MAXIMUM STANDARDIZED CUMULANT DECONVOLUTION OF NON-GAUSSIAN LINEAR PROCESSES

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A linear process is represented as a driving white noise convolved with a system response sequence. The concept of natural peakedness of a system response sequence is defined and its properties are investigated. Utilizing natural peakedness, the convergence theory of maximum standardized cumulant deconvolution is established and the uniqueness theorem of non-Gaussian linear process representations is proved. In addition, autoregressive models on a countable abelian group are defined and the relation between cumulant deconvolution and autoregressive models is given.

1. Introduction. Let $G$ represent a countable abelian group, let $w = \{w_t\}_{t \in G}$ be a square-summable sequence and let $u = \{u_t\}_{t \in G}$ be an independent and identically distributed random series with $E u_t = 0$, $E u_t^2 = \sigma^2$ and $E|u_t|^m < \infty$ for some $m > 2$.

$$x_t = (w * u)_t = \sum_{s \in G} u_s w_{t-s} \tag{1.1}$$

is called a linear process; $u$ and $w$ are called the driving noise and the system response sequence, respectively.

Throughout this paper we shall make the following assumptions:

$$0 < \sum_{t \in G} |w_t|^2 < \infty \tag{1.2}$$

and the Fourier transform of $w$,

$$w(\gamma) = \sum_{s \in G} w_s \gamma(-s), \quad \gamma \in \Gamma,$$

satisfies

$$w(\gamma) \neq 0, \quad d\gamma \text{ a.s.}, \tag{1.3}$$

where $\Gamma$ is the dual group of $G$ which consists of all complex functions $\gamma(t)$ on $G$ satisfying $|\gamma(t)| = 1$ and $\gamma(s + t) = \gamma(s) \gamma(t)$, $s, t \in G$; $d\gamma$ denotes the Haar measure on the group $\Gamma$ [see Rudin (1962)].

We note that when $G$ is the set of integers $\mathbb{Z}$, $x$ is a linear time series, and when $G$ is the set $\mathbb{Z}^2$ of pairs of integers, $x$ is a linear random field.

The objective of linear process decomposition is to estimate the driving noise (deconvolution) and to estimate the system response sequence (system identification) from $x_t$. In this paper we shall study a kind of deconvolution which is called maximum standardized cumulant deconvolution.

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The $m$-th cumulant of the random variable $x_t$ is defined by

$$c_m(x_t) = \left. \left( -i \frac{d}{dx} \right)^m \log(E e^{i x_t}) \right|_{s=0}.$$

Set

$$\hat{x}_t = \frac{x_t - E x_t}{(E x_t^2)^{1/2}}.$$

The standardized $m$-th cumulant is defined by

$$k_m(x_t) = c_m(\hat{x}_t) = \frac{c_m(x_t)}{(c_2(x_t))^{m/2}}.$$

For $x_t$ satisfying (1.1),

$$k_m(x_t) = k_m(u_t) \frac{\sum_i (u_i)^m}{(\sum_i u_i^2)^{m/2}}, \quad m > 2.$$  \hspace{1cm} (1.4)

Here we write $\Sigma_i$ in place of $\Sigma_{\tau \in G}$. This result can be found in Granger (1976).

Now we define maximum standardized cumulant deconvolution operator.

Let $S_1 \subset S_2 \subset \cdots \subset S_n \subset \cdots$ denote an increasing sequence of finite subsets of $G$ with the property that for every finite subset $F$ of $G$, there exists a positive integer $n_0$ and a $t_0 \in G$ such that $F \subseteq t_0 + S_{n_0}$. Because $G$ is countable, such sequences always exist. For instance, we can take $S_n = \{0, 1, \ldots, n\}$ when $G = \mathbb{Z}$ and $S_n = \{(i, j): 0 \leq i, j \leq n\}$ when $G = \mathbb{Z}^2$.

We denote by $g^{(n)} = \{g^{(n)}_t\}$ a sequence satisfying

$$g^{(n)}_t = 0, \quad t \not\in S_n. \hspace{1cm} (1.5)$$

$h^{(n)}$ is (not necessarily uniquely) defined as the maximum standardized $m$-th cumulant deconvolution operator of $x_t$ if it satisfies

$$\left| k_m((h^{(n)} \ast x)_t) \right| = \max_{g^{(n)}} \left| k_m((g^{(n)} \ast x)_t) \right|. \hspace{1cm} (1.6)$$

We briefly explain why $h^{(n)}$ exists. By the formulas (1.4), (2.1) and (2.14), for any nonzero constant $\alpha$, the $m$-th standardized cumulants of $g^{(n)}$ and $\alpha g^{(n)}$ have the same absolute value. So we can assume that $g^{(n)}$ satisfies

$$\sum_{t \in S_n} |g^{(n)}_t|^2 = 1.$$

Then $|k_m((g^{(n)} \ast x)_t)|$ is a continuous function with respect to $g^{(n)}_t$, $t \in S_n$ (note that $S_n$ is a finite subset). Hence, there exists at least one $h^{(n)}$ such that (1.6) holds. Of course, it is possible that $h^{(n)}$ is not unique. We shall pay more attention to the limit property of such a deconvolution operator. When $m = 4,$
by (1.4), we have

\[ k_4(x_t) = k_4(u_t) \frac{\sum_i w_i^4}{(\sum_i w_i^2)^2}. \]

Then (1.6) becomes

\[ k_4((h^{(n)} * x)_t) = \begin{cases} 
\max_{g^{(n)}} k_4((g^{(n)} * x)_t), & \text{when } c_4(x_t) > 0, \\
\min_{g^{(n)}} k_4((g^{(n)} * x)_t), & \text{when } c_4(x_t) < 0. 
\end{cases} \]

In this case, a maximum standardized cumulant deconvolution operator is just a kurtosis deconvolution operator [see Cheng (1988)]. In fact, minimum entropy deconvolution is a special case of the maximum standardized cumulant deconvolution [see Cheng (1988) and Wiggins (1978)].

The objective of this paper is to establish the convergence theory of maximum standardized cumulant deconvolution and to prove the uniqueness theorem of non-Gaussian linear process representations.

In Section 2 we present the concept of natural peakedness of a system response sequence and study its properties. Section 3 proves the uniqueness theorem of non-Gaussian linear processes. In Section 4 we give the convergence theorem of maximum standardized cumulant deconvolution; in addition, we define autoregressive models on a countable abelian group and discuss the relation between cumulant deconvolution and autoregressive models.

2. Natural peakedness of a system response sequence. We define the natural peakedness of system response sequence \( w \),

\[ q(w) = \left( \frac{\sum_i (w_i)^m}{(\sum_i w_i^2)^{m/2}} \right), \quad m > 2. \quad (2.1) \]

Let

\[ \|w\|_2 = \left( \sum_i w_i^2 \right)^{1/2} \quad (2.2) \]

\[ \|w\|_m = \left( \sum_i |w_i|^m \right)^{1/m}, \quad m > 2. \quad (2.3) \]

In order to derive the properties of natural peakedness, we introduce the absolute peakedness, defined as

\[ p(w) = \left( \frac{\|w\|_m}{\|w\|_2} \right)^m. \quad (2.4) \]

Lemma 2.1. For any constant \( a \neq 0 \) and any \( t_0 \in G \),

\[ p(w) = p(aw) = p(\delta^{(t_0)} * w), \quad (2.5) \]

where \( \delta^{(t_0)} = \{ \delta^{(t_0)}_t \}, \delta^{(t_0)}_t = 1 \text{ for } t = t_0 \text{ and zero for } t \neq t_0. \]
The proof of Lemma 2.1 is immediate.

**Lemma 2.2.**

\begin{equation}
0 < p(w) \leq 1.
\end{equation}

If \( p(w) = 1 \), then \( w = a \delta_{t_0} \), where \( a \) is a nonzero constant and \( t_0 \) is an element of \( G \).

**Proof.** Set

\[ \tilde{w} = \frac{1}{w_{t_0}} w, \]

where \( |w_{t_0}| = \max_{t \in G} |w_t| \). From Lemma 2.1, we have

\[ p(w) = p(\tilde{w}). \]

Since

\[ |\tilde{w}_t| \leq 1, \quad |\tilde{w}_{t_0}| = 1, \quad \|\tilde{w}\|_m^m \leq \|\tilde{w}\|_2^2, \]

we get

\[ 0 < p(\tilde{w}) \leq \|\tilde{w}\|_2^{-2(m-2)} \leq 1. \]

If \( p(w) = 1 \), then \( \|\tilde{w}\|_2 = 1 \). This means \( \tilde{w}_t = 0 \) for \( t \neq t_0 \). Thus, \( \tilde{w} = \delta_{t_0} \) and \( w = w_{t_0} \delta_{t_0} \). \( \square \)

**Lemma 2.3.** Let \( w^{(n)} \) and \( w \) be system response sequences and let

\begin{equation}
\|w^{(n)} - w\|_2 \to 0, \quad n \to \infty.
\end{equation}

Then

\begin{equation}
p(w^{(n)}) \to p(w), \quad n \to \infty.
\end{equation}

**Proof.** By the property of norms,

\[ \|w^{(n)} - w\|_2 \to 0, \quad n \to \infty. \]

It follows from Lemma 2.1 that

\[ \|w^{(n)} - w\|_m \leq \|w^{(n)} - w\|_2; \]

then

\[ \|w^{(n)}\|_m - \|w\|_m \leq \|w^{(n)} - w\|_m \to 0, \quad n \to \infty. \]

Hence

\[ p(w^{(n)}) = \frac{\|w^{(n)}\|_m^m}{\|w\|_m^m} \to \frac{\|w\|_m^m}{\|w\|_m^m} = p(w), \quad n \to \infty. \]

**Lemma 2.4.** Let \( w^{(n)} \) be system response sequences. In order that

\begin{equation}
p(w^{(n)}) \to 1, \quad n \to \infty,
\end{equation}

\[ P(w^{(n)}) \to 1, \quad n \to \infty. \]
it is necessary and sufficient that

\[(2.10) \quad \|\tilde{w}_i^{(n)} - \delta^{(t_n)}\|_2 \to 0, \quad n \to \infty,\]

where

\[(2.11) \quad \tilde{w}_i^{(n)} = \frac{1}{w_{t_n}^{(n)}} w_{t_n}^{(n)}, \quad |w_{t_n}^{(n)}| = \max_{t \in G} |w_t^{(n)}| .\]

**Proof.** Necessity: It follows from the proof of Lemma 2.2 that

\[p(w^{(n)}) = p(\tilde{w}^{(n)}) \leq \|\tilde{w}^{(n)}\|^{-(m-2)} \leq 1 .\]

Since (2.9) holds, we obtain

\[\|\tilde{w}^{(n)}\|_2^2 = 1 + \sum_{t \neq t_n} |\tilde{w}_t^{(n)}|^2 \to 1, \quad n \to \infty .\]

Therefore,

\[(2.12) \quad \|\tilde{w}^{(n)} - \delta^{(t_n)}\|_2^2 = \sum_{t \neq t_n} |\tilde{w}_t^{(n)}|^2 \to 0, \quad n \to \infty .\]

Sufficiency: Lemma 2.3 yields this immediately. □

We now discuss the property of natural peakedness. The following lemma is evident.

**Lemma 2.5.**

\[(2.13) \quad 0 \leq q(w) \leq p(w) \leq 1,\]

\[(2.14) \quad q(aw) = q(w) = q(\delta^{(t_0)} w) ,\]

where \(a\) is a nonzero constant and \(t_0\) is an element of \(G\).

**Lemma 2.6.** Let \(w^{(n)}\) be system response sequences. In order that

\[(2.15) \quad q(w^{(n)}) \to 1, \quad n \to \infty,\]

it is necessary and sufficient that

\[(2.16) \quad p(w^{(n)}) \to 1, \quad n \to \infty.\]

**Proof.** From (2.13), necessity holds. Sufficiency follows from (2.1), (2.14) and (2.12). □

The following theorem follows immediately from Lemmas 2.5 and 2.6.

**Theorem 2.1.** Let \(w^{(n)}\) be system response sequences. In order that

\[(2.17) \quad q(w^{(n)}) \to 1, \quad n \to \infty,\]
it is necessary and sufficient that

\[(2.18) \quad \|\hat{\delta}^{(n)} - \delta^{(n)}\|_2 \to 0, \quad n \to \infty,\]

where \(\hat{\omega}\) and \(t_n\) are defined in (2.11).

3. The uniqueness theorem of non-Gaussian linear processes. In this section we prove the uniqueness theorem of non-Gaussian linear processes.

Let \(H\) denote the Hilbert space of all random variables with finite variances and with inner product defined by covariance. Let \(x, u,\) and \(u'\) be random processes on \(G,\) and let \(H_{x}, H_{u}\) and \(H_{u'}\) denote the linear closed subspaces of \(H\) generated by \(x, u,\) and \(u', t \in G,\) respectively.

**Lemma 3.1.** Let \(x\) satisfy (1.1). Then

\[H_{x} = H_{u} .\]

**Proof.** It is obvious that \(H_{x} \subset H_{u}.\) So it suffices to show that any \(y = \sum_{s \in G} v_{s} u_{s}\) in \(H_{u}\) with the property

\[(3.1) \quad \sigma^{-2} E_{t} y = \sum_{s \in G} v_{t-s} u_{s} = (w * v)_{t} = 0, \quad t \in G,\]

must be 0. We denote by \(W(\gamma)\) and \(V(\gamma)\) the Fourier transforms or the Plancherel transforms of \(w\) and \(v,\) respectively.

As the Plancherel transform, we have

\[(3.2) \quad (w * v)_{t} = \int_{T} y(t) W(\gamma) V(\gamma) \, d\gamma .\]

(3.1) and (3.2) yield

\[W(\gamma) V(\gamma) = 0, \quad (d\gamma \text{ a.s.})\]

[see Rudin (1962) pages 26 and 27]. Applying (1.3), it follows that \(V(\gamma) = 0, \text{ d}\gamma \text{ a.s.}\) Hence \(v_{s} = 0,\) for all \(s \in G,\) and \(y = 0.\)

**Theorem 3.1 (The uniqueness theorem).** Let

\[(3.3) \quad x_{t} = (w * u)_{t} = (w' * u')_{t}, \quad t \in G,\]

where \(\{u_{t}\} \text{ and } \{u'_{t}\}\) are \(i.i.d.\) and \(w\) and \(w'\) are system response sequences satisfying (1.2) and (1.3). If \(c_{m}(x_{t}) \neq 0\) for some \(m > 2,\) then

\[(3.4) \quad u'_{t} = au_{t-t_{0}}, \quad w'_{t} = \frac{1}{a} w_{t+t_{0}},\]

where \(a\) is a nonzero constant and \(t_{0}\) is an element of \(G.\)

**Proof.** By Lemma 3.1, we have

\[H_{u} = H_{x} = H_{u'}.\]
Hence, there exist sequences $c = \{c_i\}$ and $d = \{d_i\}$ such that
\begin{equation}
3.5 \quad u' = c \ast u, \quad u = d \ast u'.
\end{equation}
According to the relation (1.4) and the definition (2.1), we have
\begin{equation}
|k_m(u'_i)| = |k_m(u_i)|q(c), \quad |k_m(u'_i)| = |k_m(u_i)|q(d).
\end{equation}
Thus,
\begin{equation}
3.6 \quad q(c)q(d) = 1.
\end{equation}
It follows from (3.6) and (2.13) that
\begin{equation}
q(c) = p(c) = 1.
\end{equation}
Applying Lemma 2.2 yields
\begin{equation}
3.7 \quad c = a\delta^{(t_0)}.
\end{equation}
(3.7), (3.5) and (3.3) imply (3.4). \Box

Theorem 3.1 shows that if we ignore the scale and shift, the representation (1.1) of non-Gaussian linear processes is essentially unique.

**Corollary 3.1.** Let $\{u_i\}$ and $\{x_i\}$ be i.i.d and $x = w \ast u$. If $k_m(x_i) \neq 0$ for some $m > 2$, then
\begin{equation}
w = a\delta^{(t_0)},
\end{equation}
where $a$ is a nonzero constant and $t_0$ is an element of $G$.

The proof of the corollary is immediate.

Donoho (1981) discusses the problem of uniqueness, using the concept of a partial order which describes the relation between probability distributions of random variables. Rosenblatt (1985, 1986) studies the uniqueness under the additional assumption that $\Sigma |t| |\omega_i| < \infty$ (when $G = \mathbb{Z}$). When $G$ is any countable abelian group, we cannot make the additional assumption. Under the condition that $x_i$ has moments of all orders, Findley (1986) gives a different proof of the uniqueness result in the case $G = \mathbb{Z}$. He seems to have payed more attention to the property of Gaussian distributions (the $m$-th cumulant is zero, for all $m \geq 3$) and overlooked the fact that his proof only needs the condition, as does ours. At any rate, from the proof of Theorem 3.1, we see that natural peakedness is a simple and powerful instrument.

### 4. Maximum standardized cumulant deconvolution

We have defined the maximum standardized $m$-th cumulant deconvolution operator in Section 1. Now we give the convergence theorem of maximum standardized cumulant deconvolution.

**Theorem 4.1.** Let $c_m(x_i) \neq 0$, for some $m > 2$, and let $h^{(n)}$ be maximum standardized $m$-th cumulant deconvolution operators of $x$. Let $t_n$ be an element of $G$ such that $|(h^{(n)} \ast w)_{t_n}| = \max_{t \in G} |(h^{(n)} \ast w)_{t}|$, and set $\alpha_n = 1/(h^{(n)} \ast w)_{t_n}$. 


Then
\begin{equation}
\lim_{n \to \infty} E\left| a_n (\delta^{(-t_0)} \ast h^{(n)} \ast x)_t - u_t \right|^2 = 0.
\end{equation}

PROOF. From (1.4), (1.6) and (2.1), \( h^{(n)} \) satisfies
\begin{equation}
q(h^{(n)} \ast w) = \max_{g^{(n)}} q(g^{(n)} \ast w).
\end{equation}

It follows from Lemma 3.1 that there exist \( l_n \in G \) and sequences \( \tilde{g}^{(n)} = (\tilde{g}_t^{(n)})_{t \in G} \) satisfying \( \tilde{g}_t^{(n)} = 0 \) if \( t \not\in l_n + S_n \), such that
\begin{equation}
\begin{aligned}
& E\left| (\tilde{g}^{(n)} \ast x)_t - u_t \right|^2 = E\left| (\tilde{g}^{(n)} \ast w \ast u)_t - (\delta^{(0)} \ast u)_t \right|^2 \\
& Eu_t^2 \| \tilde{g}^{(n)} \ast w - \delta^{(0)} \|_2^2 \to 0, \quad n \to \infty.
\end{aligned}
\end{equation}

Applying Lemma 2.3, we have
\[ \lim_{n \to \infty} p(\tilde{g}^{(n)} \ast w) = 1. \]
By Lemma 2.6,
\[ \lim_{n \to \infty} q(\tilde{g}^{(n)} \ast w) = 1. \]

Note that \( \delta^{(t_0)} \ast \tilde{g}^{(n)} \) satisfies \( (\delta^{(t_0)} \ast \tilde{g}^{(n)})_t = 0 \) if \( t \not\in G \). From (2.13) and (4.2) we get
\[ q(\tilde{g}^{(n)} \ast w) = q(\delta^{(t_0)} \ast \tilde{g}^{(n)} \ast w) \leq q(h^{(n)} \ast w) \leq 1. \]

Therefore,
\[ \lim_{n \to \infty} q(h^{(n)} \ast w) = 1. \]

From Theorem 2.1,
\begin{equation}
\lim_{n \to \infty} \left\| a_n (\delta^{(-t_0)} \ast h^{(n)} \ast w) - \delta^{(0)} \right\|_2 = 0.
\end{equation}

(4.4) and (4.3) yield (4.1). \( \square \)

Theorem 4.1 shows that when some \( m \)-th cumulant of the process is not equal to zero, we can extract the driving noise and the system response sequence only from a non-Gaussian linear process.

We now turn to the autoregressive model on \( G \).

A linear process \( x \) satisfying (1.1) is regarded as obeying an autoregressive model on \( G \) if there exists a finite set \( F \subset G \) and a sequence \( a = (a_t) \) satisfying \( a_t = 0 \) if \( t \not\in F \) such that \( a \ast w = \delta^{(0)} \). Such an \( x \) is denoted by \( AR(F); a = (a_t) \) is said to be the sequence of autoregressive coefficients of \( x \).

We take an integer \( n_0 \) and \( t_0 \in G \) such that \( F \subset t_0 + S_{n_0} \). The following theorem gives the relation between cumulant deconvolution operator and autoregressive coefficients.
Theorem 4.2. Let $x$ be AR($F$), let $a = \{a_i\}$ be the autoregressive coefficients of $x$ and let $t_0$ and $n_0$ satisfy $F \subseteq t_0 + S_{n_0}$. Let $c_{m}(x) \neq 0$ for some $m > 2$ and let $h^{(n_0)}$ be the maximum standardized $m$-th cumulant deconvolution operator. Then

$$h^{(n_0)} = \lambda \delta^{(t_1)} * a,$$

where $t_1$ is an element of $G$ and $\lambda$ is a nonzero constant.

Proof. From the definition of AR($F$), we know that $a * w = \delta^{(0)}$. Then,

$$q(a * w) = 1.$$  

We note that $\delta^{(t_0)} * a$ satisfies $(\delta^{(t_0)} * a)_t = 0$ if $t \notin S_{n_0}$. By (4.2) and (2.13),

$$q(a * w) = q(\delta^{(t_0)} * a * w) \leq q(h^{(n_0)} * w) \leq 1.$$  

Hence,

$$q(h^{(n_0)} * w) = 1.$$  

It follows from (2.13) and Lemma 2.2 that

$$h^{(n_0)} * w = \lambda \delta^{(t_1)}.$$  

Since

$$\delta^{(t_1)} = \delta^{(t_1)} * \delta^{(0)} = \delta^{(t_1)} * a * w,$$

we have

$$h^{(n_0)} * w = \lambda \delta^{(t_1)} * a * w.$$  

The condition (1.3) and the relation (4.6) yield (4.5). \qed

Theorem 4.2 shows that maximum standardized cumulant deconvolution operators for autoregressive processes are just rescaled and shifted versions of the autoregressive coefficients.

Finally, we point out that maximum standardized cumulant deconvolution is a nonlinear problem. It is possible to find a good algorithm by combining maximum standardized cumulant deconvolution and an autoregressive model and choosing a suitable initialization procedure.

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References


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