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QUANTILE ESTIMATION WITH A COMPLEX SURVEY DESIGN[1]

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Estimation of the finite population distribution function and related statistics, such as the median and interquartile range, is considered. Large-sample properties of estimators constructed from stratified cluster samples, and properties of large-sample confidence intervals, are established. The results are obtained within the context of a sequence of finite populations generated from a superpopulation.

1. Introduction. Researchers are often interested in estimating cumulative distribution functions from analytical survey data. For example, Sedransk and Sedransk (1979) examined the feasibility of using estimated cumulative distribution functions to compare patient care at radiation therapy facilities in a large-scale national survey of cancer patient medical records. Also of interest are functions of the cumulative distribution function, such as quantiles and the interquartile range. For example, median earnings are regularly reported for wage and salary workers by the Bureau of Labor Statistics in the periodical *News*. The medians are computed from a subsample of the Current Population Survey, a stratified multistage sample.

Although large-scale surveys generally use some form of stratified cluster sampling, much of the literature on quantile estimation for finite populations is restricted to simple random sampling or to stratified random sampling. Thompson (1936) and Wilks (1962) have given design-based exact confidence intervals for the sample median under simple random sampling from a finite population. Sedransk and Meyer (1978) and Blesseos (1976) investigated exact confidence interval procedures for quantiles when sampling is from a population divided into a small number of strata. In general, the design-based approach to the construction of confidence intervals with known confidence coefficients is not practical for stratified survey designs having more than two strata. McCarthy (1965) and Smith and Sedransk (1983) give some lower bounds for confidence coefficients applicable to larger stratified samples.

Bickel and Krieger (1989) investigated bootstrap confidence bands for the distribution function of a finite population estimated from a stratified random sample. Under certain conditions, confidence bands for the distribution function can be used to define confidence sets for quantiles.

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Woodruff (1952) proposed using a weighted sample median to estimate the population median, where the weight assigned to each observation is proportional to the inverse of its selection probability. Woodruff also suggested a confidence interval procedure for the median that relied upon the asymptotic normality of the sample cumulative distribution function. Gross (1980), following Maritz and Jarrett (1978), presented an estimator of the variance of the weighted sample median for stratified sampling without replacement, and gave some large sample results for stratified cluster sampling. For stratified random sampling, Binder (1982) derived expressions for confidence coefficients of Bayesian prediction intervals for population quantiles. In large samples, the Bayesian intervals are similar to those proposed by Woodruff (1952).

To introduce the problem of quantile estimation for complex designs, consider simple random sampling from a population with a continuous cumulative distribution function. The sample cumulative distribution function at a point x, denoted by $F_n(x)$, is the fraction of elements in the sample whose values are less than or equal to x. The variance of $F_n(x)$, denoted by $\Omega_n(x)$, is $n^{-1}F(x) \cdot [1 - F(x)]$, where $F(x)$ is the population cumulative distribution function and n is the sample size. Under the assumption that $F(x)$ is continuous, $\Omega_n(x)$ is continuous. The unbiased estimator of the variance is $\hat{\Omega}_n(x) = (n-1)^{-1}F_n(x)[1 - F_n(x)]$. Both $F_n(x)$ and $\hat{\Omega}_n(x)$ are step functions. The jumps in $F_n(x)$ are of height n^{-1}, and the jumps in $(n-1)\hat{\Omega}_n(x)$ are less than n^{-1} in height. The change in $(n-1)\hat{\Omega}_n(x)$ from x_1 and x_2 is bounded by the change in $F_n(x)$ from x_1 to x_2. Also, because $F_n(x)$ converges to $F(x)$ in probability, $(n-1)\hat{\Omega}_n(x)$ converges to $n\Omega_n(x)$ in probability. Thus, under simple random sampling, $F_n(x)$ and $(n-1)\hat{\Omega}_n(x)$ are very well-behaved step functions that converge in probability to the respective continuous functions.

In stratified cluster sampling, stable local behavior of $F_n(x)$ and $(n-1) \cdot \hat{\Omega}_n(x)$ is not guaranteed. For example, with stratified cluster sampling, it is possible for the estimated variance to be zero for x in the support of $F(x)$. It is also possible for jumps in $F_n(x)$ to be much larger than n^{-1} and for jumps in the estimated variance to be large relative to the jumps in the estimated distribution function. Therefore, to obtain limiting results for the estimated quantiles with a complex survey design, restrictions on the sample design are required to place bounds on the local behavior of $F_n(x)$ and $\hat{\Omega}_n(x)$.

In Section 2, the ratio estimator of the finite population distribution function is used to define a quantile estimator for stratified cluster sampling. Conditions on stratified cluster designs sufficient for asymptotic normality of the estimated cumulative distribution function are given in Section 2. The limiting distribution of a vector of quantiles is derived by establishing a representation for the quantiles in terms of the empirical distribution function in Section 3. Theorem 4 of that section provides a large sample justification for the confidence interval for a quantile proposed by Woodruff (1952). A new confidence interval procedure for quantiles based upon monotone bounds for the distribution function is introduced and justified in Theorem 5 of Section 4. Results of a Monte Carlo study are given in Section 5.

2. Sample distribution function. In this section, limiting normality of the sample cumulative distribution function for stratified cluster samples is established. It is assumed that the sequence of finite populations is generated by an infinite population, called the superpopulation. The superpopulation has subject matter relevance because practitioners often make generalizations beyond the finite population.

Let $(\xi_r)_{r=1}^{\infty}$ be a sequence of stratified finite populations, with $L_r \geq L_{r-1}$ strata. Suppose the finite population in stratum h of ξ_r is a random sample of size $N_{rh} \geq N_{r-1,h}$ clusters selected from an infinite superpopulation. Associated with the jth element in the ith cluster of stratum h is a k-dimensional column vector of characteristics. Let $\mathbf{Y}_{rhij}$ denote this vector, where $h = 1, \dots, L_r$, $i = 1, \dots, N_{rh}$, $j = 1, \dots, M_{rhi}$ and M_{rhi} is the number of elements in cluster rhi. Let the vector of cluster totals be

$$\mathbf{Y}_{rhi.} = \sum_{j=1}^{M_{rhi}} \mathbf{Y}_{rhij},$$

where the cluster totals in the superpopulation generating stratum rh have mean vector $\boldsymbol{\mu}_{rh}$ and covariance matrix $\boldsymbol{\Sigma}_{rh}$. If the first element of $\mathbf{Y}_{rhij}$ is always equal to one, then the first element of $\mathbf{Y}_{rhi.}$ is M_{rhi}.

Let a stratified random sample of clusters be selected without replacement from the rth finite population, where n_{rh} clusters are selected in stratum rh, $n_{rh} \geq 2$, for $h = 1, 2, \dots, L_r$ and $n_{rh} \geq n_{r-1,h}$ for $h = 1, 2, \dots, L_{r-1}$. The total number of clusters (primary sampling units) in the overall sample from ξ_r is n_r. For the rth population, let

$$(2.1) \qquad (\bar{\mathbf{y}}'_{rn}, \bar{\mathbf{Y}}'_{rN}) = \sum_{h=1}^{L_r} W_{rh} \left(n_{rh}^{-1} \sum_{i=1}^{n_{rh}} \mathbf{Y}'_{rhi.}, N_{rh}^{-1} \sum_{i=1}^{N_{rh}} \mathbf{Y}'_{rhi.} \right),$$

where $W_{rh} = N_{rh} N_r^{-1}$ and, for notational convenience, we assume that the first n_{rh} items in stratum h are included in the sample. Here $\bar{\mathbf{y}}_{rn}$ is the sample mean per cluster and $\bar{\mathbf{Y}}_{rN}$ is the finite population mean per cluster.

We investigate the asymptotic properties of $(\bar{\mathbf{y}}'_{rn}, \bar{\mathbf{Y}}'_{rN})$ under the following conditions.

CONDITION 1. The cluster totals have absolute $2 + \delta$ moments $(\delta > 0)$ which are uniformly bounded by a finite constant.

CONDITION 2.

(a)
$$\sup_{1 \leq h \leq L_r} n_r W_{rh}^2 n_{rh}^{-2} \to 0 \quad \text{as } r \to \infty.$$

(b)
$$\sum_{h=1}^{L_r} n_r W_{rh}^2 n_{rh}^{-1} = O(1).$$

CONDITION 3.

$$\lim_{r\to\infty} n_r \mathbf{V}\{[(\bar{\mathbf{y}}_{rn} - \boldsymbol{\mu}_r)', (\bar{\mathbf{y}}_{rn} - \bar{\mathbf{Y}}_{rN})']'\} = \boldsymbol{\Gamma},$$

where $\boldsymbol{\Gamma}$ is positive definite and $\boldsymbol{\mu}_r^* = E(\bar{\mathbf{y}}_{rn})$ is the superpopulation mean.

The conditions are sufficient restrictions on the superpopulation distributions and the sampling design to admit the Lindeberg condition. Condition 2 determines how the sample clusters in the sequence of samples are allocated to each stratum relative to the total number of clusters in the finite population. Condition 2 also assures that the total number of selected clusters increases without bound as $r \to \infty$ and that no single observation is important in the sum. Finally, in Condition 3 it is assumed that the limit of the covariance matrix for $[(\bar{\mathbf{y}}_{rn} - \boldsymbol{\mu}_r)', (\bar{\mathbf{y}}_{rn} - \bar{\mathbf{Y}}_{rN})']'$, when multiplied by the normalizing factor n_r, exists with a determinant bounded away from zero. The assumption of a limit is not required for all results, but it facilitates a number of proofs, with no loss of subject matter generality.

The multivariate central limit theorem of Theorem 1 is appropriate for generalizations both to the finite population and to the infinite population. A proof is available in Francisco (1987). Theorem 1 can be extended to multistage designs by placing suitable restrictions on the subsampling.

THEOREM 1. *Let the sequence of finite populations and samples be as described. Under regularity Conditions* 1–3,

$$\hat{\mathbf{V}}_r^{-1/2}[(\bar{\mathbf{y}}_{rn} - \boldsymbol{\mu}_r)', (\bar{\mathbf{y}}_{rn} - \bar{\mathbf{Y}}_{rN})']' \to_L N(\mathbf{0}, \mathbf{I})$$

as $r \to \infty$, *where* $f_{rh} = n_{rh} N_{rh}^{-1}$,

$$\hat{\mathbf{V}}_r = \sum_{h=1}^{L_r} W_{rh}^2 n_{rh}^{-1} \begin{pmatrix} \hat{\boldsymbol{\Sigma}}_{rh} & (1 - f_{rh})\hat{\boldsymbol{\Sigma}}_{rh} \\ (1 - f_{rh})\hat{\boldsymbol{\Sigma}}_{rh} & (1 - f_{rh})\hat{\boldsymbol{\Sigma}}_{rh} \end{pmatrix},$$

$$\hat{\boldsymbol{\Sigma}}_{rh} = (n_{rh} - 1)^{-1} \sum_{i=1}^{n_{rh}} (\mathbf{Y}_{rhi.} - \bar{\mathbf{y}}_{rh..})(\mathbf{Y}_{rhi.} - \bar{\mathbf{y}}_{rh..})',$$

$$\bar{\mathbf{y}}_{rh..} = n_{rh}^{-1} \sum_{i=1}^{n_{rh}} \mathbf{Y}_{rhi.}.$$

Related asymptotic results have been given by Fuller (1975), Krewski and Rao (1981) and Bickel and Freedman (1984). Fuller (1975) gave a multivariate central limit theorem using a sequence of finite populations similar to the type employed in Theorem 1. Krewski and Rao (1981) considered multi-stage survey designs in which clusters are selected with replacement within strata and the number of strata increases without bound. Using a Lindeberg-type condition, Bickel and Freedman (1984) established the asymptotic normality of $\bar{y}_{rn}$ for a sequence of stratified finite populations with very few restrictions on stratum sizes and stratum sampling rates. Our Conditions 1–3 are sufficient for Condition 3 of Bickel and Freedman (1984).

Theorem 1 is now used to obtain the limiting distribution for the estimated distribution function. Define the finite population distribution function for Y by

$$F_{rN}(x) = M_r^{-1} \sum_{h=1}^{L_r} \sum_{i=1}^{N_{rh}} \sum_{j=1}^{M_{rhi}} I\{Y_{rhij} \le x\}, \tag{2.2}$$

where M_r is the total number of elements in the rth finite population and

$$I\{Y_{rhij} \le x\} = \begin{cases} 1, & \text{if } Y_{rhij} \le x, \\ 0, & \text{otherwise}. \end{cases}$$

The estimator of the cumulative distribution function is

$$F_{rn}(x) = \hat{M}_r^{-1} N_r \sum_{h=1}^{L_r} W_{rh} n_{rh}^{-1} \sum_{i=1}^{n_{rh}} \sum_{j=1}^{M_{rhi}} I\{y_{rhij} \le x\}, \tag{2.3}$$

where $\hat{M}_r$ is the unbiased estimator of M_r.

For convenience in presenting the asymptotic results for $F_{rn}(x)$, we assume that a common overall superpopulation distribution function, denoted by $F(x)$, holds for all ξ_r. That is, we assume

$$E\left\{ \sum_{h=1}^{L_r} \sum_{i=1}^{N_{rh}} \sum_{j=1}^{M_{rhi}} \left[I\{Y_{rhij} \le x\} - F(x) \right] \right\} = 0, \tag{2.4}$$

for all r and all x in the support of $F(x)$, where the expectation is with respect to the superpopulation model. The estimator of the distribution function given in (2.3) is a ratio of quantities of the form shown in (2.1). Hence it is a continuous differentiable function of sample means, and the limiting distribution of the estimator follows from Theorem 1.

THEOREM 2. *Let the sequence of populations and samples be as described. Let the vector* $(1, I\{Y_{rhij} \le x\})'$ *satisfy Condition* 1 *and let Conditions* 2 *and* 3 *hold. Let* $F(x)$ *satisfy* (2.4). *Then, for fixed* x^0 *in the interior of the support of* $F(x)$,

$$[\hat{s}_r(x^0)]^{-1}[F_{rn}(x^0) - F(x^0)] \to_L N(0,1),$$

$$[\hat{V}\{F_{rn}(x^0) - F_{rN}(x^0)\}]^{-1/2}[F_{rn}(x^0) - F_{rN}(x^0)] \to_L N(0,1)$$

as $r \to \infty$, *where* $\hat{s}_r^2(x^0) = \hat{V}\{F_{rn}(x^0)\}$,

$$\hat{V}\{F_{rn}(x)\} = \sum_{h=1}^{L_r} (n_{rh} - 1)^{-1} n_{rh} \sum_{i=1}^{n_{rh}} (d_{rhi.} - \bar{d}_{rh..})^2,$$

$$\hat{V}\{F_{rn}(x) - F_{rN}(x)\} = \sum_{h=1}^{L_r} (1 - f_{rh})(n_{rh} - 1)^{-1} n_{rh} \sum_{i=1}^{n_{rh}} (d_{rhi.} - \bar{d}_{rh..})^2,$$

$$d_{rhi.} = \hat{M}_r^{-1} N_{rh} n_{rh}^{-1} \sum_{j=1}^{M_{rhi}} \left[I\{Y_{rhij} \le x\} - F_{rn}(x) \right]$$

and $\bar{d}_{rh..}$ *is defined by analogy to* $\bar{y}_{rh..}$.

The estimator $\hat{V}\{F_{rn}(x)\}$ of Theorem 2 is a variance estimator for a combined ratio estimator of the mean per element. It is a Taylor series estimator of the variance of the approximate distribution of $[F_{rn}(x) - F(x)]$.

The test inversion confidence set for a superpopulation quantile is defined in Corollary 1. For a given sample, the set need not be a closed interval and can be the entire real line. An additional condition is required before we give the corollary.

CONDITION 4. The cumulative distribution function $F(x)$ is continuous and has a continuous, positive derivative in a neighborhood of x^0.

COROLLARY 1. *Let the assumptions of Theorem 2 hold for x^0, where $F(x^0) = \gamma^0$. In addition, assume Condition 4 and let*

$$\Gamma_{r0} = \{x\colon F_{rn}(x) + t_\alpha \hat{s}_r(x) > \gamma^0 \text{ and } F_{rn}(x) - t_\alpha \hat{s}_r(x) < \gamma^0\},$$

where t_α is defined by $\Phi(t_\alpha) = 1 - \alpha 2^{-1}$ and $\Phi(\cdot)$ is the distribution function of a standard normal random variable. Then, as $r \to \infty$, the probability that x^0 is in Γ_{r0} converges to $1 - \alpha$.

3. Quantiles. While Theorem 2 and its corollary provide a method for constructing a confidence set for a given superpopulation quantile, additional conditions are needed to justify the confidence procedure proposed by Woodruff (1952). In this section we give results under which Woodruff's procedure attains the stated confidence level. The results can also be used to construct confidence intervals for functions of quantiles such as the interquartile range.

Let $q(\gamma) = F^{-1}(\gamma)$ be the quantile function. The γth quantile of Y for finite population ξ_r is

$$q_{rN}(\gamma) = \inf\{x\colon F_{rN}(x) \geq \gamma\}, \tag{3.1}$$

for $0 < \gamma < 1$. An estimator of $q_{rN}(\gamma)$ is the γth sample quantile

$$\hat{q}_{rn}(\gamma) = \inf\{x\colon F_{rn}(x) \geq \gamma\}. \tag{3.2}$$

Let $\mathbf{x}^0$ be a vector of k fixed, distinct quantiles for $0 < \gamma_i^0 < 1$, $i = 1, \dots, k$, and $\gamma_i^0 \neq \gamma_j^0$, for $i \neq j$. The corresponding set of sample quantiles for the rth sample in the sequence is denoted by

$$\hat{\mathbf{x}}_r = [\hat{q}_{rn}(\gamma_1^0), \hat{q}_{rn}(\gamma_2^0), \dots, \hat{q}_{rn}(\gamma_k^0)]' = (\hat{x}_{r1}, \dots, \hat{x}_{rk})'.$$

Let $\hat{\mathbf{\Omega}}_r(\mathbf{x})$ be the estimator of $\mathbf{\Omega}_r(\mathbf{x})$ and let $\tilde{\mathbf{\Omega}}_{rN}(\mathbf{x})$ be the estimator of $\mathbf{\Omega}_{rN}(\mathbf{x})$, where

$$\hat{\mathbf{\Omega}}_r(\mathbf{x}) = \hat{\mathbf{V}}\{[F_{rn}(x_1) - F(x_1), \dots, F_{rn}(x_k) - F(x_k)]'\}$$

and

$$\mathbf{\Omega}_{rN}(\mathbf{x}) = \mathbf{V}\{[F_{rn}(x_1) - F_{rN}(x_1), \dots, F_{rn}(x_k) - F_{rN}(x_k)]'\}.$$

Let $A_1, A_2, \dots, A_k$ be intervals of finite positive length in the interior of the support of $F(x)$ that contain $x_1^0, x_2^0, \dots, x_k^0$, respectively, as interior points. Let $A = A_1 \times A_2 \times \cdots \times A_k$, and let B be the union of $A_1, A_2, \dots, A_k$. We assume that Condition 4 holds on A and introduce three additional regularity conditions.

CONDITION 5. For x in B, $n_r V\{F_{rn}(x)\}$ and $n_r V\{F_{rn}(x) - F_{rN}(x)\}$ are positive and continuous in x.

CONDITION 6. For some $0 < C < \infty$, $V\{F_{rn}(x + \delta) - F_{rn}(x)\} \le C n_r^{-1} |\delta|$, for all r and for all x and $x + \delta$ in B.

CONDITION 7. The covariance matrices $\mathbf{\Omega}_r(\mathbf{x}^0)$ and $\mathbf{\Omega}_{rN}(\mathbf{x}^0)$ are positive definite. Furthermore, for every $\varepsilon > 0$, there exists an M_ε depending only on ε, such that

$$P\left\{\sup_{\mathbf{x}\in A} n_r \left\| \hat{\mathbf{\Omega}}_r(\mathbf{x}) - \mathbf{\Omega}_r(\mathbf{x}) \right\| \ge M_\varepsilon n_r^{-1/2}\right\} \le \varepsilon$$

and

$$P\left\{\sup_{\mathbf{x}\in A} n_r \left\| \tilde{\mathbf{\Omega}}_{rN}(\mathbf{x}) - \mathbf{\Omega}_{rN}(\mathbf{x}) \right\| \ge M_\varepsilon n_r^{-1/2}\right\} \le \varepsilon,$$

where $\|\mathbf{C}\|$ denotes the largest absolute value of the elements of the matrix $\mathbf{C}$.

Conditions such as 5 and 7 are required for the Woodruff procedure because the variance of the cumulative distribution function at the true population quantile is approximated with the estimated variance at the estimated quantile. Condition 7 guarantees that the estimated variance is converging to the true variance in probability at a sufficiently fast rate. Taken together, Conditions 5 and 7 guarantee that the variance estimators at the true quantile and at the estimated quantile become close to each other and close to the true variance at the true quantile as the sample size increases. These conditions will hold for most sampling schemes.

Condition 6 is an assumption about the correlation between estimators of the distribution function at different points. It is used in the development of an approximation to the difference $F_{rn}(x + \delta) - F_{rn}(x)$, where δ is a function of the standard error of $F_{rn}(x)$. Even though the error in $F_{rn}(x)$ is $O_p(n_r^{-1/2})$, this is not sufficient for the required local properties of the estimated distribution function. We note that all assumptions are satisfied by simple random sampling.

It is now demonstrated that the sample quantile $\hat{q}_{rn}(\gamma_1^0)$ can be expressed asymptotically as a linear function of the empirical distribution function evaluated at $q(\gamma_1^0)$. The theorem is a weak version of the result known as the Bahadur representation of the sample quantile [Bahadur (1966)]. The method of proof parallels that used by Ghosh (1971) for simple random sampling.

THEOREM 3. *Let Conditions* 1 *through* 7 *hold for x in the interval A_1 containing $q(\gamma_1^0)$ as an interior point. Then the sample quantile,*

$$(3.3)\quad \hat{q}_{rn}(\gamma) = q(\gamma) - [f(q(\gamma))]^{-1}[F_{rn}(q(\gamma)) - F(q(\gamma))] + R_{rn}^*(\gamma),$$

with $R_{rn}^*(\gamma) = o_p(n_r^{-1/2})$ uniformly in γ for γ in H_1, where $H_1 = \{\gamma\colon F(x) = \gamma$ and $x \in A_1\}$.

PROOF. Let $T_{rn}(\gamma) = n_r^{1/2}(\hat{q}_{rn}(\gamma) - q(\gamma))$. Then $T_{rn}(\gamma) \le c$ is equivalent to $Z_{rn}(\gamma) \le c_{rn}(\gamma)$, where

$$Z_{rn}(\gamma) = n_r^{1/2}\{F(q(\gamma) + cn_r^{-1/2}) - F_{rn}(q(\gamma) + cn_r^{-1/2})\}[f(q(\gamma))]^{-1},$$

(3.4)

$$c_{rn}(\gamma) = n_r^{1/2}\{F(q(\gamma) + cn_r^{-1/2}) - F_{rn}(\hat{q}_{rn}(\gamma))\}[f(q(\gamma))]^{-1}$$

and c is a fixed constant. By Conditions 4 and 7,

$$F(q(\gamma) + cn_r^{-1/2}) = F(q(\gamma)) + cn_r^{-1/2}f(q(\gamma)) + o_p(n_r^{-1/2}),$$

where the equality holds for all γ such that γ and $\gamma + cn_r^{-1/2}$ are in H_1. Using Condition 2(a), Condition 4 and (3.2), it can be shown that

$$F(q(\gamma)) - F_{rn}(\hat{q}_{rn}(\gamma)) = o(n_r^{-1/2}),$$

and $c_{rn}(\gamma) \to c$ as $r \to \infty$, for all γ in H_1. Now,

$$Z_{rn}(\gamma) - G_{rn}(\gamma) = n_r^{1/2}\big[F(q(\gamma) + cn_r^{-1/2}) - F(q(\gamma)) - \{F_{rn}(q(\gamma) + cn_r^{-1/2}) - F_{rn}(q(\gamma))\}\big][f(q(\gamma))]^{-1},$$

where $G_{rn}(\gamma) = n_r^{1/2}[F(q(\gamma)) - F_{rn}(q(\gamma))][f(q(\gamma))]^{-1}$. By Condition 6, we have

$$F_{rn}(x + cn_r^{-1/2}) - F_{rn}(x) = F(x + cn_r^{-1/2}) - F(x) + O_p(n_r^{-3/4}),$$

uniformly for all x and $x + cn_r^{-1/2}$ in A_1. Thus

$$Z_{rn}(\gamma) - G_{rn}(\gamma) = O_p(n_r^{-1/4}) \quad \text{and} \quad Z_{rn}(\gamma) - G_{rn}(\gamma) \to_P 0,$$

uniformly for γ in H_1. Hence, because $c_{rn}(\gamma) \to c$ as $r \to \infty$, we have for every $\varepsilon > 0$,

$$\lim_{r\to\infty} P\{T_{rn}(\gamma) \le c, G_{rn}(\gamma) \ge c + \varepsilon\} = \lim_{r\to\infty} P\{Z_{rn}(\gamma) \le c_{rn}(\gamma), G_{rn}(\gamma) \ge c + \varepsilon\} = 0,$$

uniformly for γ in H_1. This establishes the first condition of Lemma 1 of Ghosh (1971). The second condition of that lemma is obtained using similar arguments. Therefore,

$$T_{rn}(\gamma) - G_{rn}(\gamma) = n_r^{1/2}R_{rn}^*(\gamma) \to_P 0$$

as $r \to \infty$, uniformly in γ for γ in H_1. □

The asymptotic representation of $\hat{q}_{rn}(\gamma)$ given in Theorem 3 is used in the following theorem to prove the asymptotic multivariate normality of $\hat{\mathbf{x}}_r$.

THEOREM 4. *Let $\hat{\mathbf{x}}_r = (\hat{x}_{r1}, \hat{x}_{r2}, \ldots, \hat{x}_{rk})'$ be the vector of estimated quantiles associated with $(\gamma_1^0, \gamma_2^0, \ldots, \gamma_k^0)'$. Let the vector of true quantiles $\mathbf{x}^0$ be in the interior of the support of $F(x)$. Let Conditions 1 through 7 hold for $\mathbf{x}$ in A. Let $\hat{\mathbf{D}}_r$ be the diagonal matrix with $\hat{d}_{ri}$ on the diagonal, where, for $1 \le i \le k$,*

$$\hat{d}_{ri} = [2t_\alpha \hat{s}_{ri}]^{-1}\big[\hat{q}_{rn}(\gamma_i^0 + t_\alpha \hat{s}_{ri}) - \hat{q}_{rn}(\gamma_i^0 - t_\alpha \hat{s}_{ri})\big],$$

$\hat{s}_{ri} = \hat{s}_r(\hat{x}_{ri})$ is the square root of the ith diagonal element of $\hat{\mathbf{\Omega}}_r(\mathbf{x})$ and t_α is

defined in Corollary 1. *It is understood that $\hat{q}_{rn}(\tilde{\gamma})$ is the smallest observed x if $\tilde{\gamma} \le 0$ and that $\hat{q}_{rn}(\overset{*}{\gamma})$ is the largest observed x if $\overset{*}{\gamma} \ge 1$. Then*

$$(3.5) \qquad \left[\hat{\mathbf{D}}_r \hat{\boldsymbol{\Omega}}_r(\hat{\mathbf{x}}_r)\hat{\mathbf{D}}_r\right]^{-1/2}(\hat{\mathbf{x}}_r - \mathbf{x}^0) \to_L N(\mathbf{0}, \mathbf{I})$$

as $r \to \infty$, where $\hat{\boldsymbol{\Omega}}_r(\hat{\mathbf{x}}_r)$ is the estimated covariance matrix of the vector obtained by evaluating the sample distribution function at the k elements of $\hat{\mathbf{x}}_r$.

Also

$$(3.6) \qquad \left[\tilde{\mathbf{D}}_r \tilde{\boldsymbol{\Omega}}_{rN}(\hat{\mathbf{x}}_r)\tilde{\mathbf{D}}_r\right]^{-1/2}(\hat{\mathbf{x}}_r - \mathbf{x}_{rN}) \to_L N(\mathbf{0}, \mathbf{I}),$$

where

$$\mathbf{x}_{rN} = \left[q_{rN}(\gamma_1^0), q_{rN}(\gamma_2^0), \dots, q_{rN}(\gamma_k^0)\right]',$$

$$\tilde{\mathbf{D}}_r = \operatorname{diag}(\tilde{d}_{r1}, \dots, \tilde{d}_{rk}),$$

$$\tilde{d}_{ri} = [2t_\alpha \tilde{s}_{ri}]^{-1}\left[\hat{q}_{rn}(\gamma_i^0 + t_\alpha \tilde{s}_{ri}) - \hat{q}_{rn}(\gamma_i^0 - t_\alpha \tilde{s}_{ri})\right]$$

and $\tilde{s}_{ri} = \tilde{s}_r(\hat{x}_{ri})$ is the square root of the ith diagonal element of $\tilde{\boldsymbol{\Omega}}_{rN}(\mathbf{x})$.

PROOF. We prove (3.5). A proof of (3.6) is given by Francisco (1987). Representation (3.3) of Theorem 3 for a sample quantile yields

$$(3.7) \qquad \begin{aligned} &\hat{q}_{rn}(\gamma_i^0 + t_\alpha \hat{s}_{ri}) - \hat{q}_{rn}(\gamma_i^0 - t_\alpha \hat{s}_{ri}) \\ &\quad = q(\gamma_i^0 + t_\alpha \hat{s}_{ri}) - q(\gamma_i^0 - t_\alpha \hat{s}_{ri}) \\ &\qquad - \left[f(x_i^0 + \varepsilon_{1ri})\right]^{-1}\left[F_{rn}(x_i^0 + \varepsilon_{1ri}) - F(x_i^0 + \varepsilon_{1ri})\right] \\ &\qquad + \left[f(x_i^0 - \varepsilon_{2ri})\right]^{-1}\left[F_{rn}(x_i^0 - \varepsilon_{2ri}) - F(x_i^0 - \varepsilon_{2ri})\right] + o_p(n_r^{-1/2}), \end{aligned}$$

where $\varepsilon_{1ri} = q(\gamma_i^0 + t_\alpha \hat{s}_{ri}) - q(\gamma_i^0)$ and $\varepsilon_{2ri} = q(\gamma_i^0) - q(\gamma_i^0 - t_\alpha \hat{s}_{ri})$. By Conditions 3, 4, 5 and 7, $\varepsilon_{jri} = O_p(n_r^{-1/2})$ for $j = 1, 2$, where the bounds are uniform for x in B. Thus, from Condition 4 it follows that the third and fourth terms on the right of (3.7) can be written as

$$\begin{aligned} -\left[f(x_i^0)\right]^{-1}&\left[F_{rn}(x_i^0 + \varepsilon_{1ri}) - F_{rn}(x_i^0 - \varepsilon_{2ri})\right. \\ &\left. -\left\{F(x_i^0 + \varepsilon_{1ri}) - F(x_i^0 - \varepsilon_{2ri})\right\}\right] + o_p(n_r^{-1/2}). \end{aligned}$$

Using Theorem 5.4.4 of Fuller (1976), it can be shown that the expectation of the sample distribution function differs from the true distribution function by a quantity that is $O(n_r^{-1})$. By the expectation result, Condition 4 and Condition 6, we have

$$F_{rn}(x_i^0 + \varepsilon_{1ri}) - F_{rn}(x_i^0 - \epsilon_{2ri}) = f(x_i^0)(\varepsilon_{1ri} + \varepsilon_{2ri}) + o_p(n_r^{-1/2}).$$

Substituting this expression into (3.7) and using a Taylor expansion of $F(x)$, it follows that

$$(3.8) \qquad \begin{aligned} &\hat{q}_{rn}(\gamma_i^0 + t_\alpha \hat{s}_{ri}) - \hat{q}_{rn}(\gamma_i^0 - t_\alpha \hat{s}_{ri}) \\ &\qquad = 2t_\alpha \hat{s}_{ri}\left[f(x_i^0)\right]^{-1} + o_p(n_r^{-1/2}). \end{aligned}$$

From Conditions 5 and 7, we have $\hat{s}_{ri}^{-1} = O_p(n_r^{1/2})$. Hence, from (3.8), $\hat{\mathbf{D}}_r \to_P \mathbf{D}$ as $r \to \infty$, where the iith element of the diagonal matrix $\mathbf{D}$ is $[f(x_i^0)]^{-1}$. By Condition 7, $n_r(\hat{s}_{ri}^2 - s_{ri}^2) = O_p(n_r^{-1/2})$, where $s_{ri}^2 = V\{F_{rn}(\hat{x}_{ri})\}$, and by Theorem 3, $\hat{x}_i - x_i^0 = O_p(n_r^{-1/2})$. Therefore, $n_r \hat{\mathbf{D}}_r \hat{\mathbf{\Omega}}_r(\hat{\mathbf{x}}_r)\hat{\mathbf{D}}_r$ converges in probability to $n_r \mathbf{D}\mathbf{\Omega}_r(\mathbf{x}^0)\mathbf{D}$. The results follow, because under the conditions,

$$\left[\mathbf{D}\mathbf{\Omega}_r(\mathbf{x}^0)\mathbf{D}\right]^{-1/2}(\hat{\mathbf{x}}_r - \mathbf{x}^0) \to_L N(\mathbf{0}, \mathbf{I}). \qquad \square$$

REMARK 1. The proof of Theorem 4 furnishes a justification for the confidence interval procedure of Woodruff (1952). From (3.5), as $r \to \infty$,

$$P\{\hat{q}_{rn}(\gamma_i^0) - t_\alpha \hat{s}_{ri}\hat{d}_{ri} \le x_i^0 \le \hat{q}_{rn}(\gamma_i^0) + t_\alpha \hat{s}_{ri}\hat{d}_{ri}\} \to 1 - \alpha.$$

By the expansions used in the proof,

$$\hat{q}_{rn}(\gamma_i^0) \pm t_\alpha \hat{s}_{ri}\hat{d}_{ri} = \hat{q}_{rn}(\gamma_i^0 \pm t_\alpha \hat{s}_{ri}) + o_p(n_r^{-1/2}).$$

Therefore, $P\{x_i^0 \in \mathbf{I}_W(x_i^0)\} \to 1 - \alpha$, where

$$\mathbf{I}_W(x_i^0) = \left[\hat{q}_{rn}(\gamma_i^0 - t_\alpha \hat{s}_{ri}), \hat{q}_{rn}(\gamma_i^0 + t_\alpha \hat{s}_{ri})\right]. \tag{3.9}$$

Similarly from (3.6), as $r \to \infty$, $P\{x_{rNi} \in \mathbf{I}_{WN}(x_{rNi})\} \to 1 - \alpha$, where

$$\mathbf{I}_{WN}(x_{rNi}) = \left[\hat{q}_{rn}(\gamma_i^0 - t_\alpha \tilde{s}_{ri}), \hat{q}_{rn}(\gamma_i^0 + t_\alpha \tilde{s}_{ri})\right]. \tag{3.10}$$

The interval $\mathbf{I}_{WN}(x_{rNi})$ is the approximate $(1 - \alpha)$ confidence interval for $x_{rNi} = q_{rN}(\gamma_i^0)$, proposed by Woodruff.

REMARK 2. The asymptotic theory of Theorem 4 provides a procedure for estimating the covariance matrix of a set of quantiles. One divides the Woodruff $(1 - \alpha)$ confidence interval for each quantile by $2t_\alpha$ to obtain an estimate of the standard error of the quantile. Rao and Wu (1987) studied the standard error estimated in this manner. Their Monte Carlo results and Monte Carlo results of Section 5 suggest that the 95% interval works well as a basis for the standard error. The estimated covariance matrix is completed by noting that the estimated correlation between two estimated quantiles is equal to the estimated correlation between the two corresponding estimated distribution function values. Using the estimated large sample covariance matrix of $[\hat{q}_{rn}(0.25), \hat{q}_{rn}(0.75)]'$, one can estimate the large sample variance and construct approximate confidence intervals for the interquartile range.

4. Quantile confidence intervals based on test inversion. Fewer assumptions are required for the construction of the test inversion confidence set of Corollary 1 than for construction of the Woodruff (1952) confidence interval. However, the test inversion confidence set can contain disjoint subsets. Given a smooth distribution function and a smooth variance function, it is natural to construct pointwise confidence bounds for the cumulative distribution function that are monotone nondecreasing. The monotone confidence bounds for the distribution function can then be used to construct a large sample confidence interval for a quantile.

Three conditions beyond the conditions employed to justify the Woodruff procedure are used to extend the confidence set procedure of Corollary 1. Condition 10 of Theorem 5 is a smoothness condition on the variance of the sample distribution function and Condition 8 insures that the estimated variance function is well behaved. Under simple random sampling, the estimated variance of $F_{rn}(x)$ is a step function with steps whose absolute value is less than n^{-2}. As noted in the introduction, a change in the estimated variance is bounded by the change in the estimated distribution function. Condition 8 imposes a similar restriction on the estimated variance for the general case. Also, Condition 1 is replaced by Condition 9, a stronger condition on the stratum sampling rates. The stronger condition places tighter bounds on the height of possible jumps in $F_{rn}(x)$. Under these assumptions, the Woodruff interval and the inversion interval for the finite population quantile are asymptotically equivalent. Before giving the theorem, we will state Conditions 8, 9 and 10.

CONDITION 8. The maximum absolute value of a change in the estimated variance of $F_{rn}(x)$ and in the estimated variance of $F_{rn}(x) - F_{rN}(x)$ from a point x_1 to a point x_2 is bounded by a multiple of $n_r^{-1}|\gamma_2 - \gamma_1|$ for all r and all x_1 and x_2 that are elements of one of the sets, A_i, where $F_{rn}(x_1) = \gamma_1$ and $F_{rn}(x_2) = \gamma_2$.

CONDITION 9. $0 < c_L < n_r W_{rh} n_{rh}^{-1} < c_U < \infty$, where c_L and c_U are fixed numbers.

CONDITION 10. For x in B, $n_r V\{F_{rn}(x)\}$ and $n_r V\{F_{rn}(x) - F_{rN}(x)\}$ have derivatives that are continuous in x and bounded for all r.

THEOREM 5. *Let*

$$F_{rnU}(x_{r(i)}) = F_{rn}(x_{r(i)}) + t_\alpha\left[\hat{V}\{F_{rn}(x_{r(i)})\}\right]^{1/2}, \qquad \text{for } i = 1,$$
$$= \max\left\{F_{rnU}(x_{r(i-1)}),\ F_{rn}(x_{r(i)}) + t_\alpha\left[\hat{V}\{F_{rn}(x_{r(i)})\}\right]^{1/2}\right\}, \qquad \text{for } i > 1,$$
$$F_{rnL}(x_{r(i)}) = F_{rn}(x_{r(i)}) - t_\alpha\left[\hat{V}\{F_{rn}(x_{r(i)})\}\right]^{1/2}, \qquad \text{for } i = m_r,$$
$$= \min\left\{F_{rnL}(x_{r(i+1)}),\ F_{rn}(x_{r(i)}) - t_\alpha\left[\hat{V}\{F_{rn}(x_{r(i)})\}\right]^{1/2}\right\}, \qquad \text{for } i < m_r,$$

where $x_{r(1)}, x_{r(2)}, \ldots, x_{r(m)}$ are the ordered observed values and $m_r = m$ is the total number of elements in the sample. It is understood that $F_{rnU}(x) = 1$, if the right side of the definition equals or exceeds one and that $F_{rnL}(x) = 0$, if the right side of the definition is less than or equal to zero. Let the inverses of $F_{rnU}(x)$ and $F_{rnL}(x)$ be defined as in (3.2). *If no value of $F_{rnU}(x)$ is less than γ, then $F_{rnU}^{-1}(\gamma) = -\infty$. If no value of $F_{rnL}(x)$ is greater than γ, then $F_{rnL}^{-1}(\gamma) = \infty$.*

Assume Condition 1 *and Conditions* 3 *through* 10. *Let* $x_1^0 = q(\gamma_1^0) = F^{-1}(\gamma_1^0)$ *be the* γ_1^0th *superpopulation quantile, where* $0 < \gamma_1^0 < 1$. *Let* $x_{rN1} = q_{rN}(\gamma_1^0)$ *be the finite population quantile. Let*

$$\mathbf{I}_{WN}(x_{rN1}) = \{\hat{q}_{rn}[\gamma_1^0 - t_\alpha \tilde{s}_r(\hat{x}_{r1})], \hat{q}[\gamma_1^0 + t_\alpha \tilde{s}_r(\hat{x}_{r1})]\},$$

where $\hat{x}_{r1} = \hat{q}_{rn}(\gamma_1^0)$, *be the vector defining the Woodruff confidence interval for* $q_{rN}(\gamma_1^0)$. *Let* $\mathbf{I}_{TN}(x_{rN1})$ *be the vector defining the test inversion confidence interval for* $q_{rN}(\gamma_1^0)$. $\mathbf{I}_{TN}(x_{rN1})$ *is defined by functions of the type* $F_{rnU}(x_{r(i)})$ *and* $F_{rnL}(x_{r(i)})$ *with* $\hat{V}\{F_{rn}(x) - F_{rN}(x)\}$ *replacing* $\hat{V}\{F_{rn}(x)\}$. *Then, as* $r \to \infty$,

$$P\{F_{rnU}^{-1}(\gamma_1^0) \le x_1^0 \le F_{rnL}^{-1}(\gamma_1^0)\} \to 1 - \alpha, \tag{4.1}$$

$$P\{q_{rN}(\gamma_1^0) \in \mathbf{I}_{TN}(x_{rN1})\} \to 1 - \alpha \tag{4.2}$$

and

$$n_r^{1/2}[\mathbf{I}_{WN}(x_{rN1}) - \mathbf{I}_{TN}(x_{rN1})] \to_P 0. \tag{4.3}$$

PROOF. We prove (4.1). Statement (4.3) implies (4.2) and a proof of (4.3) is given in Francisco (1987). By Conditions 4, 5 and 10 there exists an interval A_1 and an r_0 such that x_1^0 is an interior point of A_1 and $F(x) + t_\alpha[V\{F_{rn}(x)\}]^{1/2}$ is continuous with a continuous positive derivative on the interval A_1 for $r > r_0$. Hence, if $r > r_0$, $x_{r(i+1)} \in A_1$ and $x_{r(i)} \in A_1$, then

$$F(x_{r(i+1)}) + t_\alpha\left[V\{F_{rn}(x_{r(i+1)})\}\right]^{1/2} - F(x_{r(i)}) - t_\alpha\left[V\{F_{rn}(x_{r(i)})\}\right]^{1/2} > 0,$$

with probability 1. By Condition 9, $W_{rh}n_{rh}^{-1} = O(n_r^{-1})$. Therefore, the difference between $F_{rn}(x_{r(i+1)})$ and $F_{rn}(x_{r(i)})$ is $O(n_r^{-1})$.

By Condition 8, for $\hat{q}_{rn}(\gamma_1)$ and $\hat{q}_{rn}(\gamma_2)$ in A_1,

$$\left|\left[\hat{V}\{F_{rn}[\hat{q}_{rn}(\gamma_1)]\}\right]^{1/2} - \left[\hat{V}\{F_{rn}[\hat{q}_{rn}(\gamma_2)]\}\right]^{1/2}\right| < Mn_r^{-1/2}|\gamma_2 - \gamma_1|,$$

where $M < \infty$. Therefore, for $\gamma_2 > \gamma_1$, $\hat{q}_{rn}(\gamma_1) \in A_1$ and $\hat{q}_{rn}(\gamma_2) \in A_1$, the inequality

$$\begin{aligned} &F_{rn}[\hat{q}_{rn}(\gamma_2)] + t_\alpha\left[\hat{V}\{F_{rn}[\hat{q}_{rn}(\gamma_2)]\}\right]^{1/2} \\ &\qquad < F_{rn}[\hat{q}_{rn}(\gamma_1)] + t_\alpha\left[\hat{V}\{F_{rn}[\hat{q}_{rn}(\gamma_1)]\}\right]^{1/2} \end{aligned}$$

is possible only if

$$\gamma_2 - \gamma_1 < t_\alpha M n_r^{-1/2}|\gamma_2 - \gamma_1| \le t_\alpha^2 M^2 n_r^{-1}.$$

Let $x_{r(b)}$ be the largest order statistic less than or equal to x_1^0. Then $F_{rn}(x_{r(b)}) = F_{rn}(x_1^0)$. By Condition 9, a finite number of observations are required to move $F_{rn}(x)$ through an interval of length $t_\alpha^2 M^2 n_r^{-1}$. Therefore, only a finite number of the values of

$$\hat{A}_{rn}(b - j) = F_{rn}(x_{r(b-j)}) + t_\alpha\left[\hat{V}\{F_{rn}(x_{r(b-j)})\}\right]^{1/2},$$

say $j = 0, 1, \dots, q$, are possible candidate values for $F_{rnU}(x_{r(b)})$. Let $\varepsilon_1 > 0$ be given. By Conditions 5, 7 and 9, there exist $M_\varepsilon < \infty$ and r_1 such that

$$P\left\{\left|\left(\hat{V}\left[F_{rn}(x_{r(b)})\right]\right)^{1/2} - \left(V\left[F_{rn}(x_{r(b)})\right]\right)^{1/2}\right| > M_\varepsilon n_r^{-1}\right\} \leq \varepsilon_1,$$

for $r > r_1$. Using Conditions 4 and 6, it can be shown that

$$F_{rn}(x_{r(b)}) - F_{rn}(x_{r(b-j)}) = F(x_{r(b)}) - F(x_{r(b-j)}) + O_p(n_r^{-1}),$$

for $j = 1, 2, \ldots, q$. It follows that

$$\begin{aligned}\hat{G}_{rj} - G_{rj} &= \hat{A}_{rn}(b - q + j) - \hat{A}_{rn}(b - q) - \left[A_{rn}(b - q + j) - A_{rn}(b - q)\right] \\ &= O_p(n_r^{-1}),\end{aligned}$$

where

$$\begin{aligned}\hat{G}_{rj} &= \hat{A}_{rn}(b - q + j) - \hat{A}_{rn}(b - q), \\ G_{rj} &= A_{rn}(b - q + j) - A_{rn}(b - q)\end{aligned}$$

and

$$A_{rn}(b - j) = F(x_{r(b-j)}) + t_\alpha\left[V\{F_{rn}(x_{r(b-j)})\}\right]^{1/2}.$$

Now $G_{rq} \geq \cdots \geq G_{r1}$ and the $\hat{G}_{ri}$ are estimators of G_{ri} with an error that is $O_p(n_r^{-1})$. It follows that the maximum of the $\hat{G}_{ri}$, denoted by $\tilde{G}_{rq}$, is an estimator of G_{rq} with an error that is $O_p(n_r^{-1})$. Therefore,

$$F_{rnU}(x_{r(b)}) - F_{rn}(x_{r(b)}) = \hat{A}_{rn}(b - q) + \tilde{G}_{rq} - F_{rn}(x_{r(b)})$$

is an estimator of $t_\alpha[V\{F_{rn}(x_{(b)})\}]^{1/2}$ and of $t_\alpha[V\{F_{rn}(x_1^0)\}]^{1/2}$, with an error that is $O_p(n_r^{-1})$. An analogous argument holds for the lower bound. Hence, for a fixed x_1^0,

$$P\{F_{rnL}(x_1^0) \leq F(x_1^0) \leq F_{rnU}(x_1^0)\} \to 1 - \alpha$$

and $P\{F_{rnU}^{-1}(\gamma_1^0) \leq x_1^0 \leq F_{rnL}^{-1}(\gamma_1^0)\} \to 1 - \alpha$ *as* $r \to \infty$. □

5. Monte Carlo study. A Monte Carlo simulation was conducted to evaluate the performance of the statistics investigated in previous sections. A set of 1,000 finite populations of size 500 was generated. Each finite population had ten strata with stratum sizes of 40, 40, 50, 50, 60, 60, 70, 50, 50 and 30. The observations in the strata were generated as simple random samples from 10 lognormal distributions. The superpopulation means and standard deviations for the 10 strata were (4.69, 1.44), (8.00, 3.33), (8.85, 3.68), (24.05, 15.83), (13.80, 7.36), (6.55, 2.73), (5.18, 1.59), (6.55, 2.73), (24.05, 15.83) and (61.56, 58.29), respectively.

One stratified random sample of size 100 was selected from each finite population. The sample was composed of 10 elements selected from each of the 10 strata. The simple random samples in each stratum satisfy the conditions of Theorem 5 because the stratified superpopulation has a mixed lognormal distribution with a density. The stratified sample also satisfies the conditions because the range of values is $(0, \infty)$ for all strata, and no stratum sample dominates the sum.

The 25th, 50th and 75th quantiles, the interquartile range and variance estimates for each estimator were computed for each sample. Two different procedures for calculating 95% confidence intervals for quantiles were used. The first method for computing confidence intervals was the large-sample confidence interval of Woodruff (1952) described in (3.9). The second procedure was a smoothed version of the large-sample test inversion procedure of Theorem 5. In this procedure, the upper and lower bounds were monotone nondecreasing continuous functions composed of linear segments. The procedure has been implemented in a program for the personal computer. See Fuller, Kennedy, Schnell, Sullivan and Park (1986). The normal approximation and the variance estimator constructed using the procedures described in Remark 2 in conjunction with the test inversion confidence interval were used to determine the 95% confidence interval for the interquartile range.

Monte Carlo averages of the finite population parameters agreed well with the corresponding superpopulation values. Averages of the finite population quantiles for γ-values of 0.25, 0.50 and 0.75 were within one Monte Carlo standard error of the respective superpopulation values. The same was true for the interquartile range. The approximate variance of the sample quantiles, as obtained from the representation of Theorem 3, also agreed well with the Monte Carlo variance of the sample quantiles.

Coverage probabilities for the test inversion and the Woodruff confidence interval procedures were similar for the three superpopulation quantiles, and the obtained confidence coefficients were near the nominal level of 95%. See Table 1. Coverage probabilities of Woodruff confidence intervals for the three finite population quantiles averaged between 90% and 93%, while the test

TABLE 1
Coverage probabilities of 95% confidence intervals from stratified random samples of size 100

	Estimated coverage probability		**Average length**	
Parameter	**Test inversion procedure**	**Woodruff procedure**	**Test inversion procedure**	**Woodruff procedure**
	Superpopulation			
$q(0.25)$	0.963	0.952	1.26	1.32
$q(0.50)$	0.966	0.956	2.17	2.19
$q(0.75)$	0.953	0.949	6.22	6.41
Interquartile range	0.950		6.46	
	Finite Population			
$q_N(0.25)$	0.955	0.931	1.14	1.20
$q_N(0.50)$	0.964	0.915	1.95	1.96
$q_N(0.75)$	0.958	0.900	5.56	5.70
Interquartile range	0.943		5.77	

inversion procedure produced somewhat shorter intervals with coverage probabilities closer to the 95% nominal level.

We conjecture that the test inversion procedure performed better because it uses information about the variance at points close to the estimated quantile. The Woodruff procedure uses only the estimated variance at the estimated quantile. The confidence interval for the interquartile range had end points equal to the estimate plus or minus two standard errors, where the standard error is the square root of the estimated variance. For both the superpopulation and finite populations, the coverage probabilities were quite close to the nominal value.

Kovar, Rao and Wu (1988) report a Monte Carlo study of confidence interval procedures for the median. The Woodruff method performed well in a population sample configuration that displayed smoothness characteristics consistent with Conditions 1–7. In a heavily stratified situation that violated smoothness Conditions 5–7, the Woodruff method performed poorly. Our limited investigation suggests that the test inversion procedure outperforms the Woodruff procedure for such nonregular cases.

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