SOME POSET STATISTICS

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Statistics or functions are discussed that measure agreement between certain types of partially ordered data. These poset statistics are a generalization of two familiar classes of functions: the arrangement increasing functions and the decreasing reflection functions; those functions measure agreement between linearly ordered data. Specifically, the statistics in question are functions $h(\mathbf{X}_1, \mathbf{X}_2)$ of two matrix arguments, each having N rows and they measure the agreement of the ordering of the N rows of the two matrices. An example is used to illustrate and motivate the discussion. One statistic in this class is applied to the example; it generalizes Wilcoxon's rank sum statistic, Spearman's rank correlation and Page's statistic for ordered alternatives.

1. An example of a statistical poset. This note concerns a class of functions that measure agreement between certain types of partially ordered data that occur in statistics. The partially ordered sets (or posets) discussed here are among the simplest types of posets; they are direct products of linearly ordered sets.

Consider the following example. Morton, Saah, Silberg, Owens, Roberts and Saah (1982) studied lead in the blood of the children of employees in a factory in Oklahoma which used lead in the manufacture of batteries. There were $N = 34$ children from 34 different families. The thought was that employees might bring lead home in their clothes and hair, thereby exposing their children. Morton, Saah, Silberg, Owens, Roberts and Saah classified employees in two ways. First, employees had varied jobs and therefore varied exposures to lead on the job; so employees were classified as having a low (0), medium (1) or high (2) exposure to lead. Second, employees were observed to vary in their hygiene and were classified into three categories, good (0), fair (1) and poor (2). Indicate the classification by the ordered pair (exposure, hygiene). The quantity of lead in each child's blood was also measured. Their data are given in Figure 1. If parental exposures are responsible for the lead in a child's blood, then we would anticipate agreement between children's lead levels and the partial ordering of the categories given by $(a_1, a_2) \lesssim (b_1, b_2)$ if and only if $a_1 \le b_1$ and $a_2 \le b_2$.

The original study applied the Kruskal–Wallis test twice to the data in Figure 1, once for level of exposure and once for hygiene. Although this did not lead to an error of interpretation in this case, it is arguably not the best
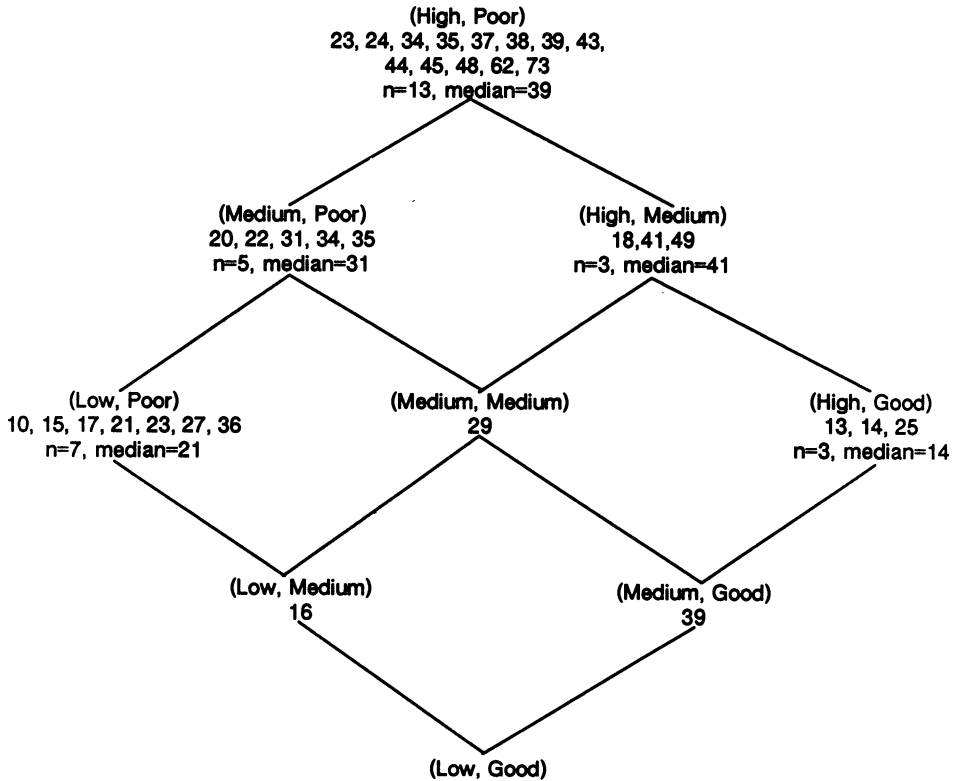

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FIG. 1. *Blood lead levels of children classified by parent's exposure and hygiene. Labels are (exposure, hygiene).*

approach. A practical byproduct of the general discussion in Sections 2 and 3 is a better test for this problem and a property that good tests should have.

Let $\mathbf{Z}$ be the 34×2 matrix in which the first column indicates the level of the parent's exposure and the second column indicates the level of hygiene. Let $\mathbf{Y}$ be the 34×1 matrix indicating the children's blood lead levels. What does it mean to speak of agreement between arrangements of $\mathbf{Z}$ and $\mathbf{Y}$?

Let $\tilde{\delta}_{ij}$ be the N-dimensional column vector whose ith coordinate is $1/\sqrt{2}$, whose jth coordinate is $-1/\sqrt{2}$ and whose other coordinates are 0. Then the $N \times N$ matrix $\mathbf{I} - 2\tilde{\delta}_{ij}\tilde{\delta}_{ij}^{\mathsf{T}}$ acts on $\mathbb{R}^N$ by interchanging the ith and jth coordinates.

Consider $\mathbf{a} = \mathbf{Z}^{\mathsf{T}}\tilde{\delta}_{ij}\tilde{\delta}_{ij}^{\mathsf{T}}\mathbf{Y}$. Then $\mathbf{a}$ has mth coordinate $a_m = \frac{1}{2}(Z_{im} - Z_{jm})(Y_i - Y_j)$, for $m = 1, 2$. Compare child i and child j. If $a_1 \geq 0$ and $a_2 \geq 0$ with at least one strict inequality, then the child with the higher blood lead level also has both the higher parental exposure and the poorer parental hygiene. In the case of these two children, the partial order on the rows of $\mathbf{Z}$ agrees with the linear order on $\mathbf{Y}$. On the other hand, if $a_1 \leq 0$ and $a_2 \leq 0$ with at least one inequality strict, then the child with the lower blood lead level has both the higher parental exposure and the poorer parental hygiene, so the two

orders disagree. Finally, if one a_m is negative and the other is positive, then there is neither clear agreement nor clear disagreement between the two orders. A function which measures agreement should be large when there are many choices of i and j such that $a_1 > 0$ and $a_2 > 0$.

Let $h(\mathbf{Z}, \mathbf{Y})$ be a real-valued function which is unchanged by simultaneously reordering the rows of $\mathbf{Z}$ and $\mathbf{Y}$ in the same way, that is, a statistic unchanged by renumbering the 34 children. Suppose that we have selected a pair (i, j) of children such that $a_1 \geq 0$ and $a_2 \geq 0$, so this pair is not out of order. If $h(\mathbf{Z}, \mathbf{Y})$ measures the agreement between $\mathbf{Z}$ and $\mathbf{Y}$, then knocking this pair out of order should not increase the function, that is, we should have $h(\mathbf{Z}, \mathbf{Y}) \geq h(\mathbf{Z}, [\mathbf{I} - 2\tilde{\boldsymbol{\delta}}_{ij}\tilde{\boldsymbol{\delta}}_{ij}^{\mathsf{T}}]\mathbf{Y})$. This is the principle underlying the general discussion in Section 2 and it has a long history.

2. Functions measuring agreement between two matrices.

Let $B = \{\mathbf{x} \in \mathbb{R}^N : \mathbf{x}^{\mathsf{T}}\mathbf{x} = 1\}$. For any $\mathbf{x} \in B$, the $N \times N$ orthogonal matrix $\mathbf{S_x} = \mathbf{I} - 2\mathbf{x}\mathbf{x}^{\mathsf{T}}$ is a reflection; it acts on $\mathbb{R}^N$ by reflecting points through the hyperplane orthogonal to $\mathbf{x}$. A finite reflection group G with root system $\Delta \subset B$ is a group of finitely many $N \times N$ orthogonal matrices such that (i) G is the smallest group containing the reflections $\{\mathbf{S_\delta}: \delta \in \Delta\}$ and (ii) a reflection $\mathbf{I} - 2\mathbf{x}\mathbf{x}^{\mathsf{T}}$ is in G only if $\mathbf{x} \in \Delta$. The group π_N of all $N \times N$ permutation matrices is a finite reflection group with root system $\Delta = \{\tilde{\boldsymbol{\delta}}_{ij}, 1 \leq i \neq j \leq N\}$, where $\tilde{\boldsymbol{\delta}}_{ij}$ was defined in Section 1. Other finite reflection groups that arise in statistics include the group of coordinate sign changes, the group of coordinate sign changes and permutations, the group of permutations within blocks or subclasses and various direct products of these groups. For discussion of finite reflection groups, see Eaton (1982, 1987) and Grove and Benson (1985) and the references given there. The group of sign changes and permutations is applied to sensitivity analysis for the signed-rank and related statistics in Rosenbaum (1987, Section 4).

Let G be a finite reflection group with root system Δ. A set D of $N \times M$ real matrices is G-invariant if $\mathbf{X} \in D$ implies $\mathbf{g}\mathbf{X} \in D$ for all $\mathbf{g} \in G$. For $k = 1, 2$, let D_k be a G-invariant set of $N \times M_k$ real matrices, with $N \geq 2$ and $M_k \geq 1$ and let $D_1 \times D_2$ be the direct product of these two sets. A function $h \colon D_1 \times D_2 \to \mathbb{R}$ is G-invariant if $h(\mathbf{X}_1, \mathbf{X}_2) = h(\mathbf{g}\mathbf{X}_1, \mathbf{g}\mathbf{X}_2)$ for all $\mathbf{g} \in G$. Write $\mathbf{A} \geq \mathbf{0}$ if $\mathbf{A}$ is a nonnegative matrix, that is, if $a_{ij} \geq 0$ for each i and j.

DEFINITION. A G-invariant function $h \colon D_1 \times D_2 \to \mathbb{R}$ is an (M_1, M_2)-decreasing reflection function (or dr function) if for all $\delta \in \Delta$ and all $\mathbf{X}_k \in D_k$, $k = 1, 2$,

$$(2.1) \qquad \mathbf{X}_1^{\mathsf{T}}\boldsymbol{\delta}\boldsymbol{\delta}^{\mathsf{T}}\mathbf{X}_2 \geq \mathbf{0} \quad \text{implies} \quad h(\mathbf{X}_1, \mathbf{X}_2) \geq h(\mathbf{X}_1, \mathbf{S_\delta}\mathbf{X}_2).$$

In words, when G is the group π_N of permutation matrices, if two specific rows of $\mathbf{X}_1$ and $\mathbf{X}_2$ are arranged in the same way, then deranging these two rows of $\mathbf{X}_2$ yielding $\mathbf{S_\delta}\mathbf{X}_2$ will reduce (i.e., not increase) h. A $(1, 1)$-dr function is a decreasing reflection function as defined in Eaton (1982, 1987). The

$(1, 1)$-dr functions for the group π_N of permutations are the arrangement increasing functions, discussed with various names by Eaton (1967), Hollander, Proschan and Sethuraman (1977), Marshall and Olkin (1979, Section 6F) and D'Abadie and Proschan (1984). Related ideas are discussed by Savage (1957).

Hollander, Proschan and Sethuraman (1977) develop many properties of arrangement increasing functions and the reader can easily verify that most of these properties also hold for the larger class of (M_1, M_2)-dr functions, sometimes with minor modifications. Two properties in particular deserve mention. First, they show that $h(\mathbf{w}, \mathbf{v}) = \prod f(w_n, v_n)$ is arrangement increasing when $f(\cdot, \cdot)$ is TP_2. Analogously, $h(\mathbf{X}_1, \mathbf{X}_2) = \prod f(\mathbf{x}_{n1}, \mathbf{x}_{n2})$ is (M_1, M_2)-dr with respect to π_N when $f(\cdot, \cdot)$ is MTP_2 in the sense of Karlin and Rinott (1980), where $\mathbf{x}_{nk}$ is the nth row of $\mathbf{X}_k$.

The second property that deserves mention is the extension of the composition theorems in Hollander, Proschan and Sethuraman [(1977), Theorem 3.3] and Eaton [(1982), Theorem 4.3; (1987), Section 3.4)]. Let $h_1: D_1 \times D_2 \to \mathbb{R}$ be an $(M_1, 1)$-dr function and let $h_2: D_2 \times D_3 \to \mathbb{R}$ be a $(1, M_3)$-dr function, where D_k is a G-invariant set for $k = 1, 2, 3$. Let μ_2 be a σ-finite, G-invariant measure on D_2. If, for all $(\mathbf{X}_1, \mathbf{X}_3) \in D_1 \times D_3$, the integral $h_3(\mathbf{X}_1, \mathbf{X}_3) = \int_{D_2} h_1(\mathbf{X}_1, \mathbf{Z}) h_2(\mathbf{Z}, \mathbf{X}_3) \mu_2(d\mathbf{Z})$ is well-defined and finite, then $h_3(\mathbf{X}_1, \mathbf{X}_3)$ is called the composition of h_1 and h_2. The composition theorem asserts that the composition $h_3(\mathbf{X}_1, \mathbf{X}_3)$ is (M_1, M_3)-dr. The proof of the composition theorem for $M_1 \geq 1$ and $M_3 \geq 1$ is identical to the proof for $(1, 1)$-dr functions given in Eaton [(1982), Section 4; (1987), Section 3.4].

3. Rank-scores.
Is it possible to assign rank scores to the rows of a matrix? That is: Is it possible to assign numerical ranks or scores to the rows of an $N \times M$ matrix $\mathbf{X}$ in a manner that is consistent with the ordering implied by G? Let D be a G-invariant set of $N \times M$ matrices, and let $\boldsymbol{\rho}: D \to \mathbb{R}^N$. We say that $\boldsymbol{\rho}(\cdot)$ is G-equivariant if $\boldsymbol{\rho}(\mathbf{gX}) = \mathbf{g}\boldsymbol{\rho}(\mathbf{X})$ for all $\mathbf{X} \in D$ and all $\mathbf{g} \in G$. An M-tuple $(a_1, \ldots, a_M)$ is said to have constant sign if all of its coordinates have the same sign, that is, if $a_i \geq 0$ for $i = 1, \ldots, M$ or $a_i \leq 0$ for $i = 1, \ldots, M$. A G-equivariant function $\boldsymbol{\rho}(\cdot)$ is called a rank-score function if for every $\mathbf{X} \in D$ and $\delta \in \Delta$, we have $\boldsymbol{\rho}(\mathbf{X})^\mathsf{T} \delta \delta^\mathsf{T} \mathbf{X} \geq \mathbf{0}$ whenever $\delta^\mathsf{T} \mathbf{X}$ has constant sign. In words, this says that if the ordering of certain rows of $\mathbf{X}$ is clear (in the sense that $\delta^\mathsf{T} \mathbf{X}$ has constant sign), then the ordering of the corresponding rank scores $\boldsymbol{\rho}(\mathbf{X})$ agrees with the ordering of $\mathbf{X}$. When $M = 1$ and $G = \pi_N$, conventional ranks are rank-score functions, providing average ranks are used for ties. For $M \geq 2$ and $G = \pi_N$, a simple rank-score function $\boldsymbol{\rho}^*(\mathbf{X})$ is obtained by separately ranking the M columns of $\mathbf{X}$, with average ranks in case of ties and using the sum of the ranks in a row as the score for that row.

A rank-score function is intended to assign scores to the rows of $\mathbf{X}$ in a manner consistent with the order implied by G, that is, the order measured by the $(1, M)$-dr functions. The following proposition says the rank-score functions are precisely the functions that do this. The proof is elementary and is omitted.

PROPOSITION 1. *A G-equivariant function $\rho\colon D \to \mathbb{R}^N$ is a rank score function if and only if for every $(1, M)$-dr function $h\colon \mathbb{R}^N \times D \to \mathbb{R}$, and for every $\mathbf{X} \in D$ and $\delta \in \Delta$ such that $\delta^\mathsf{T}\mathbf{X}$ has constant sign, we have $h\{\rho(\mathbf{X}), \mathbf{X}\} \geq h\{\rho(\mathbf{X}), \mathbf{S}_\delta \mathbf{X}\}$.*

The following proposition states that statistics $\rho_1(\mathbf{X}_1)^\mathsf{T}\rho_2(\mathbf{X}_2)$ which are sums of rank products are (M_1, M_2)-dr functions. When $M_1 = M_2 = 1$, many familiar nonparametric statistics are essentially of this form for suitable $\mathbf{X}$'s and suitable choices of the rank-score functions. In particular, this is true of Wilcoxon's rank sum statistic, Spearman's rank correlation and Page's (1963) statistic, though in the last case the group G is not π_N but rather the group of permutations within blocks. For $M_1 \geq 1$ or $M_2 \geq 1$, one statistic of this form is $\rho_1^*(\mathbf{X}_1)^\mathsf{T}\rho_2^*(\mathbf{X}_2)$. As a second example, suppose there are $M_1 \geq 1$ outcome measures in $\mathbf{X}_1$ and there are two treatment groups indicated by the binary coordinates of the $M_2 = 1$ column of $\mathbf{X}_2$; then $\rho_1^*(\mathbf{X}_1)^\mathsf{T}\mathbf{X}_2$ is the sum of M_1 Wilcoxon rank sum statistics and is also of the form covered by Proposition 2.

PROPOSITION 2. *Let $\rho_k\colon D_k \to \mathbb{R}^N$ be a rank-score function for $k = 1, 2$. Then $\rho_1(\mathbf{X}_1)^\mathsf{T}\rho_2(\mathbf{X}_2)$ is an (M_1, M_2)-dr function on $D_1 \times D_2$.*

PROOF. By equivariance, for all $\mathbf{g} \in G$,

$$\rho_1(\mathbf{g}\mathbf{X}_1)^\mathsf{T}\rho_2(\mathbf{g}\mathbf{X}_2) = \rho_1(\mathbf{X}_1)^\mathsf{T}\mathbf{g}^\mathsf{T}\mathbf{g}\rho_2(\mathbf{X}_2) = \rho_1(\mathbf{X}_1)^\mathsf{T}\rho_2(\mathbf{X}_2),$$

since each $\mathbf{g}$ is an orthogonal matrix; so $\rho_1(\mathbf{X}_1)^\mathsf{T}\rho_2(\mathbf{X}_2)$ is G-invariant. Let $\delta \in \Delta$. Assume that $\mathbf{X}_1^\mathsf{T}\delta\delta^\mathsf{T}\mathbf{X}_2 \geq \mathbf{0}$; this implies that $\delta^\mathsf{T}\mathbf{X}_1$ and $\delta^\mathsf{T}\mathbf{X}_2$ have the same constant sign. Now $\rho_1(\mathbf{X}_1)^\mathsf{T}\rho_2(\mathbf{S}_\delta\mathbf{X}_2) = \rho_1(\mathbf{X}_1)^\mathsf{T}\mathbf{S}_\delta\rho_2(\mathbf{X}_2) = \rho_1(\mathbf{X}_1)^\mathsf{T}\rho_2(\mathbf{X}_2) - 2\rho_1(\mathbf{X}_1)^\mathsf{T}\delta\delta^\mathsf{T}\rho_2(\mathbf{X}_2)$, so it suffices to show that $\rho_1(\mathbf{X}_1)^\mathsf{T}\delta\delta^\mathsf{T}\rho_2(\mathbf{X}_2) \geq 0$. Since, for $i = 1, 2$, $\rho_i(\mathbf{X}_i)$ is a rank-score function, it follows that the sign of $\rho_i(\mathbf{X}_i)^\mathsf{T}\delta$ is the same as the common constant sign of both $\delta^\mathsf{T}\mathbf{X}_1$ and $\delta^\mathsf{T}\mathbf{X}_2$, proving the result. □

4. A return to the example. The following presumptions seem to underlie the study by Morton, Saah, Silberg, Owens, Roberts and Saah (1982) in Section 1. Over a period of years, the parent of the ith child brings home a quantity x_i of lead, some of which finds its way into the child's bloodstream. At least in this study, the quantity x_i could not be measured directly. We anticipate that generally higher quantities x_i of lead will be brought home by parents with higher exposures and poorer hygiene, though there may well be individual exceptions to this pattern. We also anticipate that a higher quantity x_i brought home will generally yield a higher level y_i in the child's bloodstream, though again there may be individual exceptions. We might go further and conjecture that parental exposure z_{i1} and hygiene z_{i2} are relevant to the child's lead level only through the quantity x_i of lead brought home. If x_i were observed, we would examine the relationship between x_i and y_i and be done, but this is not possible. Instead, we must examine the relationship between observable quantities, namely y_i and (z_{i1}, z_{i2}). We seek a test whose power

increases with increasing similarity of the ordering of the unobservable x_i and the observable (z_{i1}, z_{i2}). Let us formalize this.

Let $h(\mathbf{Z}, \mathbf{Y})$ be a $(2, 1)$-dr statistic with respect to π_N selected to measure the relationship between the y_i and the (z_{i1}, z_{i2}). Suppose $\mathrm{pr}(y_i | x_i, z_{i1}, z_{i2}) = \mathrm{pr}(y_i | x_i)$, so that, as before, the child's lead level depends on the parent's exposure and hygiene only through the amount of lead brought home. Suppose that $\mathrm{pr}(y_i | x_i)$ is TP_2, so that, as before, y_i and x_i are either positively related or independent, the null hypothesis being independence. Finally, suppose the y_i's are conditionally independent given the (x_i, z_{i1}, z_{i2}). Then $\mathrm{pr}(\mathbf{Y} | \mathbf{X}, \mathbf{Z}) = \mathrm{pr}(\mathbf{Y} | \mathbf{X}) = \prod \mathrm{pr}(y_i | x_i)$ is $(1, 1)$-dr. Write $\check{\mathbf{Y}}$ for the order statistic $(y_{(1)}, \ldots, y_{(N)})^{\mathsf{T}}$, where $y_{(1)} \leq \cdots \leq y_{(N)}$. It is easily checked that the permutation distribution $\mathrm{pr}(\mathbf{Y} | \mathbf{X}, \mathbf{Z}, \check{\mathbf{Y}}) = \mathrm{pr}(\mathbf{Y} | \mathbf{X}, \check{\mathbf{Y}})$ is $(1, 1)$-dr when viewed as a function of $\mathbf{Y}$ and $\mathbf{X}$; note that this distribution is nonzero only on the orbit of $\check{\mathbf{Y}}$, namely $\mathrm{Orb}(\check{\mathbf{Y}}) = \{\mathbf{g}\check{\mathbf{Y}} : \mathbf{g} \in \pi_N\}$. Write $|A|$ for the cardinality of a set A, so $|\mathrm{Orb}(\check{\mathbf{Y}})|$ is the number of distinct rearrangements of $\check{\mathbf{Y}}$, which may be less than $N!$ if there are ties. Under the null hypothesis, $\mathrm{pr}(\mathbf{Y} | \mathbf{X}, \check{\mathbf{Y}})$ is constant on $\mathrm{Orb}(\check{\mathbf{Y}})$, that is, $\mathrm{pr}(\mathbf{Y} = \mathbf{g}\check{\mathbf{Y}} | \mathbf{X}, \check{\mathbf{Y}}) = \mathrm{pr}(\mathbf{Y} = \check{\mathbf{Y}} | \mathbf{X}, \check{\mathbf{Y}}) = 1/|\mathrm{Orb}(\check{\mathbf{Y}})|$ for all $\mathbf{g} \in \pi_N$, so under the null hypothesis, the conditional distribution is a known permutation distribution. Let $k_{\check{\mathbf{Y}}}$ be the smallest value of $h(\mathbf{Z}, \mathbf{Y})$ for $\mathbf{Y} \in \mathrm{Orb}(\check{\mathbf{Y}})$ such that

$$\alpha \geq \frac{\left| \{ \mathbf{Y} \in \mathrm{Orb}(\check{\mathbf{Y}}) : h(\mathbf{Z}, \mathbf{Y}) \geq k_{\check{\mathbf{Y}}} \} \right|}{|\mathrm{Orb}(\check{\mathbf{Y}})|},$$

so a level α conditional test based on $h(\mathbf{Z}, \mathbf{Y})$ rejects when $h(\mathbf{Z}, \mathbf{Y}) \geq k_{\check{\mathbf{Y}}}$. Let the indicator $[h(\mathbf{Z}, \mathbf{Y}) \geq k_{\check{\mathbf{Y}}}]$ equal 1 if $h(\mathbf{Z}, \mathbf{Y}) \geq k_{\check{\mathbf{Y}}}$ and equal 0 otherwise. Since $h(\mathbf{Z}, \mathbf{Y})$ is $(2, 1)$-dr on $\mathrm{Orb}(\check{\mathbf{Y}})$, so is $[h(\mathbf{Z}, \mathbf{Y}) \geq k_{\check{\mathbf{Y}}}]$. Then the conditional power of the test is

$$\beta(\mathbf{Z}, \mathbf{X}; \check{\mathbf{Y}}) = \sum_{\mathbf{Y} \in \mathrm{Orb}(\check{\mathbf{Y}})} [h(\mathbf{Z}, \mathbf{Y}) \geq k_{\check{\mathbf{Y}}}] \mathrm{pr}(\mathbf{Y} | \mathbf{X}, \check{\mathbf{Y}}),$$

which is $(2, 1)$-dr in $\mathbf{Z}$ and $\mathbf{X}$ for each $\check{\mathbf{Y}}$ by the composition theorem. The marginal power $\beta(\mathbf{Z}, \mathbf{X}) = \int \beta(\mathbf{Z}, \mathbf{X}; \check{\mathbf{Y}}) \mathrm{pr}(\check{\mathbf{Y}} | \mathbf{X}) d\check{\mathbf{Y}}$ is also $(2, 1)$-dr. This means that, when the null hypothesis of independence is false, the power of the test increases steadily as the ordering of the quantities x_i of lead brought home is permuted to resemble the ordering suggested by exposure z_{i1} and hygiene z_{i2}. In other words, the test will be particularly sensitive when exposure and hygiene do a good job of sorting parents by their unobserved x_i's. Following D'Abadie and Proschan (1984) in the case of $(1, 1)$-dr functions, a power function that is (M_1, M_2)-dr with respect to G is said to be isotonic in the G-order. Isotonic power is an attractive property in a problem such as this. It is a property of all $(2, 1)$-dr statistics, but not of the Kruskal–Wallis statistics used in the original study.

A simple $(2, 1)$-dr statistic $h(\mathbf{Z}, \mathbf{Y})$ is the sum of rank products $\boldsymbol{\rho}_1^*(\mathbf{Z})^{\mathsf{T}} \boldsymbol{\rho}_2^*(\mathbf{Y}) = 22{,}784$. Write $\boldsymbol{\rho}_Z = \boldsymbol{\rho}_1^*(\mathbf{Z})$ and $\boldsymbol{\rho}_Y = \boldsymbol{\rho}_2^*(\mathbf{Y})$ and let $\bar{\rho}_Z$ and $\bar{\rho}_Y$ be their means, for example, $\bar{\rho}_Z = (1/N) \mathbf{1}^{\mathsf{T}} \boldsymbol{\rho}_Z$. By familiar arguments for linear rank statistics, under the null hypothesis, the conditional moments of the

test statistic given $\mathbf{Z}, \check{\mathbf{Y}}$ are $E(\boldsymbol{\rho}_Z^T \boldsymbol{\rho}_Y) = N\bar{\rho}_Z\bar{\rho}_Y = 20{,}825$ and $\mathrm{var}(\boldsymbol{\rho}_Z^T \boldsymbol{\rho}_Y) = \{1/(N-1)\}\{\sum(\rho_{Zi} - \bar{\rho}_Z)^2\}\{\sum(\rho_{Yj} - \bar{\rho}_Y)^2\} = 377{,}296$, leading to a deviate of $(22{,}784 - 20{,}825)/\sqrt{377{,}296} = 3.2$. The central limit theorem for linear rank statistics leads to an approximate one-sided significance level less than 0.001.

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