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EDGEWORTH SERIES FOR LATTICE DISTRIBUTIONS[1]

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This paper investigates the use of Edgeworth expansions for approximating the distribution function of the normalized sum of n independent and identically distributed lattice-valued random variables. We prove that the continuity-corrected Edgeworth series, using Sheppard-adjusted cumulants, is accurate to the same order in n as the usual Edgeworth approximationfor continuous random variables. Finally, as a partial justification of the Sheppard adjustments, it is shown that if a continuous random variable Y is rounded into a discrete part D and a truncation error U, such that $Y = D + U$, then under suitable limiting conditions the truncation error is approximately uniformly distributed and independent of Y, but not independent of D.

1. Introduction. Suppose that $X_1, \ldots, X_n$ are independent and identically distributed random variables having finite cumulants $\kappa_1, \ldots, \kappa_r$ up to order $r \geq 2$. Define the standardized sum

$$S_n = \frac{X_1 + \cdots + X_n - n\kappa_1}{\sqrt{n}},$$

so that S_n has mean zero, variance κ_2 and higher-order cumulants $\kappa_\nu^n = \kappa_\nu / n^{\nu/2-1}$ for $2 \leq \nu \leq r$. Let $\kappa^n = (0, \kappa_2^n, \ldots, \kappa_r^n)$ represent these cumulants of S_n. We aim in this paper to produce a simple approximation to the cumulative distribution function

$$F_n(t) = \mathrm{pr}(S_n \leq t)$$

in the case where the components X_i are discrete random variables taking values on a lattice with span d. The approximation is required to have properties similar to those of the usual Edgeworth series, namely that, for large n, the error should be of order $o(n^{1-r/2})$ uniformly in t.

Evidently $F_n(t)$ is discontinuous with jumps of order $O(n^{-1/2})$ at the support points of S_n. Consequently, we should not expect any continuous function to approximate $F_n(t)$ with uniform accuracy in any nontrivial interval of t-values. Esseen's (1945) approximation is the sum of a continuous part, which is an ordinary (continuous) Edgeworth series, and a discontinuous part that is periodic with period equal to $d/\sqrt{n}$. To be specific, Esseen has shown that, for large n,

$$F_n(t) = E_{n,r}(t;\kappa) + D_{n,r}(t;\kappa) + o(n^{1-r/2}) \tag{1}$$

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uniformly in t. In Esseen's approximation $E_{n,r}(t;\kappa)$ is the usual integrated Edgeworth series with cumulants $\kappa_2, \kappa_3/n^{1/2}, \ldots, \kappa_r/n^{r/2-1}$. The discontinuous part $D_{n,r}(t;\kappa)$, which is periodic, is given in Section 2.

The purpose of the present paper is to show that the discontinuous part in Esseen's approximation can be incorporated into the continuous part by means of a suitable adjustment of the cumulants. To be precise, we show that if t is a lattice point for S_n, then

$$E_{n,r}(t^+;\gamma) = E_{n,r}(t^+;\kappa) + D_{n,r}(t^+;\kappa) + o(n^{1-r/2}), \tag{2}$$

where $t^+ = t + d/2\sqrt{n}$ is the continuity-corrected point and

$$\gamma_s = \begin{cases} \kappa_s & \text{for } s > 1 \text{ and odd,} \\ \kappa_s - \varepsilon_s d^s/n & \text{for } s \text{ even,} \end{cases}$$

are the Sheppard-corrected cumulants. The adjustment ε_s is the sth cumulant of a uniform random variable U on $[-\frac{1}{2}, \frac{1}{2}]$. In particular, $\varepsilon_2 = \frac{1}{12}$ and $\varepsilon_4 = -\frac{1}{120}$. The continuity correction is in fact the Sheppard correction of order 1.

It is most important to emphasize at the outset that the adjusted cumulants are *not* simply related to the cumulants of the sum $S_n + dU/\sqrt{n}$ if S_n and U are regarded as independent. This sum has an absolutely continuous distribution, but, because of the discontinuities in the density, it cannot be approximated with the required asymptotic accuracy by means of a continuous Edgeworth series. It may at first seem rather counter-intuitive, but in fact the Sheppard-adjusted cumulants

$$\gamma_s^n = \frac{\gamma_s}{n^{s/2-1}} = \frac{\kappa_s}{n^{s/2-1}} - \frac{\varepsilon_s d^s}{n^{s/2}}$$

are the cumulants of S_n *minus* the cumulants of $dU/\sqrt{n}$. Let γ^n represent the collection of these cumulants. Thus, the naive adjustments, apart from the continuity correction, are of the right magnitude, but in precisely the wrong direction. The correct adjustment, at least to order $O(n^{-1})$, would be obtained if we were to take $\operatorname{cov}(S_n, U) = -\operatorname{var}(U)d/n^{1/2}$, but the justification for this assumption is not immediately apparent. An argument is presented in Section 3, justifying this choice.

2. Edgeworth series. Let the random variables X_i and S_n be defined as in Section 1. We begin by formally defining the Edgeworth series to an arbitrary order. Let $P[z]$ be the set of formal power series $\sum_{j=0}^\infty \alpha_j z^j$, in which the coefficients α_j are polynomials in $1/\sqrt{n}$. Consider this to be the set of conceivable moment generating functions, convergent or not; the α_j are the possible cumulants. Let $\psi_{r,t}$: $P[z] \to \mathbf{R}$ be the function that maps $\sum_{j=0}^\infty \alpha_j z^j$ to the series

$$\alpha_0 \Phi(t, \kappa_2) - \phi(t, \kappa_2) \sum_{j=1}^{\infty} \alpha_j h_{j-1}(t, \kappa_2),$$

evaluated at t and discarding all terms of order $o(n^{1-r/2})$. Here $\Phi(t;\kappa_2)$ and $\phi(t;\kappa_2)$ are the normal integral and density respectively, both using mean zero and variance κ_2. The Hermite polynomials $h_r(t;\kappa_2)$ are given by

$$h_r(t;\kappa_2) = \kappa_2^{-r/2} h_r\big(t/\kappa_2^{1/2};1\big),$$

where $h_r(t;1)$ is the standard Hermite polynominal $(-1)^r\phi^{(r)}(t;1)/\phi(t;1)$. Differentiation is with respect to t. Effectively, we apply the Fourier inversion operation termwise, and discard terms that are small enough. This function represents evaluation of the Edgeworth series:

$$E_{1,r}(t,\kappa^n) = E_{n,r}(t,\kappa) = \psi_{r,t}\left(\exp\left[\sum_{\nu=3}^{r}\frac{1}{\nu!}\kappa_\nu^n z^\nu\right]\right).$$

Bhattacharya and Rao [(1976), page 215] show that under Cramér's condition,

$$F_n(t) = E_{n,r}(t;\kappa) + o(n^{1-r/2}), \tag{3}$$

uniformly in t. Unfortunately, Cramér's condition does not apply in the present case, since the random variables considered here are confined to a lattice.

Assume for the sake of algebraic simplification that X_i are integer-valued with lattice span $d = 1$ and that $\kappa_1 = 0$. It follows then that S_n is supported on the points $0, \pm n^{-1/2}, \pm 2n^{-1/2}, \dots$. Esseen's series for $D_{n,r}(t;\kappa)$ may then be written in the form

$$D_{n,r}(t;\kappa) = \sum_{\nu=1}^{r-2} g_\nu n^{-\nu/2} Q_\nu(n^{1/2}t) E_{n,r}^{(\nu)}(t;\kappa), \tag{4}$$

where $E_{n,r}^{(\nu)}(t;\kappa)$ is the νth derivative of the Edgeworth series, and

$$g_\nu = \begin{cases} +1 & \text{if } \nu = 4k+1 \text{ or } 4k+2,\\ -1 & \text{if } \nu = 4k-1 \text{ or } 4k,\end{cases}$$

$$Q_{2\nu}(x) = \sum_{j=1}^{\infty}\frac{\cos(2\pi jx)}{2^{2\nu-1}(\pi j)^{2\nu}},$$

$$Q_{2\nu+1}(x) = \sum_{j=1}^{\infty}\frac{\sin(2\pi jx)}{2^{2\nu}(\pi j)^{2\nu+1}}.$$

Evidently, $Q_{2\nu}(x)$ is symmetric about $x = \frac{1}{2}$, whereas $Q_{2\nu+1}(x)$ is antisymmetric and $Q_{2\nu+1}(\frac{1}{2}) = 0$. Also, $g_{2\nu} = (-1)^{\nu+1}$. At continuity-corrected points only the even-numbered terms contribute, giving

$$D_{n,r}(t^+;\kappa) = \sum_{\nu=1}^{[r/2-1]} (-1)^{\nu+1} n^{-\nu} Q_{2\nu}\big(\tfrac{1}{2}\big) E_{n,r}^{(2\nu)}(t^+;\kappa). \tag{5}$$

Note that this series decreases in whole powers of n. Bhattacharya and Rao [(1976), page 238] give a proof of Esseen's result that if X is confined to the lattice with unit spacing almost surely, then

$$F_n(t) = E_{n,r}(t;\kappa) + D_{n,r}(t;\kappa) + o(n^{1-r/2}) \tag{6}$$

uniformly for $t \in \mathbf{R}$, where $D_{n,r}$ is defined in (4).

We now prove the claim made in (2) for general r. That is, we define the Edgeworth series using Sheppard-corrected cumulants:

$$E_{1,r}(t,\gamma^n) = \psi_{r,t}\left(\exp\left[\sum_{\nu=3}^{r}\frac{1}{\nu!}\gamma_\nu^n z^\nu\right]\right),$$

the Edgeworth series with the $\kappa_1^n,\dots,\kappa_r^n$ replaced by $\gamma_1^n,\dots,\gamma_r^n$, and the resulting terms of order $o(n^{1-r/2})$ deleted. We claim that, when evaluated at continuity-corrected points, expression (6) differs from $E_{1,r}(x;\gamma^n)$ by an amount no larger than $o(n^{1-r/2})$.

THEOREM. *$E_{1,r}(t^+,\gamma^n) = E_{1,r}(t^+;\kappa^n) + D_{n,r}(t^+;\kappa) + o(n^{1-r/2})$ uniformly over continuity-corrected lattice points t^+.*

PROOF. Note that $\psi_{r,t}$ has the following linearity property: $\psi_{r,t}(\alpha p(z) + \beta q(z)) = \alpha\psi_{r,t}(p(z)) + \beta\psi_{r,t}(q(z))$. Also, $\psi_{r,t}(z^s p(z)) = (-1)^s D^s \psi_{r,t}(p(z))$. Then,

$$\begin{aligned}
&E_{1,r}(t^+;\kappa^n) + D_{n,r}(t^+;\kappa)\\
&\quad= \sum_{s=0}^{\infty} g_s n^{-s/2} Q_s(t^+)(-1)^s D^s E_{n,r-s}(t^+;\kappa^n)\\
&\quad= \sum_{s=0}^{\infty} g_s n^{-s/2} Q_s(t^+)(-1)^s D^s \psi_{r-s,t^+}\left(\exp\left[\sum_{\nu=3}^{r}\frac{1}{\nu!}\kappa_\nu^n z^\nu\right]\right).
\end{aligned}$$

Now, if t^+ is a continuity-corrected point, $Q_s(t^+) = g_s B_s(\frac{1}{2})/s!$, and hence

$$\begin{aligned}
E_{1,r}(t^+;\kappa^n) + D_{n,r}(t^+;\kappa) &= \sum_{s=0}^{\infty} n^{-s/2}\frac{1}{s!}B_s(\tfrac{1}{2})\psi_{r,t^+}\left(z^s\exp\left[\sum_{\nu=3}^{r}\frac{1}{\nu!}\kappa_\nu^n z^\nu\right]\right)\\
&= \psi_{r,t^+}\left(\sum_{s=0}^{\infty} n^{-s/2}\frac{1}{s!}B_s(\tfrac{1}{2})z^s\exp\left[\sum_{\nu=3}^{r}\frac{1}{\nu!}\kappa_\nu^n z^\nu\right]\right)\\
&= \psi_{r,t^+}\left(\exp\left[\sum_{l=2}^{\infty}\frac{-B_l}{l l!}n^{-l/2}z^l\right]\exp\left[\sum_{\nu=3}^{r}\frac{1}{\nu!}\kappa_\nu^n z^\nu\right]\right),
\end{aligned}$$

since

$$\sum_{s=0}^{\infty} n^{-s/2}\frac{1}{s!}B_s(\tfrac{1}{2})z^s = \exp\left[\sum_{l=2}^{\infty}\frac{-B_l}{l l!}n^{-l/2}z^l\right].$$

Here B_s and $B_s(x)$ are Bernoulli numbers and polynomials, respectively. Sheppard's correction for the cumulant of order l is $-B_l/(ln^l)$. Let γ^n be the cumulants κ adjusted by Sheppard's correction. Hence,

$$E_{1,r}(t^+;\kappa^n) + D_{n,r}(t^+;\kappa) = \psi_{r,t^+}\left(\exp\left[\sum_{\nu=3}^{r}\frac{1}{\nu!}\gamma_\nu^n z^\nu\right]\right) = E_{1,r}(t^+,\gamma^n).$$

This completes the proof. □

3. On the effect of rounding. Let Y be a continuous random variable with density function $f(y)$ on the real line. Suppose that $D = \varepsilon\langle Y/\varepsilon\rangle$ is Y rounded to the nearest integer multiple of ε. Here $\langle y\rangle$ denotes the integer nearest to y. The standardized difference or standardized rounding error $U = (Y - D)/\varepsilon$ is a random variable on the interval $(-\frac{1}{2}, \frac{1}{2}]$. Evidently, U is a deterministic function of Y but not of D. Despite this deterministic relationship, we show here that under certain conditions on the derivatives of $f(\cdot)$, Y and U are statistically independent to a high order of approximation for small ε. By contrast, D and U are not independent beyond a crude first order approximation.

LEMMA. *Suppose that the first 2ν derivatives of $f(y)$ are integrable over $(-\infty, \infty)$. Then the joint characteristic function $\psi(\alpha, \beta)$ of (Y, U) satisfies*

$$\psi(\alpha, \beta) = \theta(\alpha)\phi(\beta) + O(\varepsilon^{2\nu}) \tag{7}$$

for small ε. Moreover, the characteristic function $\phi(\beta)$ of U satisfies

$$\phi(\beta) = \frac{\sinh(\frac{1}{2}i\beta)}{\frac{1}{2}i\beta} + O(\varepsilon^{2\nu}),$$

showing that Y and U are asymptotically independent and U is uniformly distributed to the order indicated.

As a corollary, $\operatorname{cov}(U, D) = -\varepsilon/12 + O(\varepsilon^{2\nu})$.

The Edgeworth series satisfies the conditions of the lemma for all ν with $\varepsilon = d/\sqrt{n}$. Consequently, the error in (7) is smaller than any power of n, and may be called exponentially small.

PROOF. Expression (7), apart from the order of the error, has been derived by Wold (1934). The magnitude of the error can be determined using the method of Cramér [(1945), page 360]. □

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