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MINIMUM DISTANCE ESTIMATION IN AN ADDITIVE EFFECTS OUTLIERS MODEL

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In the additive effects outliers (A.O.) model considered here one observes $Y_{j,n} = X_j + v_{j,n}$, $0 \le j \le n$, where $\{X_j\}$ is the first order autoregressive [AR(1)] process with the autoregressive parameter $|\rho| < 1$. The A.O.'s $\{v_{j,n},\ 0 \le j \le n\}$ are i.i.d. with distribution function (d.f.) $(1 - \gamma_n)I[x \ge 0] + \gamma_n L_n(x)$, $x \in \mathbb{R}$, $0 \le \gamma_n \le 1$, where the d.f.'s $\{L_n, n \ge 0\}$ are not necessarily known. This paper discusses the existence, the asymptotic normality and biases of the class of minimum distance estimators of ρ, defined by Koul, under the A.O. model. Their influence functions are computed and are shown to be directly proportional to the asymptotic biases. Thus, this class of estimators of ρ is shown to be robust against A.O. model.

1. Introduction. Let F and L_n, $n \ge 0$, be symmetric d.f.'s on the real line $\mathbb{R}$, symmetric about 0. Throughout this paper, F is assumed to have a density f. Let $\{\gamma_n, n \ge 0\}$ be a sequence of numbers in $[0, 1]$ converging to 0 as $n \to \infty$. Define

$$\beta_n(x) := (1 - \gamma_n)I[x \ge 0] + \gamma_n L_n(x), \qquad x \in \mathbb{R}, \tag{1.1}$$

where $I[A]$ denotes the indicator function of the event A. Let $\{\varepsilon_j,\ j = 0, \pm 1, \pm 2, \dots\}$ and $\{v_{j,n},\ 0 \le j \le n\}$ be independent and identically distributed (i.i.d.) F and β_n random variables (r.v.'s), respectively.

We consider the model in which one observes, at stage n, r.v.'s $Y_{j,n}$, $0 \le j \le n$, satisfying

$$Y_{j,n} = X_j + v_{j,n}, \qquad j = 0, 1, \dots, n, \tag{1.2}$$

with

$$X_j = \rho X_{j-1} + \varepsilon_j, \qquad |\rho| < 1, \qquad j = 0, \pm 1, \pm 2, \dots, \tag{1.3}$$

where $\{X_j\}$ is stationary and $E\varepsilon_0^2 < \infty$. Moreover, $\{X_j,\ j \le n\}$ is assumed to be independent of $\{v_{j,n},\ 0 \le j \le n\}$, $n \ge 0$. This paper studies the problem of estimating ρ.

Denby and Martin (1979) called the model in (1.2) and (1.3) the additive effects outliers (A.O.) model. The assumptions on $\{v_{j,n},\ 0 \le j \le n\}$ reflect the situation in which the outliers are isolated in nature. Isolated outliers are defined by Martin and Yohai (1986) as the outliers, any pair of which are separated in time by a nonoutlier. Martin and Yohai [(1986), page 796, Theorem 5.2 and Comment 5.1] also made the assumption of independence of the process $\{X_j,\ j \le n\}$ and $\{v_{j,n},\ 0 \le j \le n\}$, $n \ge 0$.

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In practice, an appropriate model for time series data with outliers may be difficult to specify. Fox (1972) and Martin and Zeh (1977) point out the importance of finding the difference between various types of outliers in order to effectively deal with them. The two types of outliers in time series analysis that have received considerable attention are A.O. and innovations outliers (I.O.). In the I.O. model one observes $\{X_j,\ 0 \le j \le n\}$ of (1.3) and large data points are consistent with the future and perhaps the past values. On the other hand, in the A.O. model outliers are generally not consistent with the past or future values of the unobservable process $\{X_j\}$. The additive effects outliers may occur due to measurement errors like keypunch errors [Denby and Martin (1979)] or roundoff errors in which case L_n is taken to be uniform d.f. on the interval $[-0.5, 0.5]$ [Machak and Rose (1984)].

Denby and Martin (1979) studied the least squares estimator, M-estimators and a class of generalized M-estimators (GM-estimators) of ρ under the above models; they took F and L_n to be $\mathscr{N}(0, \sigma_\varepsilon^2)$ and $\mathscr{N}(0, \sigma^2)$, respectively. Under their A.O. model, all of these estimators have nonvanishing asymptotic biases with a possible reduction in biases for GM-estimators.

This paper discusses the existence and the asymptotic behavior of the class of minimum distance (m.d.) estimators of Koul (1986) under the A.O. model (1.2) and (1.3). To define this class of estimators, let h be a Borel measurable function from $\mathbb{R}$ to $\mathbb{R}$ and H be a nondecreasing function on $\mathbb{R}$. Let, for x, t in $\mathbb{R}$,

$$(1.4)\qquad \begin{aligned} S_h(x,t) = n^{-1/2} \sum_{j=1}^{n} h(Y_{j-1,n})\{I[Y_{j,n} \le x + tY_{j-1,n}] \\ -I[-Y_{j,n} < x - tY_{j-1,n}]\}, \end{aligned}$$

$$(1.5)\qquad M(t) = \int S_h^2(x,t)\, dH(x).$$

Denote $\hat{\rho}_h(H)$ to be a measurable minimizer of M, if it exists. Then $\hat{\rho}_h(H)$ satisfies

$$(1.6)\qquad \inf_t M(t) = M[\hat{\rho}_h(H)].$$

A motivation for considering these estimators is as follows. In the one sample, the multiple linear regression and I.O. models, analogous Cramér–von Mises type m.d. estimators are shown to be asymptotically normal, locally asymptotically minimax and qualitatively robust against certain departures from model assumptions [see Millar (1981, 1984) and Koul (1985, 1986)]. One reason for these properties to hold is the smoothness of these m.d. estimators as functionals of d.f.'s.

This paper proves the existence of $\hat{\rho}_h(H)$. A set of sufficient conditions is given that ensures the asymptotic normality of $\sqrt{n}\,[\hat{\rho}_h(H) - \rho]$ and the boundedness of its asymptotic bias under the A.O. model. Consequently, for a large class of functions h and H, $\hat{\rho}_h(H)$ is shown to be qualitatively robust against a sequence of A.O.'s. Again, this is true mainly because these estimators are smooth functionals in a sense made clear in Remark 4.3. Finally, just as in Koul (1986), the estimators with $h(x) \equiv x$ are observed to be asymptotically efficient among a certain class of estimators $\hat{\rho}_h(H)$ under the A.O. model.

All the assumptions on $\{Y_j, 0 \leq j \leq n\}$, $\{X_j\}$, $\{v_{j,n}, 0 \leq j \leq n\}$ and $\{\varepsilon_j\}$, given in (1.1)–(1.3) will be referred to as the model assumptions. From these assumptions we see that the process $\{X_j\}$ is ergodic and X_{j-1} is independent of ε_j, $j \geq 1$. Further, for each n, the process $\{(X_j, v_{j,n}), 0 \leq j \leq n\}$ is stationary ergodic and hence so is $\{Y_{j,n}, 0 \leq j \leq n\}$. These observations will be used in the sequel repeatedly.

Throughout this paper, $o_p(1)$ $[O_p(1)]$ denotes a sequence of r.v.'s that converges to zero in probability (is tight or bounded in probability) and $|\ |_H$ denotes the $L^2(H)$-norm. Also, Z_n is taken to be a r.v. with d.f. L_n, $n \geq 0$. Note that the asymptotic bias of $\hat{\rho}_h(H)$ is defined as the mean of the asymptotic distribution of $\sqrt{n}\,[\hat{\rho}_h(H) - \rho]$.

2. Assumptions and existence. This section states the assumptions that will be used subsequently. Some sufficient conditions that imply these assumptions are discussed. It also contains a proof of the existence of $\hat{\rho}_h(H)$. We begin by stating the assumptions.

A1. $n\gamma_n^2 = O(1)$, where $\gamma_n \in [0, 1]$.

A2. H is a nondecreasing continuous function such that

$$|H(x) - H(y)| = |H(-x) - H(-y)| \qquad \forall\, x, y \in \mathbb{R}.$$

Note that H generates a unique Lebesgue–Stieltjes measure. Hence H will also be used to represent a measure in the sequel.

A3. (a) $0 < EX_0^2 < \infty$.
(b) For some $\delta > 0$, $0 < E|h(X_0)|^{2+\delta} < \infty$.

A4. $EX_0^2 h^2(X_0) < \infty$.

A5. For some $\delta > 0$,

$$\sup_n E\int |h(X_0 + z)|^{2+\delta}\, dL_n(z) < \infty.$$

A6. $xh(x) \geq 0\ \forall\, x$ or $xh(x) \leq 0\ \forall\, x$.

A7. $0 < \int f^k\, dH < \infty$, where $k = 1, 2$.

A8.

(a) $$\sup_n \int Ef(x - Z_n)\, dH(x) < \infty.$$

(b) $$\limsup_n \int Ef^2(x - Z_n)\, dH(x) < \infty.$$

A9. There exists $C_0 > 0$ such that

$$\int |f(x-u) - f(x)|\, dx \le C_0 |u|, \qquad \forall\, u \in \mathbb{R}.$$

A10.

(a) $$\lim_{s\to 0} \int E|X_0| h^2(X_0)[f(x+sX_0) - f(x)]\, dH(x) = 0,$$

(b) $$\lim_{s\to 0} \int E X_0^2 h^2(X_0)[f(x+sX_0) - f(x)]^2\, dH(x) = 0.$$

In all the assumptions to follow, $0 \le C_n \in \mathbb{R}$ are such that $C_n \to 0$.

A11.

(a) $$\limsup_n \frac{1}{2C_n} \int_{-C_n}^{C_n} \int E\left[\int |X_0| h^2(X_0) f(x + sX_0 - z)\, dL_n(z)\right] dH(x)\, ds < \infty.$$

(b) $$\limsup_n \frac{1}{2C_n} \int_{-C_n}^{C_n} \int E\left[X_0^2 h^2(X_0) \int f^2(x + sX_0 - z)\, dL_n(z)\right] dH(x)\, ds < \infty.$$

A12.

$$\sup_n \int E\left[\int |X_0 + z|\, |h(X_0 + z)| Ef(x + \rho z - jZ_n)\, dL_n(z)\right] dH(x) < \infty,$$

holds with (a) $j = 0$ and (b) $j = 1$.

A13.

$$\limsup_n \int E\left[\int (X_0 + z)^2 h^2(X_0 + z) Ef^2(x + \rho z - jZ_n)\, dL_n(z)\right] dH(x) < \infty,$$

holds with (a) $j = 0$ and (b) $j = 1$.

A14.

$$\limsup_n \frac{1}{2C_n} \int_{-C_n}^{C_n} \int E\Bigg[\int |X_0 + z| h^2(X_0 + z) \times \left\{\int f(x + \rho z + s[X_0 + z] - ju)\, dL_n(u)\right\} dL_n(z)\Bigg] dH(x)\, ds < \infty,$$

holds with (a) $j = 0$ and (b) $j = 1$.

A15.

$$\limsup_n \frac{1}{2C_n} \int_{-C_n}^{C_n} \int E\Bigg[\int (X_0 + z)^2 h^2(X_0 + z) \times \left[\int f^2(x + \rho z + s[X_0 + z] - ju)\, dL_n(u)\right] dL_n(z)\Bigg] dH(x)\, ds < \infty,$$

holds with (a) $j = 0$ and (b) $j = 1$.

A16.

$$\limsup_n \int Eh^2(Y_{0,n}) G_n(x+\rho v_{0,n})[1 - G_n(x+\rho v_{0,n})]\, dH(x) < \infty,$$

where

$$G_n(x) = (1-\gamma_n)F(x) + \gamma_n EF(x - Z_n), \qquad x \in \mathbb{R}.$$

A17.

$$\limsup_n \int E\left[\int h^2(X_0 + z)[E\{F(x+\rho z - jZ_n) - F(x - \rho z - jZ_n)\}]^2 dL_n(z)\right] dH(x) < \infty,$$

holds with (a) $j = 0$ and (b) $j = 1$.

NOTE. Most of the above assumptions are finite moment assumptions. In Remarks 2.1–2.4, various sets of sufficient conditions that imply the above assumptions are given. Remark 2.4 gives an example in which all of the above assumptions are satisfied.

REMARK 2.1. Consider the following assumptions:

S1: The function defined by $q_1(u) = \int [f(x+u) - f(x)]^2\, dH(x)$, $u \in \mathbb{R}$, is bounded and continuous at 0.

S2: The function defined by $q_2(u) = \int f(x+u)\, dH(x)$, $u \in \mathbb{R}$, is bounded and continuous at 0.

S3: $\sup_n E\int(|X_0| + |z|)^2 h^2(X_0 + z)\, dL_n(z) < \infty$.

S4: $\limsup_n EZ_n^2 < \infty$.

Under S1, S2, S3 and S4, the assumptions A1–A17 reduce to A1–A7, A9 and A16. This follows from the inequality

$$\int f^2(x+u)\, dH(x) \le 2\int [f(x+u) - f(x)]^2\, dH(x) + 2\int f^2\, dH, \tag{2.1}$$

the moment inequality, the Fubini theorem and the dominated convergence theorem (D.C.T.). For example, consider A17(b). The expression inside the lim sup of A17(b) is bounded above by

$$\int E\int h^2(X_0 + z)\left[E\left|\int_{-|z|}^{|z|} f(x + u - Z_n)\, du\right|\right]^2 dL_n(z)\, dH(x),$$

which in turn is

$$\le 2\int E\int h^2(X_0 + z)|z| E\int_{-|z|}^{|z|} f^2(x + u - Z_n)\, du\, dL_n(z)\, dH(x), \tag{2.2}$$

using the moment inequality. From S1, A7 and (2.1), $\int f^2(x + u - Z_n)\, dH(x)$ is bounded in $u - Z_n$. That the lim sup of the r.h.s. of (2.2) is finite now follows from the Fubini theorem and S3.

REMARK 2.2. In the case when $H(x) \equiv x$, all the assumptions A1–A17 reduce to A1, S3, S4, A3–A6, A7 with $k = 2$ and A9. The reduction of A1–A17 given in Remark 2.1 and the translation invariance of the Lebesgue measure is used throughout in its proof.

Note that A2 and S2 are trivially satisfied. In view of A7, to prove S1 we need only to prove that as a function of u, $\int f(x) f(x+u)\,dx$ is bounded and continuous at 0. Boundedness follows easily by the Hölder inequality. From A7, $f \in L^2(H)$. Hence from Rudin [(1974), Theorem 3.14], we have for any $\eta > 0$, there exists a continuous function ϕ_η vanishing outside a compact set such that

$$|\phi_\eta - f|_H < \eta. \tag{2.3}$$

Moreover, from the Hölder inequality it follows that

$$\begin{aligned}&\left|\int f(x) f(x+s)\,dx - \int f(x) f(x+t)\,dx\right| \\ &\quad\le 2|f|_H|\phi_\eta - f|_H + |f|_H|\phi_\eta(\cdot + s) - \phi_\eta(\cdot + t)|_H.\end{aligned} \tag{2.4}$$

The continuity of $\int f(x) f(x+u)\,dx$ as a function of u now follows from (2.3), (2.4), A7 and the uniform continuity of the function ϕ_η.

We shall now show that A16 holds. First note that for an H as in A2, A*16 implies A16 because L_n and F are symmetric about 0.

A*16.

(a) $$\int F(x)[1 - F(x)]\,dH(x) < \infty,$$

(b) $$\limsup_n \int EF(x - jZ_n)\{1 - EF(x - Z_n)\}\,dH(x) < \infty, \qquad j = 0, 1,$$

(c) $$\begin{aligned}&\limsup_n \int E\bigg[\int h^2(X_0 \pm z) F(x + \rho z) \\ &\qquad \times [1 - EF(x + \rho z - Z_n)]\,dL_n(z)\bigg]\,dH(x) < \infty,\end{aligned}$$

(d) $$\begin{aligned}&\limsup_n \int E\bigg[\int h^2(X_0 + z) EF(x + \rho z - jZ_n) \\ &\qquad \times [1 - EF(x + \rho z - jZ_n)]\,dL_n(z)\bigg]\,dH(x) < \infty, \qquad j = 0, 1.\end{aligned}$$

Since F is continuous and $E|\varepsilon_1| < \infty$, we have

$$\int_0^\infty [1 - F(x)]\,dx < \infty \quad \text{and} \quad \int_{-\infty}^0 F(x)\,dx < \infty. \tag{2.5}$$

Thus A*16(a) holds a priori when $H(x) = x$. Using the Fubini theorem and A5, we see that A*16(d) with $j = 0$ follows from A*16(a) and also to prove

A*16(b) with $j = 1$ is the same as to prove A*16(d) with $j = 1$. A*16(b) with $j = 1$ can be rewritten as

$$\begin{aligned}(2.6)\qquad &\int\int\int F(x)[1 - F(x - z + u)]\,dx\,dL_n(z)\,dL_n(u)\\ &= \int\int\int_0^{\infty} F(x)[1 - F(x - z + u)]\,dx\,dL_n(z)\,dL_n(u)\\ &\quad + \int\int\int_{-\infty}^{0} F(x)[1 - F(x - z + u)]\,dx\,dL_n(z)\,dL_n(u).\end{aligned}$$

The second term on the r.h.s. of (2.6) is bounded by the second term in (2.5). Rewrite the first term on the r.h.s. of (2.6) as

$$\begin{aligned}(2.7)\qquad &\int\int\int_0^{\infty} F(x)[1 - F(x - z + u)]I[u \geq z]\,dx\,dL_n(z)\,dL_n(u)\\ &+ \int\int\int_0^{\infty} F(x)[1 - F(x - z + u)]I[u < z]\,dx\,dL_n(z)\,dL_n(u).\end{aligned}$$

The first term in (2.7) is bounded by the first term in (2.5). Rewrite the second term in (2.7), using change of variable and splitting the range of integration, as

$$\begin{aligned}(2.8)\qquad &\int\int\int_{u-z}^{0} F(x + z - u)[1 - F(x)]I[u < z]\,dx\,dL_n(z)\,dL_n(u)\\ &+ \int\int\int_0^{\infty} F(x + z - u)[1 - F(x)]I[u < z]\,dx\,dL_n(z)\,dL_n(u).\end{aligned}$$

The first term in (2.8) is bounded by $2E|Z_n|$. Hence S4 implies that the lim sup of the first term in (2.8) is finite. The second term in (2.8) is bounded by the first term in (2.5); consequently A*16(b) and A*16(d) with $j = 1$ hold. By the Fubini theorem, the symmetry of L_n's about 0, A*16(c), can be written as

$$\limsup_n E\left\{\int h^2(X_0 + z)\,dL_n(z)\right\}\int F(x)[1 - EF(x - Z_n)]\,dH(x),$$

which is the same as proving A*16(b) with $j = 0$, in view of A5. Proceeding exactly as in (2.6)–(2.8) and using (2.5), S4 and A5, we get that A*16(c) holds.

REMARK 2.3. In the case H generates a finite measure, all the assumptions A1–A17 reduce to A1–A7, A9, A10(b), A11(b) and A15, A8(b) and A13 with lim sups replaced by sups. Proof of this remark follows from the Hölder and the moment inequalities. Moreover, in the case of finite H and bounded continuous f, A1–A17 reduce to A1–A6, A9 and

$$\limsup_n E\int\{(X_0 + z)h(X_0 + z)\}^2\,dL_n(z) < \infty.$$

REMARK 2.4. For H given by $dH = dF/(F(1 - F))$, where $f(x) = 2^{-1}\exp\{-|x|\}$, $x \in \mathbb{R}$, we shall show, via Remark 2.1, that the assumptions

A1–A17 reduce to A1, A3(b), S3, A4–A6 and

S5: $\limsup_n E \exp|Z_n| < \infty$,

S6: $\limsup_n E\int h^2(X_0 + z)\exp\{|\rho z|\}\, dL_n(z) < \infty$.

To see this, note that $F(x)[1 - F(x)] = 4^{-1}\exp\{-|x|\}[2 - \exp\{-|x|\}]$. Hence, for x, u in $\mathbb{R}$,

$$\frac{\exp\{-k|x| - j|x + u|\}}{F(x)[1 - F(x)]} \le 4\exp\{-(k-1)|x|\}[2 - \exp\{-|x|\}]^{-1}, \tag{2.9}$$

where $(k, j) = (2, 0), (3, 0), (2, 1)$. Since the r.h.s. of (2.9) is Lebesgue integrable over $\mathbb{R}$, applying the D.C.T. to the functions in (2.9), we get that S1 and A7 hold. Since the range of the function $2 - \exp\{-|x|\}$ is $[1, 2)$, from the extended D.C.T. we now see that the Lebesgue integral of the l.h.s. of (2.9) with $k = 1$, $j = 1$, is continuous at 0. These arguments show that S3 holds. From the inequality

$$|e^{-|x-u|} - e^{-|x|}| \le |u|[\exp\{-|x - u|\} + \exp\{-|x|\}], \qquad x, u \in \mathbb{R},$$

and the translation invariance of the Lebesgue measure, we see $\int |f(x-u) - f(x)|\, dx \le |u|$, $u \in \mathbb{R}$, Since A*16 implies A16, we shall verify that A*16 holds. A2 and A*16(a) are trivially satisfied from definition of H. A3(a) follows trivially from the a.s. representation of X_0 as $\sum_{j=0}^{\infty}\rho^j\varepsilon_{-j}$. S5 implies that S4 holds. Using A5, S5 and S6, lengthy but simple calculations show that A*16(b)–(d) hold.

If in addition to the restrictions of this remark we assume $h(x) \equiv x$, Z_n to be $\mathscr{N}(0, \sigma^2)$ and $\gamma_n = n^{-1/2}$ for all n. Then all the assumptions A1–A17 are satisfied. Finally, the above discussion goes through when f is taken to be the $\mathscr{N}(0, \tau^2)$, $\tau > 0$. We shall now discuss the existence of $\hat{\rho}_h(H)$.

LEMMA 2.1. *Assume that* A2 *and* A6 *hold. Then either* (i) $H(\mathbb{R}) = \infty$ *or* (ii) $H(\mathbb{R}) < \infty$ *and* $h(0) = 0$, *implies the existence of* $\hat{\rho}_h(H)$.

PROOF. The proof will be given only for the case $xh(x) \ge 0\ \forall\ x \in \mathbb{R}$; the proof in the case $xh(x) \le 0\ \forall\ x \in \mathbb{R}$ is exactly the same, with h replaced by $-h$. Define

$$c(x) := n^{-1/2}h(0)\sum_{j=1}^{n} I[Y_{j-1,n} = 0]\{I[Y_{j,n} \le x] - I[-Y_{j,n} < x]\}, \qquad x \in \mathbb{R},$$

$$\begin{aligned} d &:= n^{-1/2}\sum_{j=1}^{n} |h(Y_{j-1,n})| I[Y_{j-1,n} \ne 0], \\ b &:= \max_{1 \le j \le n} |Y_{j,n}|. \end{aligned} \tag{2.10}$$

Observe that $c(x) = 0$ for $|x| > b$ and hence c is H-integrable. Now rewrite

$$\begin{aligned} S_h(x, t) = n^{-1/2}\sum_{j=1}^{n} & h(Y_{j-1,n}) I[Y_{j-1,n} \ne 0] \\ & \times\{I[Y_{j,n} \le tY_{j-1,n} + x] \\ & -I[-Y_{j,n} < -tY_{j-1,n} + x]\} + c(x), \qquad x, t \text{ in } \mathbb{R}. \end{aligned} \tag{2.11}$$

The first term on the r.h.s. of (2.11) is bounded by d. Hence

$$c(x) - d \le S_h(x,t) \le d + c(x), \qquad t, x \in \mathbb{R}. \tag{2.12}$$

Further, $xh(x) \ge 0$ implies

$$S_h(x,t) \to c(x) \pm d \quad \text{as } t \to \pm\infty. \tag{2.13}$$

Moreover, under A2, H is continuous and by calculations similar to those in Koul [(1986), equations (2.1)–(2.3)], for all $t \in \mathbb{R}$,

$$\begin{aligned} M(t) = n^{-1} \sum_i \sum_j & h(Y_{i-1,n}) h(Y_{j-1,n}) \\ & \times \big[\big| H(Y_{i,n} - tY_{i-1,n}) - H(-Y_{j,n} + tY_{j-1,n}) \big| \\ & \quad - \big| H(Y_{i,n} - tY_{i-1,n}) - H(Y_{j,n} - tY_{j-1,n}) \big| \big]. \end{aligned} \tag{2.14}$$

Hence M is continuous on $\mathbb{R}$. Now consider

CASE 1. $H(\mathbb{R}) = \infty$. If $d = 0$, then from (2.12), $M \equiv \int c^2\, dH$. Hence a trivial measurable minimizer exists. Now let $d > 0$. Since c and c^2 are H-integrable, $\int (c(x) \pm d)^2\, dH(x) = \infty$. Therefore from (2.13) and the Fatou lemma, $M(t) \to \infty$ as $t \to \pm\infty$. This and the continuity of M ensure the existence of a measurable minimizer of M.

CASE 2. $H(\mathbb{R}) < \infty$. From (2.10), $h(0) = 0$ implies that $c \equiv 0$. Thus (2.12), (2.13) and the D.C.T. give $M(t) \to d^2 H(\mathbb{R})$ as $t \to \pm\infty$. This with (2.12) and the continuity of M ensure the existence of a measurable minimizer of M. Note that in both cases measurability of the minimizers is established from Brown and Purves [(1973), Corollary 2.1]. □

Lemma 2.1 continues to hold true even if, from A2, the condition $|H(x) - H(y)| = |H(-x) - H(-y)|$ for all $x, y \in \mathbb{R}$ is removed. Also, its proof does not require most of the model assumptions, e.g., $|\rho| < 1$.

3. Asymptotic approximation of $\hat{\rho}_h(H)$. For stating the main results of this section we need some more notation. From the model assumptions and A16, G_n is the d.f. of $v_{1,n} + \varepsilon_1$. A density of G_n is

$$g_n(x) = (1 - \gamma_n) f(x) + \gamma_n E f(x - Z_n), \qquad x \in \mathbb{R}. \tag{3.1}$$

Define

$$Q(t) = \int \big[S_h(x,\rho) + n^{1/2}(t-\rho)\{a_n(x) + \underline{a}_n(x)\} \big]^2 dH(x), \qquad t \in \mathbb{R}, \tag{3.2}$$

where $a_n(x) = EY_0 h(Y_0) g_n(x + \rho v_0)$ and $\underline{a}_n(x) = a_n(-x)$, $x \in \mathbb{R}$.

We shall first uniformly approximate M by Q, uniformity taken over shrinking neighborhoods of ρ. Using this approximation we obtain the asymptotic approximation of $\hat{\rho}_h(H)$ in terms of the minimizer of Q.

From here on we shall suppress n in the r.v.'s $v_{j,n}$, a_n and $Y_{j,n}$, etc., for the sake of convenience.

THEOREM 3.1. *Let all the model assumptions* (1.1)–(1.3) *hold. Further, let* A1–A4, A7, A8 *and* A10–A17 *hold. Then for any* $0 < b < \infty$,

$$E\left[\sup_{n^{1/2}|t-\rho|\le b} |M(t) - Q(t)|\right] = o(1). \tag{3.3}$$

PROOF. The techniques used in here are as in Koul (1986). Define, $\forall\, x, t \in \mathbb{R}$,

$$\begin{aligned} W(x,t) &= \left\{ n^{-1/2} \sum_{j=1}^{n} h(Y_{j-1}) I\left[v_j - \rho v_{j-1} + \varepsilon_j \le x + n^{-1/2} t Y_{j-1}\right] \right\} - U(x,t) \\ &\text{with } U(x,t) = n^{-1/2} \sum_{j=1}^{n} h(Y_{j-1}) G_n\left(x + n^{-1/2} t Y_{j-1} + \rho v_{j-1}\right). \end{aligned} \tag{3.4}$$

Note that the jth summand in $W(x,t)$ is conditionally centered, given (v_{j-1}, Y_{j-1}). From (1.4) and (3.4), for all $x, t \in \mathbb{R}$,

$$\begin{aligned} S_h(x, n^{-1/2}t + \rho) = {} & W(x,t) + W(-x,t) + U(x,t) \\ & + U(-x,t) - n^{-1/2} \sum_{j=1}^{n} h(Y_{j-1}). \end{aligned} \tag{3.5}$$

From (1.5) and (3.5),

$$\begin{aligned} & M(n^{-1/2}t + \rho) \\ &\quad = \int \big[W(x,t) - W(x,0) + W(-x,t) - W(-x,0) + S_h(x,\rho) \\ &\qquad + t[a(x) + a(-x)] + U(-x,t) - U(-x,0) - ta(-x) \\ &\qquad + U(x,t) - U(x,0) - ta(x) \big]^2 \, dH(x). \end{aligned}$$

From the above representation of $M(n^{-1/2}t + \rho)$, using (3.2), the inequality $(a+b)^2 \le 2a^2 + 2b^2$, $a, b \in \mathbb{R}$, the Hölder and the Minkowski inequalities, the transformation theorem for integrals and A2,

$$\begin{aligned} & \left| M(n^{-1/2}t + \rho) - Q(n^{-1/2}t + \rho) \right| \\ &\quad \le 8 |W(t) - W(0)|_H^2 + 8 |U(t) - U(0) - ta|_H^2 \\ &\qquad + 4 \left| S_h(\rho) + t[a + \underline{a}] \right|_H \\ &\qquad \times \left[|W(t) - W(0)|_H + |U(t) - U(0) - ta|_H \right], \end{aligned} \tag{3.6}$$

where $W(t)$, $S_h(\rho)$ and $U(t)$ are functions $W(x,t)$, $S_h(x,t)$ and $U(x,t)$ with their integrating variables suppressed. The proof of the theorem follows from (3.6) and from the statements:

$$E \sup_{|t|\le b} |U(t) - U(0) - ta|_H^2 = o(1). \tag{i}$$

$$E \sup_{|t|\le b} |W(t) - W(0)|_H^2 = o(1). \tag{ii}$$

$$\limsup_{n} E \sup_{|t|\le b} \left| S_h(\rho) + t[a + \underline{a}] \right|_H^2 < \infty. \tag{iii}$$

For the proof of (i), (ii) and (iii) above, see the Appendix. □

In order to prove Theorem 3.2, define $\tilde{\rho}_h(H)$ as a measurable minimizer of Q. Then

$$(3.7)\quad \sqrt{n}\,(\tilde{\rho}_h(H) - \rho) = -\int \frac{S_h(\rho)[\underline{a} + a]\,dH}{|a + \underline{a}|_H^2} = -2\int \frac{S_h(\rho)a}{|a + \underline{a}|_H^2}\,dH.$$

The first equality in (3.7) follows from the definition of $\tilde{\rho}_h(H)$ and (3.2); the second from (1.4) and the fact that $S_h(\cdot, \rho)$ is even and A2.

THEOREM 3.2. *In addition to the assumptions of Theorem* 3.1, *assume that* A6 *holds. Then*

$$(3.8)\qquad n^{1/2}[\hat{\rho}_h(H) - \rho] = n^{1/2}[\tilde{\rho}_h(H) - \rho] + o_p(1).$$

PROOF. The line of proof is as follows:

(i) For any $\eta > 0$ and $0 < z < \infty$, there exists an N and b, $0 < b < \infty$, depending on η and z such that

$$P\Big(\inf_{|t|>b} M(n^{-1/2}t + \rho) \geq z\Big) \geq 1 - \eta \qquad \forall\, n \geq N.$$

(ii) $\sqrt{n}\,[\tilde{\rho}_h(H) - \rho] = O_p(1)$ and $\sqrt{n}\,[\hat{\rho}_h(H) - \rho] = O_p(1)$.
(iii) $M[\hat{\rho}_h(H)] = Q[\hat{\rho}_h(H)] + o_p(1)$ and $M[\hat{\rho}_h(H)] = Q[\tilde{\rho}_h(H)] + o_p(1)$.

Proof of (i) follows exactly as in Koul and de Wet [(1983), Corollary 5.1] or Koul [(1985), Lemma 3.1]. Proof of (ii) follows from (i), (5.16) and the reasoning given in Koul [(1985), Theorem 3.1]. Proof of (iii) follows from (ii) and Theorem 3.1. From (iii), $Q[\hat{\rho}_h(H)] - Q[\tilde{\rho}_h(H)] = o_p(1)$, which in turn, using (3.2) and (3.7) simplifies to give

$$(3.9)\qquad n[\hat{\rho}_h(H) - \tilde{\rho}_h(H)]^2 |a + \underline{a}|_H^2 = o_p(1).$$

From (3.1), the definition of $a, \underline{a}$ in (3.2) and the symmetry of the function g_n w.r.t. the y-axis, we get

$$\begin{aligned}
|a + \underline{a}|_H^2 &= 4(1 - \gamma_n)^2[EX_0h(X_0)]^2\int g_n^2\,dH \\
&\quad + \gamma_n^2\int\Big[E\int(X_0 + z)h(X_0 + z) \\
&\qquad\qquad \times[g_n(x + \rho z) + g_n(-x + \rho z)]\,dL_n(z)\Big]^2\,dH(x) \\
(3.10)\qquad &\quad + 4\gamma_n(1 - \gamma_n)EX_0h(X_0) \\
&\qquad \times\int g_n(x)E\Big[\int(X_0 + z)h(X_0 + z) \\
&\qquad\qquad \times[g_n(x + \rho z) + g_n(-x + \rho z)]\,dL_n(z)\Big]\,dH(x).
\end{aligned}$$

From A1, A2, A4, A7, A8(b), A13, the moment and the Hölder inequalities, the second and third terms on the r.h.s. of (3.10) go to zero and the first term converges to

$$2q = 4\left[EX_0 h(X_0)\right]^2 \int f^2\, dH. \tag{3.11}$$

From A3, A6 and A7, $q > 0$. Hence from (3.9) the proof is complete. □

NOTE. In order to study the limiting distribution of $\sqrt{n}\,[\hat{\rho}_h(H) - \rho]$, we need to appropriately center (3.8). In view of the Theorem 3.2, for fixed h and H, the asymptotic behavior of $\hat{\rho}_h(H)$ will not be affected if for each $n \geq 0$, $\hat{\rho}_h(H)$ is replaced by any convex combination of the measurable minimizers of (1.5).

4. Asymptotic normality and influence function of $\hat{\rho}_h(H)$. In this section we show that $\hat{\rho}_h(H)$ is asymptotically normally distributed when appropriately centered. We also discuss the sufficient conditions under which the asymptotic bias of $\hat{\rho}_h(H)$ is vanishing or nonvanishing. Finally, we compute the influence function and show that it is directly proportional to the asymptotic bias of $\hat{\rho}_h(H)$.

To study the limiting distribution of $\sqrt{n}\,[\hat{\rho}_h(H) - \rho]$ when centered, from (3.7), (3.10), (3.11) and Theorem 3.2, it suffices to study the limiting distribution of $-q^{-1}\int S_h(\rho) a\, dH$, i.e., that of

$$-q^{-1} n^{-1/2} \sum_{j=1}^{n} h(Y_{j-1}) \psi_n(Y_j - \rho Y_{j-1}), \tag{4.1}$$

when centered, where

$$\psi_n(x) = \int_{-\infty}^{x} a\, dH - \int_{-\infty}^{-x} a\, dH. \tag{4.2}$$

Let

$$\begin{aligned} \mu_n &= Eh(Y_0)\psi_n(Y_1 - \rho Y_0) \quad \text{and} \\ \xi_{j,n} &= h(Y_{j-1})\psi_n(Y_j - \rho Y_{j-1}) - \mu_n, \qquad 1 \leq j \leq n. \end{aligned} \tag{4.3}$$

From the model assumptions, (4.2) and with a as in (3.2), one can rewrite

$$\begin{aligned} \psi_n(y) = {} & (1 - \gamma_n) EX_0 h(X_0) \left[\int_{-\infty}^{y} g_n\, dH - \int_{-\infty}^{-y} g_n\, dH \right] \\ & + \gamma_n \left[\int_{-\infty}^{y} E \int (X_0 + z) h(X_0 + z) g_n(x + \rho z)\, dL_n(z)\, dH(x) \right. \\ & \left. - \int_{-\infty}^{-y} E \int (X_0 + z) h(X_0 + z) g_n(x + \rho z)\, dL_n(z)\, dH(x) \right]. \end{aligned} \tag{4.4}$$

From (4.4), A2, A4, A7, A8(a), A12 and $\gamma_n \in [0, 1]$, we see that ψ is uniformly bounded. Hence under additional assumptions A3(b) and A5, $\xi_{j,n}$, $1 \leq j \leq n$, are real valued r.v.'s and $\sup\{|\mu_n| < \infty,\ n \geq 0\}$.

THEOREM 4.1. *Let* A1–A17 *and all the model assumptions* (1.1) *to* (1.3) *hold. Then*

$$\sqrt{n}\,[\hat{\rho}_h(H) - \rho + \mu_n q^{-1}] \to \mathscr{N}(0, \sigma_h^2) \quad \text{in distribution}, \tag{4.5}$$

where

$$\sigma_h^2 = q^{-2}[EX_0 h(X_0)]^2 Eh^2(X_0) E\psi^2(\varepsilon_1), \tag{4.6}$$

$$\psi(y) = \int_{-\infty}^{y} f\,dH - \int_{-\infty}^{-y} f\,dH, \qquad \forall\, y \in \mathbb{R}, \tag{4.7}$$

q is as in (3.11) *and μ_n as in* (4.3).

PROOF. In Dhar [(1990), Lemma 3.2] take $\theta_n(x, y) = h(x)\psi_n(y - \rho x)$, $x, y \in \mathbb{R}$, Y_j, X_j and v_j as in the model assumptions (1.1)–(1.3) with $\omega(x, y) = x + y$. We see that the set process $\{\xi_n, n \geq 1\}$, where $\xi_n = \{\xi_{j,n}, 1 \leq j \leq n\}$ and $\xi_{j,n}$ as in (4.3), is α_ξ-mixing provided the sequence $\{X_j\}$ is α_X-mixing. But X_j can be a.s. represented as $\sum_{k=0}^{\infty} \rho^k \varepsilon_{j-k}$; hence, using A3(a), A9 and Pham and Tran [(1985), Theorem 2.1] with $\delta = 2$, $A(k) = \rho^k$, we get that $\{X_j\}$ is strongly α_X-mixing with $\alpha_X(n) \leq C_\rho |\rho|^{2n/3}$ $\forall\, n \geq 1$.

Also note from A1, A7, A8(a) and (3.1) that

$$\int_{-\infty}^{x} g_n\, dH \to \int_{-\infty}^{x} f\,dH \quad \text{uniformly in } x. \tag{4.8}$$

Thus the r.h.s. of (4.4) converges to $EX_0 h(X_0)\psi(y)$, uniformly in y, by (4.8), A1, A3 and A12. By A7, ψ is bounded and hence ψ_n is uniformly bounded. Since F is symmetric about 0 and ψ is an odd function, $E\psi(\varepsilon_1) = 0$. Thus letting θ_n, ω, Y_j, X_j, v_j and $\xi_{j,n}$ be as above and

$$\theta(x, y) = h(x)\psi(y - \rho x) EX_0 h(X_0), \qquad x, y \in \mathbb{R},$$

in Dhar [(1990), Lemma 3.3], we see $\tau^2 = \sigma_h^2 q^2$. From (4.6), (4.7), A3, A6 and A7, $\sigma_h^2 q^2 > 0$. We now see that all the conditions of Dhar [(1990), Theorem 3.4] are satisfied. Hence the central limit theorem holds for ξ defined by (4.3). Thus from (3.7), (3.10), (3.11) and Theorem 3.2, (4.5) holds. □

Theorem 4.2 discusses the asymptotic bias of $\hat{\rho}_h(H)$.

THEOREM 4.2. *Let all the model assumptions* (1.1)–(1.3) *and* A1, A2, A4, A5, A7, A8(b), A13 *and* A16–A17 *hold. Then*

(a) *h continuous on $\mathbb{R}$, $Z_n \to Z$ in distribution and $\sqrt{n}\,\gamma_n \to \gamma_c$, $0 < \gamma_c < \infty$ imply $\sqrt{n}\,\mu_n \to \mu$, with*

$$\begin{aligned} \mu &= \gamma_c[EX_0 h(X_0)] \\ &\quad \times \int f(x) E \int h(X_0 + z)\{F(x - \rho z) - F(x + \rho z)\}\, dL(z)\, dH(x), \end{aligned} \tag{4.9}$$

where L is the d.f. of the r.v. Z.

(b) *Either $Z_n \to 0$ in probability or $\sqrt{n}\,\gamma_n \to 0$ imply $\sqrt{n}\,\mu_n \to 0$.*

PROOF. From (3.7) and (4.1)–(4.3), $\sqrt{n}\,\mu_n = E\int S_h(\rho)a\,dH$. For large enough n, we shall justify the interchange of expectation and integral above using the Fubini theorem. Accordingly,

$$\int E|S_h(\rho)|\,|a|\,dH \le \left[\int ES_h^2(\rho)\,dH\right]^{1/2}\left[\int a^2\,dH\right]^{1/2}. \tag{4.10}$$

Inequality (4.10) follows from the Hölder and the moment inequalities. That the lim sup of the r.h.s. of (4.10) is finite follows from (5.16) and the same reasoning as in (3.10) to (3.11). Thus, by the stationarity of the process (v_{j-1}, Y_{j-1}), the independence of (v_{j-1}, Y_{j-1}) and $v_j + \varepsilon_j$, $1 \le j \le n$, and (1.1), for large enough n, we get

$$\begin{aligned}\sqrt{n}\,\mu_n = \sqrt{n}\,\gamma_n \int a(x)E\int h(X_0+z)[G_n(x-\rho z)\\ -G_n(x+\rho z)]\,dL_n(z)\,dH(x).\end{aligned} \tag{4.11}$$

Using the definition of G_n and (3.1), (4.11) can be rewritten as

$$\begin{aligned}&\sqrt{n}\,\gamma_n(1-\gamma_n)^3 EX_0h(X_0)\int f(x)E\int h(X_0+z)\\ &\qquad\times[F(x-\rho z)-F(x+\rho z)]\,dL_n(z)\,dH(x)\\ &+\sqrt{n}\,\gamma_n^2(1-\gamma_n)^2[EX_0h(X_0)]\int EF(x-Z_n)E\int h(X_0+z)\\ &\qquad\times[F(x-\rho z)-F(x+\rho z)]\,dL_n(z)\,dH(x)\\ &+\sqrt{n}\,\gamma_n^2(1-\gamma_n)\int\left[E\int(X_0+z)h(X_0+z)g_n(x+\rho z)\,dL_n(z)\right]\\ &\qquad\times\left[E\int h(X_0+z)[F(x-\rho z)-F(x+\rho z)]\,dL_n(z)\right]dH(x)\\ &+\sqrt{n}\,\gamma_n^2\int a(x)E\int h(X_0+z)\\ &\qquad\times E[F(x-\rho z-Z_n)-F(x+\rho z-Z_n)]\,dL_n(z)\,dH(x).\end{aligned} \tag{4.12}$$

The fourth term in (4.12) converges to zero by A1 and A17(b), the same reasoning as in (3.10) to (3.11) and the Hölder inequality. The third term in (4.12) converges to zero by A1, A13, A17(a), the Hölder and the moment inequalities. That the second term in (4.12) converges to zero follows from A1, A4, A8(b), A17(a), the Hölder and the moment inequalities. For each $x \in \mathbb{R}$,

$$
\begin{aligned}
&E\int h(X_0+z)[F(x-\rho z)-F(x+\rho z)]\,dL_n(z) \\
(4.13)\qquad &\to E\int h(X_0+z)[F(x-\rho z)-F(x+\rho z)]\,dL(z),
\end{aligned}
$$

we get that (4.13) holds from A5, $Z_n \to Z$ in distribution the continuity of F and the D.C.T.

The proof of (a) and (b) now follows from the convergence of the first term in (4.12) to the appropriate quantity, which itself follows from A1, A4, A5, A7, (4.13) and the D.C.T. □

REMARK 4.1. From Theorems 4.1 and 4.2(a) or (b), we see that $\sqrt{n}[\hat{\rho}_h(H)-\rho] \to \mathscr{N}(-\mu q^{-1}, \sigma_h^2)$ or $\mathscr{N}(0, \sigma_h^2)$ in distribution.

REMARK 4.2. Koul (1986) has shown that the function $h(x) \propto x$, $x \in \mathbb{R}$, minimizes σ_h^2. Let $\hat{\rho}_x(H)$ be the estimator corresponding to $h(x) \equiv x$ and measure H. Then, the asymptotic bias of $\hat{\rho}_x(H)$ under Theorem 4.2(a) looks like

$$\left[2EX_0^2\int f^2\,dH\right]^{-1}\gamma_c\int f(x)EZ[F(x+\rho Z)-F(x-\rho Z)]\,dH(x),$$

which has the same sign as the sign of ρ.

REMARK 4.3 (Influence function). We shall define a functional T on a subset, say P_0, of the set of all stationary ergodic measures on $(\mathbb{R}^{-\infty,\infty}, \mathscr{B})$, where $\mathbb{R}^{-\infty,\infty}$ is a collection of all sequences of the type $\tilde{y} = (\dots, y_{-1}, y_0, y_1, y_2, \dots)$ and $\mathscr{B}$ is the Borel σ-field on $\mathbb{R}^{-\infty,\infty}$. Proceeding as in Martin and Yohai [(1986), equations 3.1–3.3], the functional T, $\nu \in P_0$, is defined as $T(\nu)$ satisfying the equation

$$
\begin{aligned}
&\frac{d}{dt}\int\bigg[\int h(y_0)\{I[y_1 \le x + ty_0] \\
(4.14)\qquad &\qquad -I[-y_1 < x - ty_0]\}\,d\nu(\tilde{y})\bigg]^2 dH(x) = 0,
\end{aligned}
$$

where P_0 is the set of all stationary ergodic measures on $(\mathbb{R}^{-\infty,\infty}, \mathscr{B})$, such that the integral in (4.14) exists and is differentiable. Further, $T(\nu)$ minimizes the double integral in (4.14) as a function of t and, of all the minimizers, it is defined as the one with the smallest magnitude. Under the assumptions A2 and A6 and following the same argument as in Lemma 2.1, we see that (4.14) has at least one solution which minimizes the double integral in (4.14), as a function of t. For the sake of completeness, let us repeat the definition of the time-series influence functional as in Martin and Yohai [(1986), Definition 4.2]. For T as a solution of (4.14) and for the measure ν_y^γ generated by the process (1.2) with $\gamma_n = \gamma$ and $L_n = L \ \forall \ n$, the influence functional (IF) of T is defined as

$$\mathrm{IF} = \lim_{\gamma \to 0} \frac{T(\nu_y^\gamma) - T(\nu_y^0)}{\gamma},$$

provided the limit exists.

In (4.14), taking $\nu = \nu_y^\gamma$ and then computing the innermost integral and differentiating within the integral sign w.r.t. t under some regularity conditions, we get that $T^\gamma \equiv T(\nu_y^\gamma)$ satisfies

$$\begin{aligned}\int \Big[&(1-\gamma) E\big(h(X_0)\{G(x + [T^\gamma - \rho]X_0) - G(x - [T^\gamma - \rho]X_0)\}\big) \\ &+ \gamma E \int h(X_0 + z)\{G(x + [T^\gamma - \rho](X_0 + z) + \rho z) \\ &\qquad - G(x - [T^\gamma - \rho](X_0 + z) - \rho z)\}\, dL(z)\Big] \\ \times \Big[&(1-\gamma) E\big(X_0 h(X_0)\{g(x + [T^\gamma - \rho]X_0) + g(x - [T^\gamma - \rho]X_0)\}\big) \\ &+ \gamma E \int h(X_0 + z)(X_0 + z)\{g(x + [T^\gamma - \rho](X_0 + z) + \rho z) \\ &\qquad + g(x - [T^\gamma - \rho](X_0 + z) + \rho z)\}\, dL(z)\Big]\, dH(x) = 0.\end{aligned} \tag{4.15}$$

Note that in (4.15), G and g both depend on γ. Setting $\gamma = 0$ in (4.15) we see that $T^0 \equiv T(\nu_y^0)$ satisfies

$$\begin{aligned}\int &E\big(h(X_0)\{F(x + [T^0 - \rho]X_0) - F(x - [T^0 - \rho]X_0)\}\big) \\ &\times E\big(X_0 h(X_0)\{f(x + [T^0 - \rho]X_0) \\ &\qquad + f(x - [T^0 - \rho]X_0)\}\big)\, dH(x) = 0.\end{aligned} \tag{4.16}$$

Assuming that the derivative of F exists and equals f we see from A6 that $T^0 = \rho$ is the only solution to (4.16), which makes the double integral in (4.14) with $\nu = \nu_y^0$ minimum. Thus, differentiating the l.h.s. of (4.15) under the integral sign w.r.t. γ, then setting $\gamma = 0$ and solving for $\partial T^\gamma / \partial \gamma|_{\gamma=0}$, we get

$$\begin{aligned}\mathrm{IF} = q^{-1} E X_0 h(X_0) \int f(x) E \int h(X_0 + z) \\ \times \{F(x + \rho z) - F(x - \rho z)\}\, dL(z)\, dH(x).\end{aligned}$$

Note that IF $= -\gamma_c^{-1}$ (asymptotic bias of $\sqrt{n}\,[\hat{\rho}_h(H) - \rho]$), where the bias is computed under the assumptions of Theorem 4.2(a).

APPENDIX

PROOF OF THEOREM 3.1(i). Define $h^+(x) = h(x)I[xh(x) \geq 0]$ and $h^-(x) = h(x) - h^+(x)$, $x \in \mathbb{R}$. Replacing h with $h^\pm$ in each of the functions a and U gives new functions, say $a^\pm$ and $U^\pm$.

From the inequality, $(a+b)^2 \leq 2a^2 + 2b^2$, $a, b \in \mathbb{R}$,

$$\begin{aligned}
&|U^\pm(t) - U^\pm(0) - a^\pm(x)|_H^2 \\
&\quad \leq 2\int\left[U^\pm(t,x) - U^\pm(0,x) - \frac{t}{n}\sum_{j=0}^{n-1} Y_j h^\pm(Y_j) g_n(x+\rho v_j)\right]^2 dH(x) \\
(5.1)\quad &\quad + 2t^2\int\left[\frac{1}{n}\sum_{j=0}^{n-1} Y_j h^\pm(Y_j) g_n(x+\rho v_j) - a^\pm(x)\right]^2 dH(x) \\
&\quad = 2\mathrm{I}(t) + 2t^2\mathrm{II},
\end{aligned}$$

$|t| \leq b$, where $\mathrm{I}(t)$ and II represent the first and the second integral on the r.h.s. of (5.1), respectively. From G_n as in A16, (3.1), (3.4) and the integral representation of F in terms of f, we get

$$\begin{aligned}
\mathrm{I}(t) = \int\Bigg[& n^{-1/2}\sum_{j=0}^{n-1} Y_j h^\pm(Y_j) \\
&\times\Big\{(1-\gamma_n)\int_0^{n^{-1/2}t}\big[f(x+sY_j+\rho v_j) - f(x+\rho v_j)\big]\,ds \\
&+\gamma_n\int\int_0^{n^{-1/2}t}\big[f(x+sY_j+\rho v_j - z) \\
&\qquad -f(x+\rho v_j - z)\big]\,ds\,dL_n(z)\Big\}\Bigg]^2 dH(x)
\end{aligned}$$

$$\begin{aligned}
(5.2)\quad &\leq 4(1-\gamma_n)^2\int n^{-1/2b}\sum_{j=0}^{n-1} Y_j^2 h^2(Y_j) \\
&\qquad\times\int_{-n^{-1/2b}}^{n^{-1/2b}}\big[f(x+sY_j+\rho v_j) - f(x+\rho v_j)\big]^2\,ds\,dH(x) \\
&\quad + 4\gamma_n^2\int n^{-1/2b}\sum_{j=0}^{n-1} Y_j^2 h^2(Y_j) \\
&\qquad\times\int_{-n^{-1/2b}}^{n^{-1/2b}}\int\big[f(x+sY_j+\rho v_j - z) \\
&\qquad\qquad -f(x+\rho v_j - z)\big]^2\,dL_n(z)\,ds\,dH(x).
\end{aligned}$$

Inequality (5.2) follows from the Cauchy–Schwarz inequality for the finite sum of real numbers and the moment inequality. Use (5.2), the Fubini theorem, the stationarity of $\{(v_j, Y_j), 0 \le j \le n\}$, (1.1) and (1.2) to get

$$
\begin{aligned}
E \sup_{|t|\le b} \mathrm{I}(t) \le{}& 4\sqrt{n}\,(1-\gamma_n)^3 b \\
& \int_{-n^{-1/2b}}^{n^{-1/2b}} \int E X_0^2 h^2(X_0) [f(x+sX_0) - f(x)]^2 \, dH(x)\, ds \\
& + 4\sqrt{n}\,\gamma_n (1-\gamma_n)^2 b \\
& \times \int_{-n^{-1/2b}}^{n^{-1/2b}} \int E\Big[\int (X_0+z)^2 h^2(X_0+z) \\
& \qquad \times [f(x+s[X_0+z]+\rho z) - f(x+\rho z)]^2 \, dL_n(z)\Big] dH(x)\, ds \\
& + 4\sqrt{n}\,\gamma_n^2 (1-\gamma_n) b \\
& \times \int_{-n^{-1/2b}}^{n^{-1/2b}} \int \Big[E X_0^2 h^2(X_0) \int [f(x+sX_0-z) \\
& \qquad - f(x-z)]^2 \, dL_n(z)\Big] dH(x)\, ds \\
& + 4\sqrt{n}\,\gamma_n^3 b \int_{-n^{-1/2b}}^{n^{-1/2b}} \int E\Big[\int (X_0+z)^2 h^2(X_0+z) \\
& \qquad \times \Big[\int [f(x+s(X_0+z)+\rho z - u) \\
& \qquad - f(x+\rho z - u)]^2 \, dL_n(u)\Big] dL_n(z)\Big] dH(x)\, ds.
\end{aligned}
\tag{5.3}
$$

A1 and the continuity property in A10(b) show that the first term on the r.h.s. of (5.3) converges to zero. The remaining terms of (5.3) go to zero by A1, A4, A8(b), A11(b), A13 and A15. Thus

$$
E \sup_{|t|\le b} \mathrm{I}(t) = o(1). \tag{5.4}
$$

Now consider

$$
\begin{aligned}
E\,\mathrm{II} ={}& \int E\left[\frac{1}{n}\sum_{j=0}^{n-1} Y_j h^{\pm}(Y_j) g_n(x+\rho v_j) - a^{\pm}(x)\right]^2 dH(x) \\
\le{}& 2(1-\gamma_n)^2 \int E\Bigg[\frac{1}{n}\sum_{j=0}^{n-1} Y_j h^{\pm}(Y_j) f(x+\rho v_j) \\
& \qquad - E Y_0 h^{\pm}(Y_0) f(x+\rho v_0)\Bigg]^2 dH(x) \\
& + 2\gamma_n^2 \int E\Bigg[\frac{1}{n}\sum_{j=0}^{n-1} Y_j h^{\pm}(Y_j) \int f(x+\rho v_j - z)\, dL_n(z) \\
& \qquad - E Y_0 h^{\pm}(Y_0) \int f(x+\rho v_0 - z)\, dL_n(z)\Bigg]^2 dH(x).
\end{aligned}
\tag{5.5}
$$

The inequality (5.5) follows from (3.1) and the inequality $(a+b)^2 \le 2a^2 + 2b^2$, $a, b \in \mathbb{R}$. From A1, A4, A8(b), A13(b), the stationarity of (v_j, Y_j), $0 \le j \le n-1$, and the Cauchy–Schwarz inequality for the finite sum of real numbers, the second term on the r.h.s. of (5.5) goes to zero as $n \to \infty$. The first term on the r.h.s. of (5.5) can be written as

$$\begin{aligned} &2n^{-1}(1-\gamma_n)^2 \int \operatorname{Var}\{Y_0 h^{\pm}(Y_0) f(x+\rho v_0)\}\, dH(x) \\ (5.6) \qquad &+ 4n^{-2}(1-\gamma_n)^2 \sum_{j=1}^{n-1} (n-j) \int \operatorname{Cov}\{Y_0 h^{\pm}(Y_0) f(x+\rho v_0), \\ &\qquad Y_j h^{\pm}(Y_j) f(x+\rho v_j)\}\, dH(x), \end{aligned}$$

which follows from the stationarity of (v_j, Y_j), $0 \le j \le n-1$. The first integral in (5.6) can be written as

$$\begin{aligned} &(1-\gamma_n) E X_0^2 \{h^{\pm}(X_0)\}^2 \int f^2\, dH - (1-\gamma_n)^2 [E X_0 h^{\pm}(X_0)]^2 \int f^2\, dH \\ &+ \gamma_n \int E\left[\int (X_0+z)^2 \{h^{\pm}(X_0+z)\}^2 f^2(x+\rho z)\, dL_n(z)\right] dH(x) \\ &- 2\gamma_n(1-\gamma_n) E X_0 h^{\pm}(X_0) \\ &\quad \times \int f(x) E\left[\int (X_0+z) h^{\pm}(X_0+z) f(x+\rho z)\, dL_n(z)\right] dH(x) \\ &- \gamma_n^2 \int \left[E \int (X_0+z) h^{\pm}(X_0+z) f(x+\rho z)\, dL_n(z)\right]^2 dH(x), \end{aligned}$$

which converges to $\operatorname{Var}[X_0 h^{\pm}(X_0)] \int f^2\, dH$. This in turn follows from A1, A4, A7, A13(a), the moment inequality and the Hölder inequality. Hence, the first term in (5.6) converges to zero. The second term in (5.6) can be written as

$$\begin{aligned} &4n^{-2}(1-\gamma_n)^4 \sum_{j=1}^{n-1} (n-j) \operatorname{Cov}\{X_0 h^{\pm}(X_0), X_j h^{\pm}(X_j)\} \int f^2\, dH \\ &+ 4n^{-2}\gamma_n(1-\gamma_n)^3 \sum_{j=1}^{n-1} (n-j) \int \operatorname{Cov}\Big[X_0 h^{\pm}(X_0) f(x), \\ &\qquad \int (X_j+z) h^{\pm}(X_j+z) f(x+\rho z)\, dL_n(z)\Big]\, dH(x) \\ (5.7) \qquad &+ 4n^{-2}\gamma_n(1-\gamma_n)^3 \sum_{j=1}^{n-1} (n-j) \int \operatorname{Cov}\Big[X_j h^{\pm}(X_j) f(x), \\ &\qquad \int (X_0+z) h^{\pm}(X_0+z) f(x+\rho z)\, dL_n(z)\Big]\, dH(x) \\ &+ 4n^{-2}\gamma_n^2(1-\gamma_n)^2 \sum_{j=1}^{n-1} (n-j) \\ &\quad \times \int \operatorname{Cov}\Big[\int (X_0+z) h^{\pm}(X_0+z) f(x+\rho z)\, dL_n(z), \\ &\qquad \int (X_j+z) h^{\pm}(X_j+z) f(x+\rho z)\, dL_n(z)\Big]\, dH(x). \end{aligned}$$

By assumptions A1, A4, A7, A13(a), the Hölder inequality and the stationarity of the X_j process, the second, third and the fourth terms in (5.7) go to zero. Since

$$(5.8)\qquad \begin{aligned}\operatorname{Var}\left\{n^{-1}\sum_{j=1}^{n} X_j h^{\pm}(X_j)\right\} &= n^{-1}\operatorname{Var}\left[X_0 h^{\pm}(X_0)\right]\\ &\quad + 2n^{-2}\sum_{j=1}^{n-1}(n-j)\operatorname{Cov}\left\{X_0 h^{\pm}(X_0), X_j h^{\pm}(X_j)\right\},\end{aligned}$$

to prove that the first term in (5.7) goes to zero, from A1, A4, A7 and (5.8) it suffices to prove that

$$(5.9)\qquad \operatorname{Var}\left\{n^{-1}\sum_{j=1}^{n} X_j h^{\pm}(X_j)\right\} \to 0 \quad \text{as } n \to \infty.$$

But from A4 and the stationary ergodic theorem,

$$(5.10)\qquad n^{-1}\sum_{j=1}^{n} X_j h^{\pm}(X_j) \to E X_0 h^{\pm}(X_0) \quad \text{a.s.}$$

Also, from A4, the sequence $\{X_j h^{\pm}(X_j)\}$ is uniformly integrable of order 2; hence, so is the sequence $\{n^{-1}\sum_{j=1}^{n} X_j h^{\pm}(X_j)\}$, which follows clearly from Chung [(1974), exercise 9, page 100]. Thus, from (5.10), (5.9) follows, which in turn gives

$$(5.11)\qquad E\mathrm{II} \to 0 \quad \text{as } n \to \infty.$$

Thus (5.4) and (5.11) applied to (5.1) prove (i). □

PROOF OF THEOREM 3.1(ii). Replace the h in W by $h^{\pm}$ and call the new r.v. $W^{\pm}$. Fix $t \in [-b, b]$. Using (3.4) write $(W^{\pm}(t) - W^{\pm}(0))^2$ as the sum of squares of terms and the cross product terms, we see that the conditional mean of any of its ijth $(i < j)$ cross product term given $\{(v_l, Y_l), 0 \le l \le j-1\}$ is zero follows from the model assumptions and the fact that $W^{\pm}$ is conditionally centered given (v_{j-1}, Y_{j-1}). Using these facts and the Fubini theorem, we get

$$(5.12)\qquad \begin{aligned} &E|W^{\pm}(t) - W^{\pm}(0)|_H^2\\ &\quad = n^{-1}\int\sum_{j=1}^{n} E\Big[\{h^{\pm}(Y_{j-1})\}^2 E\Big\{\Big(I\big[v_j + \varepsilon_j \le x + n^{-1/2}tY_{j-1} + \rho v_{j-1}\big]\\ &\qquad - I\big[v_j + \varepsilon_j \le x + \rho v_{j-1}\big] - G_n\big(x + n^{-1/2}tY_{j-1} + \rho v_{j-1}\big)\\ &\qquad\qquad + G_n(x + \rho v_{j-1})\Big)^2\Big|(v_{j-1}, Y_{j-1})\Big\}\Big]\, dH(x),\end{aligned}$$

which can be dominated by

$$2\int E\{h^{\pm}(Y_0)\}^2\left|G_n(x+n^{-1/2}tY_0+\rho v_0)-G_n(x+\rho v_0)\right|dH(x),$$

which follows by applying the inequality that the variance is less than or equal to the second moment, to the conditional variance in (5.12). Using G_n as in A16, the Fubini theorem and representation of F in terms of its density, this term can be dominated by

$$\begin{aligned}
&2(1-\gamma_n)^2\int_{-n^{-1/2b}}^{n^{-1/2b}}\int E|X_0|h^2(X_0)\,f(x+X_0s)\,dH(x)\,ds\\
&\quad+2\gamma_n(1-\gamma_n)\int_{-n^{-1/2b}}^{n^{-1/2b}}\int E\left[\int|x_0+z|h^2(X_0+z)\right.\\
&\qquad\qquad\left.\times f(x+\rho z+s[X_0+z])\,dL_n(z)\right]dH(x)\,ds\\
&\quad+2\gamma_n(1-\gamma_n)\int_{-n^{-1/2b}}^{n^{-1/2b}}\int E\left[\int|X_0|h^2(X_0)\right.\\
&\qquad\qquad\left.\times f(x-z+X_0s)\,dL_n(z)\right]dH(x)\,ds\\
&\quad+2\gamma_n^2\int_{-n^{-1/2b}}^{n^{-1/2b}}\int E\left[\int|X_0+z|h^2(X_0+z)\right.\\
&\qquad\qquad\left.\times\int f(x-u+[X_0+z]s+\rho z)\,dL_n(u)\,dL_n(z)\right]dH(x)\,ds.
\end{aligned}\tag{5.13}$$

From A1, A3(a), A4 and A10(a) the first term in (5.13) converges to zero. That the remaining terms also converge to zero follows from A1, A11(a) and A14, giving

$$E\left|W^{\pm}(t)-W^{\pm}(0)\right|_H^2\to 0,\qquad t\in\mathbb{R}.\tag{5.14}$$

Thus to complete the proof of (ii), use the monotone structure of $W^{\pm}$ and $U^{\pm}$, the compactness of $[-b,b]$, $\limsup_n E|a|_H^2<\infty$ and (i), just as in Koul and de Wet [(1983), page 929, Theorem 5.1, proof of (ii)]. The details are similar, hence deleted. □

PROOF OF THEOREM 3.1(iii). Taking $t=0$ in (3.5) gives

$$\begin{aligned}
S(x,\rho)&=W(x,0)+W(-x,0)\\
&\quad+n^{-1/2}\sum_{j=1}^{n}h(Y_{j-1})\{G_n(x+\rho v_{j-1})-G_n(x-\rho v_{j-1})\}.
\end{aligned}\tag{5.15}$$

We shall now proceed to prove that

$$\limsup_n E|S(\rho)|_H^2<\infty.\tag{5.16}$$

Using the same reasoning as in the proof of (5.12) and the stationarity of the process (v_j, Y_j), $0 \le j \le n-1$, we get

$$E\int W(0)^2\,dH = \int E\frac{1}{n}\sum_{j=1}^{n} h^2(Y_{j-1})\{I[v_j - \rho v_{j-1} + \varepsilon_j \le x] - G_n(x + \rho v_{j-1})\}^2\,dH(x) \tag{5.17}$$
$$= \int Eh^2(Y_0)\{I[v_1 - \rho v_0 + \varepsilon_1 \le x] - G_n(x + \rho v_0)\}^2\,dH(x).$$

The lim sup of the r.h.s. of (5.17) is equal to the expression in A16 and hence finite. Next, using the stationarity of (v_j, Y_j), $0 \le j \le n-1$,

$$E\int\left\{n^{-1/2}\sum_{j=0}^{n-1} h(Y_j)[G_n(x+\rho v_j) - G_n(x-\rho v_j)]\right\}^2 dH(x)$$
$$= \int Eh^2(Y_0)[G_n(x+\rho v_0) - G_n(x-\rho v_0)]^2\,dH(x) \tag{5.18}$$
$$+ 2n^{-1}\sum_{j=1}^{n-1}(n-j)\int E\big[h(Y_0)[G_n(x+\rho v_0) - G_n(x-\rho v_0)]h(Y_j) \times [G_n(x+\rho v_j) - G_n(x-\rho v_j)]\big]\,dH(x).$$

The lim sup of the first term on the r.h.s. of (5.18) is finite by (1.1), A1 and A17. The expression inside the sum in the second term on the r.h.s. of (5.18) can be written as

$$(n-j)\gamma_n^2\int E\left[\int h(X_0+z)[G_n(x+\rho z) - G_n(x-\rho z)]\,dL_n(z) \times \int h(X_j+z)[G_n(x+\rho z) - G_n(x-\rho z)]\,dL_n(z)\right]dH(x), \tag{5.19}$$

which follows from the independence of $\{X_j, j \le n\}$ and $\{v_j, 0 \le j \le n\}$ and the latter being i.i.d. β_n. Thus applying the Cauchy-Schwarz and the moment inequalities to the integrand in (5.19) and using the stationarity of $\{X_j\}$, the second term on the r.h.s. of (5.18) can be dominated by

$$(n-1)\gamma_n^2\int E\left[\int h^2(X_0+z)[G_n(x+\rho z) - G_n(x-\rho z)]^2\,dL_n(z)\right]dH(x). \tag{5.20}$$

From A1 and A17(a), the lim sup of (5.20) is finite. Hence (5.16) holds. The proof of (iii) now follows from $\limsup_n E|a|_H^2 < \infty$, which in turn follows from (3.1), A4, A7, A8(b) and A13. □

NOTE. The proof of Theorem 3.1 only needs $0 < Eh^2(X_0) < \infty$, instead of A3(b); hence, the proof of (5.9) gives an alternate way to prove Koul [(1986), equation (14), page 1211].

REMARK 5.1. One of the assumptions under which we have studied the asymptotic behavior of $\hat{\rho}_h(H)$ as an estimator of ρ is that F and L_n's are symmetric about 0. One could generalize the results by making an attempt to discard this assumption. A careful study of the results shows that due to this change, the arguments involved in the proof of (5.16) fail. The proof of Theorem 3.2 can be easily modified without this assumption of symmetry or any additional assumptions. In view of Dhar [(1990), Lemmas 3.2–3.3 and Theorem 3.4] we see no additional modifications are needed to incorporate this change. Thus it only remains to modify the arguments from (5.15)–(5.20). Note in (5.15) and (5.18) we need to replace $G_n(x - \rho v_0)$ by $1 - G_n(-x + \rho v_0)$, since G_n is not symmetric. Thus in view of (5.18), we will now need

$$\limsup_n \int Eh^2(Y_0)[G_n(x + \rho v_0) + G_n(-x + \rho v_0) - 1]^2 \, dH(x) < \infty.$$

In the case when F is symmetric about 0 but L_n's are not symmetric about 0, (5.19) can be replaced by

$$\begin{aligned}
(n - j)\gamma_n^2\Bigg((1 - \gamma_n)^2 & Eh(X_0)h(X_j) \\
& \times \int [E\{F(x - Z_n) + F(-x - Z_n) - 1\}]^2 \, dH(x) \\
& + (1 - \gamma_n) \int [E\{F(x - Z_n) + F(-x - Z_n) - 1\}] \\
& \qquad \times E\bigg\{h(X_0) \int Eh(X_j + z)[G_n(x + \rho z) \\
& \qquad\qquad + G_n(-x + \rho z) - 1] \, dL_n(z)\bigg\} \, dH(x) \\
& + (1 - \gamma_n) \int [E\{F(x - Z_n) + F(-x - Z_n) - 1\}] \\
& \qquad \times E\bigg\{h(X_j) \int Eh(X_0 + z)[G_n(x + \rho z) \\
& \qquad\qquad + G_n(-x + \rho z) - 1] \, dL_n(z)\bigg\} \, dH(x) \\
& + \int E\bigg\{\int h(X_0 + z)[G_n(x + \rho z) + G_n(-x + \rho z) - 1] \, dL_n(z) \\
& \qquad \times \int h(X_j + z)[G_n(x + \rho z) \\
& \qquad\qquad + G_n(-x + \rho z) - 1] \, dL_n(z)\bigg\} \, dH(x)\Bigg).
\end{aligned}$$

Thus, using the same arguments as in (5.20) and under proper finite moment assumptions, Theorem 3.1 will hold.

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