The Annals of Statistics
1990, Vol. 18, No. 1, 465–468

THE POWER FUNCTION OF THE STUDENTISED RANGE TEST

BY A. J. HAYTER AND W. LIU

University of Bath

In this paper we investigate the power function of the Studentised range test for comparing the means of normal populations in the one-way fixed effects analysis of variance model. The main results provide rigorous proofs of certain least favourable configurations of population means. These results are important in the calculation of the sample sizes required to guarantee power levels under certain restrictions on the ranges of the population means.

Consider the usual balanced one-way fixed effects analysis of variance model

$$X_{ij} = \mu_i + \varepsilon_{ij}, \qquad 1 \le i \le k, 1 \le j \le n,$$

where the μ_i, $1 \le i \le k$, are the k unknown population means, and the ε_{ij} are independently, identically distributed as $N(0, \sigma^2)$ random variables for some unknown error variance σ^2. Let $\bar{X}_i$ denote the sample mean of the ith population, and assume that an estimate S^2 of σ^2 is available distributed as a $\sigma^2\chi^2_v/v$ random variable for some degrees of freedom v, independent of the k sample means $\bar{X}_i$, $1 \le i \le k$.

The Studentised range test of the null hypothesis H_0: $\mu_1 = \cdots = \mu_k$ against a general alternative hypothesis operates by rejecting the null hypothesis if and only if the statistic

$$\max_{1 \le i,\, j \le k} \frac{|\bar{X}_i - \bar{X}_j|\sqrt{n}}{S}$$

exceeds a suitable critical point. It is our purpose in this paper to consider separately the two conditions

$$b_1(\boldsymbol{\theta}) = \max_{1 \le i \le k} |\theta_i - \bar{\theta}| \ge b$$

and

$$b_2(\boldsymbol{\theta}) = \max_{1 \le i,\, j \le k} |\theta_i - \theta_j| \ge b,$$

where $\boldsymbol{\theta} = (\theta_1, \dots, \theta_k) = (\mu_1/\sigma, \dots, \mu_k/\sigma)$ and $\bar{\theta}$ is the arithmetic average of the θ_i, $1 \le i \le k$, and to establish in each case the configuration of the θ_i for which the power function of the Studentised range test is minimized. These results are given below in Theorems 1 and 2, respectively, and the paper is concluded with a discussion of the motivation for this work.

If the power function of the Studentised range test is denoted by $\beta(\boldsymbol{\theta})$, then by conditioning on the value of the random variable S^2, it is apparent that for any $\boldsymbol{\theta}$, $\boldsymbol{\theta}^* \in \mathbf{R}^k$,

$$W_c(\boldsymbol{\theta}) \le W_c(\boldsymbol{\theta}^*)\ \forall\, c \quad \Rightarrow \quad \beta(\boldsymbol{\theta}^*) \le \beta(\boldsymbol{\theta}), \tag{1.1}$$

Received April 1988; revised January 1989

AMS 1980 *subject classifications.* Primary 62J15; secondary 62J10.

Key words and phrases. Multiple comparisons, analysis of variance, Studentised range, power function, least favourable configuration.

where the function $W_c(\boldsymbol{\theta})$ for $\boldsymbol{\theta} \in \mathbf{R}^k$ and $c \in \mathbf{R}$ is defined as

$$W_c(\boldsymbol{\theta}) = P\big(|Y_i - Y_j| \le c;\, 1 \le i,\, j \le k\big),$$

where the Y_i, $1 \le i \le k$, are independent normal random variables with variances $1/n$ and means θ_i, respectively.

Now for any $c \in \mathbf{R}$ we have the following four properties for the function $W_c(\boldsymbol{\theta})$.

1. $W_c(\boldsymbol{\theta}) = W_c(-\boldsymbol{\theta})$.
2. $W_c(\boldsymbol{\theta} + \lambda\mathbf{1}) = W_c(\boldsymbol{\theta})$, $\mathbf{1} = (1,\dots,1)$, $\lambda \in \mathbf{R}$.
3. $W_c(\pi(\boldsymbol{\theta})) = W_c(\boldsymbol{\theta})$, where the operator π permutes coordinates.
4. $W_c(\boldsymbol{\theta})$ is log-concave, i.e., for $0 \le \alpha \le 1$, and for all $\boldsymbol{\theta}$, $\boldsymbol{\theta}^* \in \mathbf{R}^k$,

$$W_c(\alpha\boldsymbol{\theta} + (1-\alpha)\boldsymbol{\theta}^*) \ge W_c^{\alpha}(\boldsymbol{\theta})W_c^{1-\alpha}(\boldsymbol{\theta}^*).$$

The first three properties are readily apparent. The fourth property follows from a result of Prekopa (1973) [see, for example, Eaton (1987), page 79] since the function $W_c(\boldsymbol{\theta})$ may be written as

$$W_c(\boldsymbol{\theta}) = \int_{\mathbf{R}^k} g(\boldsymbol{\theta}, \mathbf{y})\, d\mathbf{y},$$

where the function $g(\boldsymbol{\theta}, \mathbf{y})$ is log-concave in $(\boldsymbol{\theta}, \mathbf{y}) \in \mathbf{R}^{2k}$ and is given by

$$g(\boldsymbol{\theta}, \mathbf{y}) = I_\omega(\mathbf{y}) n^{k/2} \prod_{i=1}^{k} \varphi\big((y_i - \theta_i)\sqrt{n}\big),$$

where $\varphi(\cdot)$ is the density function of a standard normal random variable and $I_\omega(\mathbf{y})$ is the indicator function of the convex set

$$\omega = \big\{\mathbf{y} = (y_1,\dots,y_k)\colon |y_i - y_j| \le c;\, 1 \le i,\, j \le k\big\} \subset \mathbf{R}^k.$$

Notice that the fourth property implies by induction that for any $m \in \mathbf{N}$,

5. $W_c\big(\sum_{i=1}^m \alpha_i \boldsymbol{\theta}^{(i)}\big) \ge W_c(\boldsymbol{\theta}^{(1)})$ if $\alpha_i \ge 0$, $\sum_{i=1}^m \alpha_i = 1$ and $W_c(\boldsymbol{\theta}^{(1)}) = \cdots = W_c(\boldsymbol{\theta}^{(m)})$.

Also, notice that since $\frac{1}{2}(1+\rho)\boldsymbol{\theta} + \frac{1}{2}(1-\rho)(-\boldsymbol{\theta}) = \rho\boldsymbol{\theta}$, properties (5) and (1) above imply that

6. $$W_c(\rho\boldsymbol{\theta}) \ge W_c(\boldsymbol{\theta}) \quad \text{for } |\rho| \le 1.$$

Now we give the main results.

Theorem 1. *Let $b \ge 0$ and $\boldsymbol{\theta}^* = (0,\dots,0, kb/(k-1))$, so that $b_1(\boldsymbol{\theta}^*) = b$. Then*

$$b_1(\boldsymbol{\theta}) \ge b \quad \Rightarrow \quad \beta(\boldsymbol{\theta}) \ge \beta(\boldsymbol{\theta}^*).$$

Proof. Suppose that $b_1(\boldsymbol{\theta}) = \theta_k - \bar{\theta} = \tilde{b} \ge b$. Let $\boldsymbol{\theta}^{(i)}$, $i = 1,\dots,(k-1)!$ be the vectors obtained by permuting $\theta_1,\dots,\theta_{k-1}$ and leaving θ_k in place. Let $\bar{\theta}_k = [1/(k-1)]\sum_{i=1}^{k-1}\theta_i$, so that $\theta_k - \bar{\theta}_k = [k/(k-1)](\theta_k - \bar{\theta}) = [k/(k-1)]\tilde{b}$.

Now by properties (1)–(6) above, it follows that for any $c \in \mathbf{R}$,

$$W_c(\boldsymbol{\theta}) \le W_c\left(\sum_{i=1}^{(k-1)!} \boldsymbol{\theta}^{(i)} \Big/ (k-1)! \right) = W_c(\bar{\theta}_k, \dots, \bar{\theta}_k, \theta_k)$$

$$= W_c(0, \dots, 0, \theta_k - \bar{\theta}_k) = W_c(0, \dots, 0, [k/(k-1)]\tilde{b}) \le W_c(\boldsymbol{\theta}^*).$$

The proof is then completed by appealing to (1.1) given above. □

THEOREM 2. *Let* $b \ge 0$ *and* $\boldsymbol{\theta}^* = (-b/2, 0, \dots, 0, b/2)$, *so that* $b_2(\boldsymbol{\theta}^*) = b$. *Then*

$$b_2(\boldsymbol{\theta}) \ge b \quad \Rightarrow \quad \beta(\boldsymbol{\theta}) \ge \beta(\boldsymbol{\theta}^*).$$

PROOF. Suppose that $b_2(\boldsymbol{\theta}) = \theta_k - \theta_1 = \tilde{b} \ge b$. Let $\boldsymbol{\theta}^{(i)}$, $i = 1, \dots, (k-2)!$ be the vectors obtained by permuting $\theta_2, \dots, \theta_{k-1}$ and leaving θ_1 and θ_k in place. Let $\bar{\theta}_{1k} = [1/(k-2)]\sum_{i=2}^{k-1}\theta_i$. Then by properties (1)–(6) above it follows that for any $c \in \mathbf{R}$,

$$W_c(\boldsymbol{\theta}) \le W_c\left(\sum_{i=1}^{(k-2)!} \boldsymbol{\theta}^{(i)} \Big/ (k-2)! \right) = W_c(\theta_1, \bar{\theta}_{1k}, \dots, \bar{\theta}_{1k}, \theta_k)$$

$$\le W_c\left(\tfrac{1}{2}(\theta_1, \bar{\theta}_{1k}, \dots, \bar{\theta}_{1k}, \theta_k) + \tfrac{1}{2}(-\theta_k, -\bar{\theta}_{1k}, \dots, -\bar{\theta}_{1k}, -\theta_1)\right)$$

$$= W_c\left(-\tfrac{1}{2}\tilde{b}, 0, \dots, 0, \tfrac{1}{2}\tilde{b}\right) \le W_c(\boldsymbol{\theta}^*).$$

Again, the proof is completed by appealing to (1.1) above. □

The quantities $b_1(\boldsymbol{\theta})$ and $b_2(\boldsymbol{\theta})$ can be used as measures of variability of the population means and have been suggested before in, for example, Pearson and Hartley (1951), Scheffe (1959) Section 3.3, and Kastenbaum, Hoel and Bowman (1970). It is generally agreed that they allow an experimenter an intuitively appealing interpretation of the sensitivity of an experiment if they are used in the following manner. The experimenter may specify a positive constant b and a power level β, $\alpha \le \beta < 1$, and require that a test of size α of the null hypothesis H_0 has power no less than β whenever $b_i(\boldsymbol{\theta}) \ge b$ (where i may be chosen to be 1 or 2 depending upon which measure is more appropriate for the problem under consideration). The theorems above show that this can be done by guaranteeing the power β at the specified set of least favourable population means. The power calculations at these two specified sets of least favourable population means involve only a two-dimensional integral regardless of the number of populations k, and so may be performed very easily. Some calculations of the power level achieved for various sample sizes n and amount of variability b are given in David, Lachenbruch and Brandis (1972) and Hayter and Liu (1988).

An alternative method of testing the null hypothesis H_0 is, of course, to use the F-test. The power function of the F-test depends on the population means only through the quantity $\sum_{i=1}^{k}(\theta_i - \bar{\theta})^2$. This simple dependence has allowed the calculation of tables of the power function of the F-test and makes it easy to establish that the least favourable configurations of population means given above in Theorems 1 and 2 also provide least favourable configurations for the

F-test. It is interesting to compare the power levels of the Studentised range test and the F-test under these common least favourable configurations of population means, and generally the Studentised range test is more powerful [again see David, Lachenbruch and Brandis (1972) and Hayter and Liu (1988)].

Notice that since the experiment is designed as a one stage procedure, it is necessary to state the condition on the population means in terms of the unknown variance σ^2. This may be avoided, so that the power level may be guaranteed under the condition $b_i(\mu) \geq b$ rather than $b_i(\theta) \geq b$, by using a two stage procedure in which examination of the data obtained in the first stage indicates what further sample sizes are required in the second stage to guarantee the probability requirements [see Hochberg and Lachenbruch (1976)].

Finally, in this paper we have only considered the balanced one-way fixed effects model. It has been shown in Hayter (1984) that for an unbalanced model (where different populations may have different numbers of observations), the Studentised range test may be modified to produce a conservative test. The power function of this modified Studentised range test is complicated by the fact that it depends on the matchup between the population means and the population sample sizes. Nevertheless, for the problem of experimental design, since the experiment will in general be designed in a balanced manner, it is necessary only to investigate the power function in the balanced case. Furthermore, the consideration in this paper applies also to testing the equality of fixed effects of a certain kind in other higher order balanced orthogonal models.

Acknowledgments. The authors would like to acknowledge the substantial help of a referee and an editor leading to an improved presentation of the results in this paper.

REFERENCES

DAVID, H. A., LACHENBRUCH, P. A. and BRANDIS, H. P. (1972). The power function of range and Studentised range tests in normal samples. *Biometrika* **59** 161–168.

EATON, M. L. (1987). Lectures on topics in probability inequalities. CWI Tract 35. Centrum voor Wiskunde en Informatica/Mathematisch Centrum, Amsterdam.

HAYTER, A. J. (1984). A proof of the conjecture that the Tukey–Kramer multiple comparisons procedure is conservative. *Ann. Statist.* **12** 61–75.

HAYTER, A. J. and LIU, W. (1988). The Studentized range test: sample size determination. Technical Report AJH288, School of Math. Sciences, Univ. of Bath.

HOCHBERG, Y. and LACHENBRUCH, P. A. (1976). Two stage multiple comparison procedures based on the Studentised range. *Comm. Statist. A-Theory Methods* **5(15)** 1447–1453.

KASTENBAUM, M. A., HOEL, D. G. and BOWMAN, K. O. (1970). Sample size requirements: One-way analysis of variance. *Biometrika* **57** 421–430.

PEARSON, E. S. and HARTLEY, H. O. (1951). Charts of the power function for analysis of variance test, derived from the non-central F-distribution. *Biometrika* **38** 112–130.

PREKOPA, A. (1973). On logarithmic concave measures and functions. *Acta Sci. Mat.* (*Szeged*) **36** 335–343.

SCHEFFE, H. (1959). *The Analysis of Variance.* Wiley, New York.

SCHOOL OF MATHEMATICAL SCIENCES
UNIVERSITY OF BATH
BATH BA2 7AY
ENGLAND