } \mu \to 1.$$

Therefore

$$F^{-1}(\mu) \sim (-\log(1-\mu))^{1/\alpha} \quad \text{as } \mu \to 1.$$

Since $\alpha_1 = \alpha_2$,

$$G^{-1}(\mu) \sim (-\log(1-\mu))^{1/\alpha} \quad \text{as } \mu \to 1,$$

and therefore $F \sim_q G$. That $F \sim_D G$ may not hold follows from Example 3. □

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