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CANONICAL PARTIAL AUTOCORRELATION FUNCTION OF A MULTIVARIATE TIME SERIES

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We propose a definition of the partial autocorrelation function $\beta(\cdot)$ for multivariate stationary time series suggested by the canonical analysis of the forward and backward innovations. Here $\beta(\cdot)$ satisfies $\beta(-n) = \beta(n)'$, $n = 0, 1, \ldots,$ where $\beta(0)$ is nonnegative definite, $\{\beta(n), n = 1, 2, \ldots\}$ is a sequence of square matrices having singular values less than or equal to 1 and such that the order of $\beta(n+1)$ is equal to the rank of $I - \beta(n)\beta(n)'$, the order of $\beta(1)$ being equal to the rank of $\beta(0)$. We show that there exists a one-to-one correspondence between the set of matrix autocovariance functions $\Lambda(\cdot)$, with the positive definiteness property, and the set of canonical partial autocorrelation functions $\beta(\cdot)$ as described above.

1. Introduction and notation. In the scalar case, the parametrization of an autocovariance function $\Lambda(\cdot)$ by a sequence of partial autocorrelations $\beta(\cdot)$ was first established by Barndorff-Nielsen and Schou (1973) for autoregressive (AR) models. It was extended independently by Ramsey (1974) and Burg (1975) to the general situation. This correspondence is a well-known result of orthogonal polynomial theory [see Geronimus (1960)]. The attractive property of $\beta(\cdot)$ is that its variation domain is unconstrained, which gives rise to new estimation techniques: maximum entropy method [Burg (1975)], recursive or exact maximum likelihood estimation of AR processes [Kay (1983) and Dégerine and Pham (1987)]. See also Atal (1977), Dickinson (1978) and Dégerine (1987).

$\beta(\cdot)$ appears naturally in the well-known Levinson–Durbin algorithm for fitting AR models of increasing orders to a given $\Lambda(\cdot)$. Whittle (1963) extended this algorithm to the multivariate nondegenerate case and Inouye (1983, 1985) considered the general situation. However, the AR filters giving the forward and backward innovations are now different, because of the noncommutativity of the matrix product, and $\beta(\cdot)$ is no longer naturally defined. Morf, Vieira and Kailath (1978) proposed a normalized version of this extended algorithm and so obtained a possible definition of $\beta(\cdot)$. Sakai (1983), using a Levinson-type circular recursive algorithm, characterized $\Lambda(\cdot)$ by a set of sequences of scalar partial autocorrelation coefficients whose magnitudes are all less than 1. Estimation techniques of $\beta(\cdot)$, in the multivariate case, were proposed by Morf, Vieira, Lee and Kailath (1978) and Dickinson (1979).

Let $X(\cdot) = \{X(t), t \in \mathbb{Z}\}$ be a zero-mean real m-variate stationary time series with autocovariance function $\Lambda(\cdot)$,

$$\Lambda(t-s) = E\{X(t)X(s)'\}, \qquad (t,s) \in \mathbb{Z}^2.$$

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$\Lambda(\cdot)$ characterizes the structure of the real Hilbert space generated by $X(\cdot)$,

$$\mathscr{M} = \overline{\mathscr{L}}\{X_j(t), j = 1, \dots, m, t \in \mathbb{Z}\}.$$

Elements of $\mathscr{M}$ are zero-mean $\mathbb{R}$-valued variables, so the inner product in $\mathscr{M}$ is just defined by $\langle U, V \rangle = E\{UV\}$. When the dimension of $\mathscr{M}$ is finite, $X(\cdot)$ is said to be *linearly singular of order* d, where d is the smallest integer k for which $X(\cdot)$ satisfies the stochastic difference equation

$$\sum_{j=0}^{k} b(j) X(t-j) = 0, \qquad t \in \mathbb{Z},$$

$b(0)$ being the m-identity matrix I_m.

We define, for all t of $\mathbb{Z}$, the following subspaces of $\mathscr{M}$:

$$\mathscr{M}(t; n) = \mathscr{L}\{X_j(s), j = 1, \dots, m, t - n < s \le t\},$$
$$n \in \mathbb{N}^*, \mathscr{M}(t; 0) = \{0\}.$$

$X(t; n)$ denotes the best approximate of $X(t)$ by its nearest past of length n: Each component $X_j(t; n)$, $j = 1, \dots, m$, is the orthogonal projection of $X_j(t)$ on $\mathscr{M}(t-1; n)$. With the convention $X(t; 0) = 0$, the *nth-order forward innovation* is defined as

$$(1)\quad \varepsilon(t; n) = X(t) - X(t; n) = \sum_{k=0}^{n} b(n, k) X(t-k), \qquad n \in \mathbb{N}, t \in \mathbb{Z},$$

where $b(n; k)$, $k = 1, \dots, n$, satisfy

$$(2)\qquad \sum_{k=0}^{n} b(n, k) \Lambda(j-k) = 0, \qquad j = 1, \dots, n.$$

The *forward residual covariance matrices* $\sigma^2(n) = E\{\varepsilon(t; n)\varepsilon(t; n)'\}$, $n \in \mathbb{N}$, are given by

$$(3)\qquad \sigma^2(n) = \sum_{k=0}^{n} b(n, k) \Lambda(-k).$$

Notice that (2) and (3) are the Yule–Walker equations. In what follows the symbol * indicates the *backward* quantities obtained by reversing the time index in the considerations above.

When $\sigma^2(n)$ is singular, the *regression coefficients* $b(n, k)$, $k = 1, \dots, n$, in (1) are not uniquely defined. Nevertheless, $\varepsilon(t; n)$ is well defined and the *partial autocovariance function* $\delta(\cdot)$ is given by $\delta(0) = E\{X(t)X(t)'\}$ and

$$\delta(-n)' = \delta(n) = E\{\varepsilon(t; n-1)\varepsilon^*(t-n; n-1)'\}, \qquad n \in \mathbb{N}^*.$$

A definition of $\beta(\cdot)$ will be a normalized version of $\delta(\cdot)$:

$$\beta(n) = \left[\sigma^2(n-1)\right]^{-1/2} \delta(n) \left\{\left[\sigma^{*2}(n-1)\right]^{-1/2}\right\}', \qquad n \in \mathbb{N}^*,$$

but the difficulty is: Which matrix square root must be used? Morf, Vieira and Kailath (1978) consider, for positive-definite (p.d.) matrix R, a lower triangular matrix $R^{1/2}$ such that $R^{1/2}(R^{1/2})' = R$. We can see that any other definition of $R^{1/2}$, provided that it is unique, gives rise to a possible definition of $\beta(\cdot)$. Moreover, matrix square roots are used to normalize the successive innovations, and so to any choice corresponds a normalized innovation with a different interpretation.

In the present paper a square root $\sigma(n)$ of $\sigma^2(n)$ and a generalized inverse $\sigma(n)^-$ are defined recursively. It is seen that the matrix correlation coefficient $\beta(n) = \sigma(n-1)^- \delta(n)[\sigma^*(n-1)^-]'$ has a singular value decomposition $\beta(n) = L_n' \Delta_n \Theta_n M_n$ in which L_n, M_n and Θ_n are orthogonal matrices such that Θ_n commutes with Δ_n and can be seen as a generalized sign. Δ_n is the diag-
onal matrix associated with the canonical correlations between $\varepsilon(t; n-1)$ and $\varepsilon^*(t-n; n-1)$. We prove also that $\{(L_n, \Delta_n), n \geq 1\}$ [resp. $\{(M_n, \Delta_n), n \geq 1\}$], together with $\Lambda(0)$, is in a one-to-one correspondence with $\{\sigma^2(n), n \geq 0\}$ [resp. $\{\sigma^{*2}(n), n \geq 0\}$]. Finally we note that the normalized innovations $\eta(t; n) = \sigma(n)^- \varepsilon(t; n)$ and $\eta^*(t-n; n) = \sigma^*(n)^- \varepsilon^*(t-n; n)$ are the canonical variables between $\varepsilon(t; n)$ and $\varepsilon^*(t-n; n)$ and that the canonical correlations are also given by Δ_n.

Notice that our definition of $\beta(n)$ is always valid and stops when the rank of $\sigma^2(n)$ is equal to 0 while that of Sakai and of Morf, Vieira and Kailath stop as soon as $\sigma^2(n)$ is singular.

2. Standard partial autocorrelation functions. For a matrix A let A^- denote any generalized inverse (g-inverse) of A, that is to say any matrix A^- such that $AA^-A = A$ [cf. Rao (1965), page 24].

THEOREM 1. *Innovations and partial autocovariances satisfy the following recursions*:

$$(4)\qquad \begin{aligned} \varepsilon(t; n) &= \varepsilon(t; n-1) - \delta(n)\sigma^{*2}(n-1)^- \varepsilon^*(t-n; n-1),\\ \varepsilon^*(t; n) &= \varepsilon^*(t; n-1) - \delta(n)'\sigma^2(n-1)^- \varepsilon(t+n; n-1), \end{aligned} \qquad n \in \mathbb{N}^*, t \in \mathbb{Z},$$

where the residual covariance matrices are given by

$$(5)\qquad \begin{aligned} \sigma^2(n) &= \sigma^2(n-1) - \delta(n)\sigma^{*2}(n-1)^- \delta(n)',\\ \sigma^{*2}(n) &= \sigma^{*2}(n-1) - \delta(n)'\sigma^2(n-1)^- \delta(n), \end{aligned} \qquad n \in \mathbb{N}^*.$$

PROOF. We suppose that the recursion is valid up to order $(n-1)$, $n \in \mathbb{N}^*$. The first relation in (4) holds if

$$E\{\varepsilon(t; n-1)X(t-n)'\} = \delta(n)\sigma^{*2}(n-1)^- \sigma^{*2}(n-1),$$

which is equivalent to $\delta(n) = \delta(n)A^-A$ with $A = \sigma^{*2}(n-1)$. The matrix A being symmetric, $(A^-)'$ is a g-inverse of A and the equality comes from the definition of $\delta(n)$ if, for $\varepsilon^* = \varepsilon^*(t-n; n)$, we prove that $\varepsilon^* = A(A^-)'\varepsilon^*$ a.s. Using the symmetry of A, it is derived, as in the proof of Lemma 6 of Inouye

(1983), from

$$E\{\|\varepsilon^* - A(A^-)'\varepsilon^*\|^2\} = A - A(A^-)'A - AA^-A + A(A^-)'AA^-A = 0.$$

$\varepsilon(t;n)$ and $\varepsilon^*(t-n;n-1)$ are now uncorrelated so we have

$$\sigma^2(n) = \sigma^2(n-1) - \delta(n)A^-AA^-\,\delta(n)',$$

which gives the first equality in (5) since $\delta(n)A^-A = \delta(n)$. The second relations in (4) and (5) can be proved in a similar way. □

The Levinson–Durbin algorithm. When g-inverses are uniquely defined, using Theorem 1, the associated regression coefficients are given by

$$\begin{aligned}
&[b(n;0),\dots,b(n,n)]\\
&\quad= [b(n-1,0),\dots,b(n-1,n-1),0]\\
&\qquad - \delta(n)\sigma^{*2}(n-1)^-\,[0,b^*(n-1,n-1),\dots,b^*(n-1,0)],\\
&[b^*(n;0),\dots,b^*(n,n)]\\
&\quad= [b^*(n-1,0),\dots,b^*(n-1,n-1),0]\\
&\qquad - \delta(n)'\sigma^2(n-1)^-\,[0,b(n-1,n-1),\dots,b(n-1,0)],
\end{aligned}$$

where the covariance matrices are updated by (5) and

$$\delta(n) = \sum_{k=0}^{n-1} b(n-1,k)\Lambda(n-k) = \left\{\sum_{k=0}^{n-1} b^*(n-1,k)\Lambda(k-n)\right\}',$$

with $b(0,0) = b^*(0,0) = I$ and $\sigma^2(0) = \sigma^{*2}(0) = \Lambda(0)$ as starting values.

This algorithm was given by Whittle (1963) for regular residual covariance matrices. In the singular case if we use, for uniqueness, the Moore inverse [cf. Rao (1965) page 25], we recognize the algorithm of Inouye (1985) and the forward coefficients $b(n,k)$ are the AR parameters given by the algorithm of Inouye (1983). These two algorithms are proved directly on the regression coefficients, using the Yule–Walker equations (2) and (3). Nevertheless, the choice of the Moore inverse is not determining and one could lead a similar proof of our Theorem 1.

It is easy to see that such an algorithm provides a one-to-one correspondence between $\Lambda(\cdot)$ and $\delta(\cdot)$. But $\delta(\cdot)$ must satisfy conditions in order that (5) well defines nonnegative definite (n.n.d.) matrices. Partial autocorrelation functions eliminate this problem.

Standard orthogonalization process. We call so a process that associates to any n.n.d. matrix σ^2 two uniquely defined matrices σ and σ^-, where σ is a square root of σ^2, $\sigma\sigma' = \sigma^2$ and σ^- is a g-inverse of σ such that $\sigma^-\sigma = h$ is a diagonal idempotent matrix. Then, if ε is a zero-mean random vector with covariance matrix σ^2, $\eta = \sigma^-\varepsilon$ gives an orthogonalization of the components of ε with $E\{\eta\eta'\} = h$ and $\varepsilon = \sigma\eta$. This last equality comes from $\sigma\eta = \sigma\sigma^-\varepsilon = \varepsilon$ which is proved by using $\sigma\sigma^-\sigma^2 = \sigma\sigma^-\sigma\sigma' = \sigma\sigma' = \sigma^2$ in

$$E\{\|\varepsilon - \sigma\sigma^-\varepsilon\|^2\} = \sigma^2 - \sigma\sigma^-\sigma^2 - \sigma^2(\sigma^-)'\sigma' + \sigma\sigma^-\sigma^2(\sigma^-)'\sigma'.$$

The function defined by $\beta(0) = E\{X(t)X(t)'\}$ and

$$\beta(-n)' = \beta(n) = E\{\eta(t; n-1)\eta^*(t-n; n-1)'\}, \qquad n \in \mathbb{N}^*,$$

where

$$\eta(t; n-1) = \sigma(n-1)^- \varepsilon(t; n-1),$$
$$\eta^*(t-n; n-1) = \sigma^*(n-1)^- \varepsilon^*(t-n; n-1)$$

is called the *standard partial autocorrelation function* of $X(\cdot)$.

THEOREM 2. *A standard orthogonalization process establishes a one-to-one correspondence between the associated standard partial autocorrelation function $\beta(\cdot)$ and the autocovariance function $\delta(\cdot)$ through the recursions, $n \in \mathbb{N}^*$:*

$$\beta(n) = \sigma(n-1)^- \delta(n)\left[\sigma^*(n-1)^-\right]',$$
$$\delta(n) = \sigma(n-1)\beta(n)\sigma^*(n-1)',$$
$$\sigma^2(n) = \sigma(n-1)[I - \beta(n)\beta(n)']\sigma(n-1)',$$
$$\sigma^{*2}(n) = \sigma^*(n-1)[I - \beta(n)'\beta(n)]\sigma^*(n-1)',$$

with $\sigma^2(0) = \sigma^{*2}(0) = \beta(0) = \delta(0)$ as starting values.

PROOF. Relations between $\beta(n)$ and $\delta(n)$ are straightforward. From (5) and $\sigma(n-1)\sigma(n-1)' = \sigma^2(n-1)$ we obtain

$$\sigma^2(n) = \sigma(n-1)\big[I - \beta(n)\sigma^*(n-1)' \times\sigma^{*2}(n-1)^- \sigma^*(n-1)\beta(n)'\big]\sigma(n-1)'.$$

Putting $h = \sigma(n-1)^-\sigma(n-1)$, we have $h\eta(t; n-1) = \eta(t; n-1)$ and then $h\beta(n) = \beta(n)$. We have also $\beta(n)h^* = \beta(n)$, where $h^* = \sigma^*(n-1)^-\sigma^*(n-1)$. Neglecting $(n-1)$ arguments, we can write

$$\beta(n)\sigma^{*\prime}\sigma^{*2-}\sigma^* = \beta(n)h^*\sigma^{*\prime}\sigma^{*2-}\sigma^* = \beta(n)\sigma^{*-}\sigma^{*2}\sigma^{*2-}\sigma^*$$
$$= \beta(n)\sigma^{*-}\sigma^* = \beta(n)h^* = \beta(n)$$

and the expression of $\sigma^2(n)$ is proved. That of $\sigma^{*2}(n)$ is proved in the same way. □

The main constraint that $\beta(\cdot)$ must satisfy is that, for $n \in \mathbb{N}^*$, the *singular values* of $\beta(n)$ [positive square root of the eigenvalues of $\beta(n)\beta(n)'$] are less than or equal to 1. Furthermore, the variation domain of $\beta(\cdot)$, which depends on the orthogonalization process, can be well described as in the following examples.

(a) *Gram–Schmidt process*. Let ε be a zero-mean real m-variate random vector with covariance matrix σ^2. The Gram–Schmidt orthogonalization process of the successive components ε_j, $j = 1, \dots, m$, of ε uses the inner product $\langle \varepsilon_j, \varepsilon_k \rangle = E\{\varepsilon_j \varepsilon_k'\}$ and defines uniquely new variables η_j, $j = 1, \dots, r$, where r is the rank of σ^2. The random vector $\eta = (\eta_1, \dots, \eta_r)'$ is zero-mean with

$E\{\eta\eta'\} = I_r$. We have $\varepsilon = \sigma\eta$, σ being a unique $m \times r$ matrix. We also have $\eta = \sigma^-\varepsilon$, where the $r \times m$ matrix σ^- is unique if η is expressed only through the r components of ε which, in the orthogonalization process, gave rise to η_j, $j = 1, \dots, r$.

(b) *Principal component process*. Let $\sigma^2 = V\Delta^2V'$ be a spectral decomposition of σ^2 in which Δ^2 is the square diagonal matrix with the eigenvalues $\delta_1 \geq \cdots \geq \delta_m$ of σ^2, including the multiplicities, as diagonal elements. Δ denotes the diagonal (not necessarily square) matrix with the positive square root of nonzero elements of Δ^2 as diagonal elements ($\Delta\Delta' = \Delta^2$) and Δ^- is the matrix obtained by replacing the nonzero elements of Δ' by their reciprocals. For a root with multiplicity, the first corresponding eigenvector is such that the number of successive zero components, starting from the last one, is maximum; the next eigenvectors are selected in the same way while preserving the orthogonalization conditions. Then V is unique if the first nonzero component of each eigenvector is positive. σ and σ^- are given by $\sigma = V\Delta$ and $\sigma^- = \Delta^-V'$.

Notice that the symmetric square root $\sqrt{\sigma^2}$ and its Moore inverse $\sqrt{\sigma^2}^{\,+}$ (which is symmetric) used in Inouye (1983) does not satisfy the constraint $\sigma^-\sigma = h$.

The normalized version of recursions (4) is

$$\begin{aligned} \eta(t;n) &= \sigma(n)^-\sigma(n-1)\{\eta(t;n-1) - \beta(n)\eta^*(t-n;n-1)\}, \\ \eta^*(t;n) &= \sigma^*(n)^-\sigma^*(n-1)\{\eta^*(t;n-1) - \beta(n)'\eta(t+n;n-1)\}. \end{aligned} \tag{6}$$

The coefficients in

$$\eta(t;n) = \sum_{k=0}^{n} B(n;k)X(t-k), \qquad \eta^*(t;n) = \sum_{k=0}^{n} B^*(n;k)X(t+k)$$

are uniquely defined by $\beta(\cdot)$ and can be given by a normalized version of the Levinson–Durbin algorithm. The associated regression coefficients come from $\varepsilon(t;n) = \sigma(n)\eta(t;n)$ and agree with those of the original algorithm in which g-inverses σ^{2-} are $(\sigma^-)'\sigma^-$.

From (6) we observe that the variation domain of $\beta(\cdot)$, further denoted by $\mathscr{D}_\beta(m)$, in the orthogonalization processes (a) and (b) is as follows. $\beta(0)$ is n.n.d.; $\{\beta(n), n = 1, 2, \dots\}$ is a sequence of square matrices having singular values less than or equal to 1 and such that the order of $\beta(n+1)$ is equal to the rank of $I - \beta(n)\beta(n)'$, the order of $\beta(1)$ being equal to the rank of $\beta(0)$. So either the sequence is not finite and the rank of $\beta(n)$ is constant from some n_0 or $X(\cdot)$ is linearly singular of order d and the sequence stops with $\beta(d)$ in which all singular values are equal to 1.

The Gram–Schmidt process leads to the definition of $\beta(\cdot)$ proposed by Morf, Vieira and Kailath (1978). Using the normalized version of the Levinson–Durbin algorithm, they prove the one-to-one correspondence between $\{\Lambda(n),\ n = 0, \dots, N\}$ and $\{\beta(n), n = 0, \dots, N\}$ as long as $\sigma^2(N)$ is nonsingular.

To any orthogonalization process corresponds a normalized innovation $\eta(t; n)$ with a specified interpretation. If we use the Gram–Schmidt process with an upper triangular matrix for σ and a lower one for σ^*, then the components of the normalized innovations are the variables which appear in the circular lattice filtering of Sakai (1983). But his characterization of $\{\Lambda(n), n = 0, \dots, N\}$ is different and also stops as soon as $\sigma^2(N)$ is singular.

3. Canonical partial autocorrelation function. Let β be a real square matrix whose singular values are less than or equal to 1. A *singular value decomposition* of β can be uniquely defined as

$$\beta = L'\Delta\Theta M = L'\Theta\Delta M,$$

where L and M are the orthogonal matrices of the spectral decompositions

$$I - \beta\beta' = L'(I - \Delta^2)L, \qquad I - \beta'\beta = M'(I - \Delta^2)M,$$

using the conventions of the principal component process. Then Δ is the square diagonal matrix whose diagonal elements are the singular values of β, including the multiplicities, arranged in increasing order. Θ is given by $L\beta M' = \Delta\Theta$ with zero elements in rows and columns of Θ corresponding to the zero elements of Δ. The matrix Θ is block diagonal and commutes with Δ; each block corresponds to a different nonzero singular value and is an orthogonal transformation on the associated eigensubspaces. If the singular values are all distinct then Θ is a diagonal matrix giving the signs of the diagonal elements of $\Delta\Theta$.

We use the following notation: $(I - \Delta^2)^{1/2}$ and $(I - \Delta^2)^{1/2-}$ are the diagonal (not necessarily square) matrices defined from $(I - \Delta^2)$ as in the principal component process. In the particular case of the spectral decomposition $\sigma^2(0) = L_0'\Delta^2(0)L_0$, Δ_0 and Δ_0^- are associated with $\Delta^2(0)$ as above. Notice that all spectral decompositions are uniquely defined because of the conventions of the principal component process.

LEMMA 1. *Let $\sigma^2(n)$, $n \in \mathbb{N}$, be the sequence of forward residual covariance matrices of a stationary process $X(\cdot)$. From the spectral decomposition $\sigma^2(0) = L_0'\Delta^2(0)L_0$ we define the starting values $\sigma(0) = L_0'\Delta_0$ and $\sigma(0)^- = \Delta_0^-L_0$ of the recursion*

$$\sigma(n) = \sigma(n-1)L_n'[I - \Delta^2(n)]^{1/2},$$
$$\sigma(n)^- = [I - \Delta^2(n)]^{1/2-}L_n\sigma(n-1)^-, \qquad n \in \mathbb{N}^*,$$

where L_n and $\Delta^2(n)$ are given by the spectral decomposition

$$\sigma(n-1)^-\sigma^2(n)[\sigma(n-1)^-]' = L_n'[I - \Delta^2(n)]L_n.$$

Then, for any $n \in \mathbb{N}$, $\sigma(n)$ and $\sigma(n)^-$ are rectangular matrices of respective order $m \times r(n)$ and $r(n) \times m$, $r(n)$ being the rank of $\sigma^2(n)$, and satisfy

(i) $\sigma(n)\sigma(n)' = \sigma^2(n)$,
(ii) $\sigma(n)^-\sigma(n) = I_{r(n)}$.

PROOF. At first note that a similar statement holds for the sequence of backward residual covariance matrices $\sigma^{*2}(n)$, $n \in \mathbb{N}$. The proof is carried out simultaneously for the two sequences. Note also that properties (i) and (ii) imply that $\sigma(n)^-$, $\sigma(n)$ and $[\sigma(n)^-]'\sigma(n)^-$ are g-inverses of $\sigma(n)$, $\sigma(n)^-$ and $\sigma^2(n)$, respectively. Obviously, $\sigma(0) = L_0' \Delta_0$ and $\sigma(0)^- = \Delta_0^- L_0$ satisfy (i) and (ii) and then $\sigma^{*2}(0)^-$ is well defined since $\sigma^{*2}(0) = \sigma^2(0)$. Now we suppose that $\sigma(n-1)$ and $\sigma(n-1)^-$ satisfy (i) and (ii) and that $\sigma^{*2}(n-1)^-$ is given. Using (5), we have

$$\sigma(n-1)^- \sigma^2(n)\left[\sigma(n-1)^-\right]' = I_{r(n-1)} - \sigma(n-1)^- \delta(n)\sigma^{*2}(n-1)^- \delta(n)'\left[\sigma(n-1)^-\right]'.$$

Then the left-hand side of the above equality is a n.n.d. matrix whose spectral decomposition can be written as $L_n'[I - \Delta^2(n)]L_n$. The recursion leads to $\sigma(n)$ and $\sigma(n)^-$ and the verification of (ii) is straightforward. For (i) we have

$$\sigma(n)\sigma(n)' = \sigma(n-1)\sigma(n-1)^- \sigma^2(n)\left[\sigma(n-1)^-\right]'\sigma(n-1)'.$$

$\sigma(n-1)\sigma(n-1)^-$ is the projection, in $\mathbb{R}^m$, onto the subspace generated by the rows (or columns) of $\sigma^2(n-1)$. Writing (5) as

$$\sigma^2(n-1) = \sigma^2(n) + \delta(n)\sigma^{*2}(n-1)^- \delta(n)',$$

the kernel of $\sigma^2(n-1)$ is included in that of $\sigma^2(n)$ and we can delete $\sigma(n-1)\sigma(n-1)^-$ and its transpose in the above expression of $\sigma(n)\sigma(n)'$. The order $m \times r(n)$ of $\sigma(n)$ comes from the inclusion of kernels just noted together with our conventions. Now $\sigma^2(n)^-$ is available for the further step in the recursion on the backward sequence. □

Lemma 1 gives an orthogonalization process of special kind: $\sigma(n)$ and $\sigma(n)^-$ are defined from $\sigma(n-1)$ and $\sigma(n-1)^-$. Nevertheless, this process, further called *canonical process*, operates as a standard process. For backward quantities we use the following notation:

$$\sigma^{*2}(0) = M_0' \Delta^2(0) M_0, \qquad M_0 = L_0,$$

$$\sigma^*(n-1)^- \sigma^{*2}(n)\left[\sigma^*(n-1)^-\right]' = M_n'\left[I - \Delta^2(n)\right]M_n, \qquad n \in \mathbb{N}^*.$$

Theorem 2 proves that Δ_n, $n \in \mathbb{N}$, are the same for the backward and forward sequences.

The standard partial autocorrelation function associated with this canonical process can be defined as follows.

DEFINITION. The canonical partial autocorrelation function $\beta(\cdot)$ of a stationary process $X(\cdot)$ is defined by $\beta(0) = E\{X(t)X(t)'\} = L_0' \Delta^2(0) L_0$ and

$$\beta(-n)' = \beta(n) = E\{\eta(t;n-1)\eta^*(t-n;n-1)'\} = L_n' \Delta_n \Theta_n M_n, \qquad n \in \mathbb{N}^*,$$

where the normalized innovations are given by the recursion

$$\eta(t;n) = \left[I - \Delta^2(n)\right]^{1/2-}\{L_n \eta(t;n-1) - \Delta_n \Theta_n M_n \eta^*(t-n;n-1)\},$$

$$\eta^*(t;n) = \left[I - \Delta^2(n)\right]^{1/2-}\{M_n \eta^*(t;n-1) - \Delta_n \Theta_n L_n \eta(t+n;n-1)\},$$

with $\eta(t;0) = \eta^*(t;0) = \Delta_0^- L_0 X(t)$ as starting values.

From now $\beta(\cdot)$ denotes the canonical function. We see that the variation domain of $\beta(\cdot)$ is $\mathscr{D}_\beta(m)$ and the one-to-one correspondence between $\Lambda(\cdot)$ and $\beta(\cdot)$ can be proved using the normalized Levinson–Durbin algorithm associated with the recursion in the definition of $\beta(\cdot)$. The following constructive process proves that the application $\Lambda(\cdot) \to \beta(\cdot)$ is one-to-one from the set of matrix autocovariance functions $\Lambda(\cdot)$, with the positive definiteness property, onto $\mathscr{D}_\beta(m)$.

Let $\{Y(n), n \in \mathbb{N}\}$ be a sequence of uncorrelated zero-mean random vectors whose covariance matrices $E\{Y(n)Y(n)'\} = \sigma^2(n)$ are associated with a given $\beta(\cdot)$ in $\mathscr{D}_\beta(m)$ by use of (i) in Lemma 1. Then we can verify that the recursion

$$X(n) = X(n;n-1) + \sigma(n-1)\beta(n)\sigma^*(n-1)^- \varepsilon^*(0;n-1) + Y(n), \qquad n \in \mathbb{N}^*,$$

starting with $X(0) = Y(0)$, well defines a stationary sequence $\{X(n), n \in \mathbb{N}\}$ having $\beta(\cdot)$ as canonical partial autocorrelation function.

From Lemma 1, we see that $\{(L_n, \Delta_n), n \in \mathbb{N}\}$ [resp. $\{(M_n, \Delta_n), n \in \mathbb{N}\}$] is in a one-to-one correspondence with $\{\sigma^2(n), n \in \mathbb{N}\}$ [resp. $\{\sigma^{*2}(n), n \in \mathbb{N}\}$]. As in the scalar case, $\beta(\cdot)$ is defined, except for signs $\{\Theta_n, n \in \mathbb{N}^*\}$, by the residual covariance matrices. We observe also the following properties.

(i) The rank of $\sigma^2(n)$ is equal to that of $\sigma^{*2}(n)$, $n \in \mathbb{N}$.
(ii) The kernel of $\sigma^2(n-1)$ is included in that of $\sigma^2(n)$, $n \in \mathbb{N}^*$.
(iii) The diagonal elements of $I - \Delta^2(n)$ are the eigenvalues of $\sigma^2(n)$ with respect to the inner product defined by $\sigma^2(n-1)^-$:

$$\sigma^2(n)\sigma^2(n-1)^-\sigma(n-1)L_n' = \sigma(n-1)L_n'\left[I - \Delta^2(n)\right],$$

$$\left[\sigma(n-1)L_n'\right]\sigma^2(n-1)^-\left[\sigma(n-1)L_n'\right] = I.$$

The covariance matrices of the random vectors

$$L_n\eta(t;n-1) = L_n\sigma(n-1)^-\varepsilon(t;n-1),$$

$$M_n\eta^*(t-n;n-1) = M_n\sigma^*(n-1)^-\varepsilon^*(t-n;n-1),$$

are equal to the identity matrix and their correlation matrix is given by

$$E\left\{L_n\eta(t;n-1)\eta^*(t-n;n-1)'M_n'\right\} = L_n\beta(n)M_n' = \Delta_n\Theta_n.$$

Then Δ_n stands for the canonical correlation between $\varepsilon(t;n-1)$ and $\varepsilon^*(t-n;n-1)$, $L_n\eta(t;n-1)$ and $M_n\eta^*(t-n;n-1)$ giving the associated canonical variables. For another orthogonalization process, the singular values of the corresponding sequence $\{\tilde{\beta}(n), n = 1,2,\dots\}$ are also given by the canonical correlations but the normalized innovations differ. In our choice the components of $\eta(t;n)$ and of $\eta^*(t-n;n)$ are the canonical variables in the canonical analysis of $\{\varepsilon(t;n), \varepsilon^*(t-n;n)\}$.

THEOREM 3. *The correlation matrix between the normalized innovations* $\eta(t;n) = \sigma(n)^{-}\varepsilon(t;n)$ *and* $\eta^*(t-n;n) = \sigma^*(n)^{-}\varepsilon^*(t-n;n)$ *is given by*

$$E\{\eta(t;n)\eta^*(t-n;n)'\} = -\big[I - \Delta^2(n)\big]^{1/2-}\Delta_n\Theta_n\big[I - \Delta^2(n)\big]^{1/2}.$$

PROOF. The following equalities lead to the result:

$$\begin{aligned}
&E\{\eta(t;n)\eta^*(t-n;n)'\}\\
&\quad = \sum_{k=0}^{n} E\{B(n,k)X(t-k)\eta^*(t-n;n)'\}\\
&\quad = E\{B(n,n)X(t-n)\eta^*(t-n;n)'\}\\
&\quad = E\{B(n;n)\varepsilon^*(t-n;n)\eta^*(t-n;n)'\}\\
&\quad = E\{B(n;n)\varepsilon^*(t-n;n)\varepsilon^*(t-n;n)'\big[\sigma^*(n)^{-}\big]'\}\\
&\quad = -\big[I-\Delta^2(n)\big]^{1/2-}\Delta_n\Theta_n M_n B^*(n-1;0)\sigma^*(n)\sigma^*(n)'\big[\sigma^*(n)^{-}\big]'\\
&\quad = -\big[I-\Delta^2(n)\big]^{1/2-}\Delta_n\Theta_n M_n\sigma^*(n-1)^{-}\sigma^*(n-1)'M_n'\big[I-\Delta^2(n)\big]^{1/2}\\
&\quad = -\big[I-\Delta^2(n)\big]^{1/2-}\Delta_n\Theta_n\big[I-\Delta^2(n)\big]^{1/2}. \qquad \square
\end{aligned}$$

The above correlation matrix is the matrix obtained from $-\Delta_n\Theta_n$ by deleting rows and columns corresponding to the singular values equal to one in Δ_n.

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REFERENCES

ATAL, B. S. (1977). On determining partial correlation coefficients by the covariance method of linear prediction. Presented at the Ninety-Fourth Meeting, Acoust. Soc. Amer.

BARNDORFF-NIELSEN, O. and SCHOU, G. (1973). On the parametrization of autoregressive models by partial autocorrelations. *J. Multivariate Anal.* **3** 408–409.

BURG, J. P. (1975). Maximum entropy spectral analysis. Ph.D. dissertation, Dept. Geophys., Stanford Univ.

DÉGERINE, S. (1987). Maximum likelihood estimation of autocovariance matrices from replicated short time series. *J. Time Series Anal.* **8** 135–146.

DÉGERINE, S. and PHAM, D. T. (1987). Une nouvelle approche du calcul de l'estimateur de maximum de vraisemblance exact d'un modèle autorégressif. Presented at the Eleventh Colloque GRETSI sur le traitement du signal et des images. Nice.

DICKINSON, B. W. (1978). Autoregressive estimation using residual energies ratios. *IEEE Trans. Inform. Theory* **IT-24** 503–506.

DICKINSON, B. W. (1979). Estimation of partial correlation matrices using Cholesky decomposition. *IEEE Trans. Automat. Control* **AC-24** 302–305.

GERONIMUS, Y. L. (1960). *Polynomials Orthogonal on a Circle and Interval*. Pergamon, London.

INOUYE, Y. (1983). Modeling of multichannel time series and extrapolation of matrix-valued autocorrelation sequences. *IEEE Trans. Acoust. Speech Signal Process.* **ASSP-31** 45–55.

INOUYE, Y. (1985). Autoregressive model fitting for multichannel time series of degenerate rank: Limit properties. *IEEE Trans. Circuits and Systems* **CAS-32** 252–259.

KAY, S. M. (1983). Recursive maximum likelihood estimation of autoregressive processes. *IEEE Trans. Acoust. Speech Signal Process.* **ASSP-31** 56–65.

MORF, M., VIEIRA, A. and KAILATH, T. (1978). Covariance characterization by partial autocorrelation matrices. *Ann. Statist.* **6** 643–648.

MORF, M., VIEIRA, A., LEE, D. and KAILATH, T. (1978). Recursive multichannel maximum entropy spectral estimation. *IEEE Trans. Geosc. Electron.* **GE-16** 85–94.

RAMSEY, F. (1974). Characterization of the partial autocorrelation function. *Ann. Statist.* **2** 1296–1301.

RAO, C. R. (1965). *Linear Statistical Inference and Its Applications*. Wiley, New York.

SAKAI, H. (1983). Covariance matrices characterization by a set of scalar partial autocorrelation coefficients. *Ann. Statist.* **11** 337–340.

WHITTLE, P. (1963). On the fitting of multivariable autoregressions and the approximate canonical factorization of a spectral density matrix. *Biometrika* **50** 129–134.

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