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ESTIMATORS AND SPREAD

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Arbitrary, possibly randomized estimators of a one-dimensional parameter are considered. It is shown that suitable averages of their distribution functions are more spread out than particular distribution functions, which are defined in terms of the weight functions by which the averages are taken over the parameter space and in terms of the family of distributions for the random quantity on which the estimators are based. In this way bounds are provided for the performance of arbitrary estimators of the parameter. As consequences of this nonasymptotic spread inequality, which will be proved under mild regularity conditions, a local asymptotic minimax inequality and a generalization of the classical results on superefficiency can be derived, thus showing the strength of our spread inequality.

1. The spread inequality. We shall consider estimation of a one-dimensional parameter. Let the parameter space Θ be a measurable subset of $\mathbb{R}$ containing an open interval. Let X be a random variable taking values in $\mathscr{X}$ and having density $f_\theta(\cdot)$, $\theta \in \Theta$, with respect to some σ-finite measure μ on the measurable space $(\mathscr{X}, \mathscr{A})$. The parameter θ is estimated by a (randomized) estimator T based on X. We are interested in the distribution of $T - \theta$ under $f_\theta(\cdot)$.

Since this distribution may be anything and even degenerate, very little can be said about it. However, in general this distribution cannot be arbitrarily much concentrated for several possible values of the parameter simultaneously. Therefore, we are going to consider

$$G(y) = \int P_\theta(a(T - \theta) \le y) w(\theta)\, d\theta, \qquad y \in \mathbb{R}, \tag{1.1}$$

where a is a positive constant and where the weight function w is a density on Θ with respect to Lebesgue measure. Under appropriate regularity conditions to be specified in Theorem 1.1 and implying, that for μ almost all $x \in \mathscr{X}$ the function $\theta \mapsto f_\theta(x)w(\theta)$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$ with derivative $\dot{f}_\theta(x)w(\theta) + f_\theta(x)w'(\theta)$, we can define the distribution function H by

$$H(z) = \int P_\theta\left(\frac{1}{a}\left\{\frac{\dot{f}_\theta}{f_\theta}(X) + \frac{w'}{w}(\theta)\right\} \le z\right) w(\theta)\, d\theta, \qquad z \in \mathbb{R}. \tag{1.2}$$

Note that, if ϑ is a random variable with density $w(\theta)$ on Θ, then (X, ϑ) is a random variable with density $f_\theta(x)w(\theta)$ on $\mathscr{X} \times \Theta$ and G can be viewed as the

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distribution function of $a(T - \vartheta)$ and H as the distribution function of

$$S = \frac{1}{a}\left\{\frac{\dot{f}_\vartheta}{f_\vartheta}(X) + \frac{w'}{w}(\vartheta)\right\}, \tag{1.3}$$

a kind of normed score function taken at (X, ϑ).

We define the distribution function K by

$$K^{-1}(u) = \int_{1/2}^{u} \frac{1}{\int_s^1 H^{-1}(t)\,dt}\,ds, \qquad 0 < u \le 1, \tag{1.4}$$

where H^{-1} is the inverse distribution function $H^{-1}(t) = \inf\{z \mid H(z) \ge t\}$ of H. Denoting the first and second derivative of K by k and k', respectively, we see, that the score function of K with respect to a location parameter satisfies

$$-\frac{k'}{k}\left(K^{-1}(u)\right) = H^{-1}(u), \qquad \text{Lebesgue almost all } u \in (0,1). \tag{1.5}$$

This implies that the distribution function of the score function of K equals H, the distribution function of the normed score function for the original estimation problem, and that K is strongly unimodal.

The relations between (X, ϑ) and K and between (T, ϑ) and G suggest that G and K are related too. In fact, G is more spread out than K, that is, any two quantiles of G are further apart than the corresponding quantiles of K [cf. Bickel and Lehmann (1979)]. We will denote this by $G \ge_1 K$, a notation in accordance with Oja (1981). Our result reads as follows.

THEOREM 1.1. *Let the weight function w on $\mathbb{R}$ be absolutely continuous with respect to Lebesgue measure with Radon–Nikodym derivative w' satisfying*

$$\int_{\mathbb{R}} |w'(\theta)|\,d\theta < \infty \tag{1.6}$$

and let w concentrate its mass on Θ, that is,

$$\int_{\Theta} w(\theta)\,d\theta = 1. \tag{1.7}$$

For μ almost all $x \in \mathscr{X}$, let the function $\theta \mapsto f_\theta(x)$ be the restriction to Θ of a function, which is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$ with Radon–Nikodym derivative $\dot{f}_\theta(x)$. If $\dot{f}_\theta(x)$ is $\mu \times$ Lebesgue-measurable, if

$$\int_{\mathbb{R}}\int_{\mathscr{X}} |\dot{f}_\theta(x)| w(\theta)\,d\mu(x)\,d\theta < \infty \tag{1.8}$$

holds and if T is a possibly randomized estimator of θ, then the distribution functions G and K defined by (1.1) *and* (1.4) *are differentiable with derivatives g, respectively, k satisfying*

$$g\left(G^{-1}(s)\right) \le k\left(K^{-1}(s)\right) = \int_s^1 H^{-1}(t)\,dt, \qquad 0 < s < 1. \tag{1.9}$$

This implies $G \ge_1 K$, that is,

$$G^{-1}(v) - G^{-1}(u) \ge K^{-1}(v) - K^{-1}(u), \qquad 0 \le u \le v \le 1. \tag{1.10}$$

Proof. Since the function $\theta \mapsto f_\theta(x)w(\theta)$ is absolutely continuous on $\mathbb{R}$ for μ almost all $x \in \mathscr{X}$, we have

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \int_{\mathbb{R}}\int_{\mathscr{X}} \left| \frac{1}{\varepsilon}\{ f_{\theta+\varepsilon}(x)w(\theta+\varepsilon) - f_\theta(x)w(\theta)\} \right| d\mu(x)\, d\theta \\
&\le \limsup_{\varepsilon\to 0} \int_{\mathscr{X}}\int_{\mathbb{R}} \frac{1}{\varepsilon}\int_\theta^{\theta+\varepsilon} |\dot f_\eta(x)w(\eta) + f_\eta(x)w'(\eta)|\, d\eta\, d\theta\, d\mu(x) \qquad (1.11)\\
&= \int_{\mathbb{R}}\int_{\mathscr{X}} |\dot f_\eta(x)w(\eta) + f_\eta(x)w'(\eta)|\, d\mu(x)\, d\eta < \infty,
\end{aligned}$$

where the finiteness is implied by (1.8) and (1.6). In view of

$$\begin{aligned}
1 - G(y+\delta) &= \int_{\mathbb{R}} P_\theta\big(a(T-(\theta+\delta/a)) > y\big)w(\theta)\, d\theta \qquad (1.12)\\
&= \int_{\mathbb{R}} P_{\theta-\delta/a}\big(a(T-\theta) > y\big)w(\theta-\delta/a)\, d\theta, \qquad y \in \mathbb{R},
\end{aligned}$$

inequality (1.11) and Vitali's theorem imply that G is differentiable with derivative g satisfying

$$\begin{aligned}
g(y) &= \lim_{\delta\to 0}\int_{\mathbb{R}}\int_{\mathscr{X}} E\big(1_{(y,\infty)}(a(T-\theta))\,|\,X = x\big) \\
&\qquad \times \frac{1}{\delta}\{ f_\theta(x)w(\theta) - f_{\theta-\delta/a}(x)w(\theta-\delta/a)\}\, d\mu(x)\, d\theta \qquad (1.13)\\
&= \int_{\mathbb{R}}\int_{\mathscr{X}} E\big(1_{(y,\infty)}(a(T-\theta))\,|\,X = x\big) \\
&\qquad \times \frac{1}{a}\{ \dot f_\theta(x)w(\theta) + f_\theta(x)w'(\theta)\}\, d\mu(x)\, d\theta.
\end{aligned}$$

Because $\{\theta \in \mathbb{R} \mid f_\theta(x)w(\theta) = 0,\ \dot f_\theta(x)w(\theta) + f_\theta(x)w'(\theta) \neq 0\}$ is a Lebesgue null set for μ almost all $x \in \mathscr{X}$, we obtain from (1.13)

$$g\big(G^{-1}(s)\big) = ES1_{(G^{-1}(s),\infty)}\big(a(T-\vartheta)\big). \qquad (1.14)$$

Together with

$$1 - s = E1_{(G^{-1}(s),\infty)}\big(a(T-\vartheta)\big) \qquad (1.15)$$

and the monotonicity of H^{-1} this yields

$$\begin{aligned}
&\int_s^1 H^{-1}(t)\, dt - g\big(G^{-1}(s)\big) \\
&= \int_0^1 \{H^{-1}(t) - H^{-1}(s)\} \qquad (1.16)\\
&\qquad \times \Big\{1_{(s,1]}(t) - E\big(1_{(G^{-1}(s),\infty)}(a(T-\vartheta))\,|\,S = H^{-1}(t)\big)\Big\}\, dt \\
&\ge 0.
\end{aligned}$$

Since the function $s \mapsto \int_s^1 H^{-1}$ is concave on $[0,1]$ vanishing only at 0 and 1, it is not difficult to verify that K is a distribution function well defined by (1.4). Hence (1.16) implies (1.9) and we obtain (1.10) from (1.9) by noting that the absolutely continuous part of $G^{-1}(\cdot)$ has Radon–Nikodym derivative $1/g(G^{-1}(\cdot))$. For further details see Klaassen (1984a). □

REMARK 1.1. Let the loss function l_y be defined by

$$l_y(\cdot) = 1_{(-\infty,\, y/a]}(\cdot).$$

Note that $G(y) = El_y(T - \vartheta)$ is the Bayes risk of T for the loss function l_y and the prior distribution with density w.

REMARK 1.2. If $T - \vartheta$ is a deterministic strictly increasing function of S, equality holds in (1.16) and hence in (1.9) and (1.10). Note also, that (1.9) and (1.10) are insensitive to translations both of G and of K.

EXAMPLE 1.1. With $\mathscr{X} = \mathbb{R}$ let X have a $\mathscr{N}(\theta, 1)$ and ϑ a $\mathscr{N}(0, \sigma^2)$ distribution and let $a = 1$. Now $S = X - (1 + \sigma^{-2})\vartheta$ has a $\mathscr{N}(0, 1 + \sigma^{-2})$ distribution and it follows from (1.10) that for every estimator T the distribution of $T - \vartheta$ is more spread out than a $\mathscr{N}(0, (1 + \sigma^{-2})^{-1})$ distribution. Furthermore, Remark 1.2 shows that $(1 + \sigma^{-2})^{-1}X + c$, $c \in \mathbb{R}$, are optimal estimators in the sense of Theorem 1.1. Indeed, $(1 + \sigma^{-1})^{-1}X - \vartheta$ has a $\mathscr{N}(0, (1 + \sigma^{-2})^{-1})$ distribution. Note that this distribution tends to a standard normal distribution as $\sigma^2 \to \infty$.

More generally, if $\mathscr{X} = \mathbb{R}^n$, X is $\mathscr{N}(\theta v, \Sigma)$, $v \in \mathbb{R}^n$, Σ positive definite, and if ϑ is $\mathscr{N}(\mu, \sigma^2)$, then the distribution of $T - \vartheta$ is more spread out than a $\mathscr{N}(0, (v'\Sigma^{-1}v + \sigma^{-2})^{-1})$ distribution and $T = (v'\Sigma^{-1}X + \mu\sigma^{-2})(v'\Sigma^{-1}v + \sigma^{-2})^{-1}$ is an optimal estimator. For $\sigma^2 \to \infty$ this estimator becomes $T = v'\Sigma^{-1}X(v'\Sigma^{-1}v)^{-1}$, which is equivariant in the sense, that adding ηv to the observation results in adding η to the estimate of θ.

The last part of Example 1.1 suggests that any equivariant estimator of θ is more spread out than a $\mathscr{N}(0, (v'\Sigma^{-1}v)^{-1})$ distribution. Indeed, this is the case and we shall present a generalization of this in Section 2. In that section we will also give some other examples.

From our spread inequality (1.10) both global and local asymptotic results can be derived in a relatively simple way. In Section 3 we shall indicate some generalizations of the classical (global) results on superefficiency and of the local asymptotic minimax theorem. The spread inequality and some of its consequences have already been published without proof in Klaassen (1984b, 1985).

2. The spread of location estimators. In the situation of the preceding section we choose now $\mathscr{X} = \mathbb{R}^n$, μ Lebesgue measure, $\Theta = \mathbb{R}$ and $f_\theta(\cdot) = f(\cdot - \theta v)$, where f, a density on $\mathbb{R}^n$, and $v \in \mathbb{R}^n$ are fixed. We consider estimators T of the location parameter θ, which are based on X and which are equivariant in the sense

$$(2.1)\quad L(T \mid X = x + \eta v) = L(T + \eta \mid X = x), \quad f_\theta\text{-almost all } x \in \mathbb{R}^n,\ \theta, \eta \in \mathbb{R}.$$

Note that for such T,

$$(2.2)\qquad L(T - \vartheta \mid \vartheta) = L(T \mid \vartheta = 0), \qquad w\text{-a.s.},$$

holds and that hence G from (1.1) reduces to

$$(2.3)\qquad G_0(y) = P_0(aT \le y), \qquad y \in \mathbb{R}.$$

Let us assume that f is absolutely continuous in each component of its argument, let us denote the Radon–Nikodym derivative of f with respect to the ith component by f_i and let us assume

$$(2.4)\qquad \sum_{i=1}^{n} \int_{\mathbb{R}^n} |f_i(x)|\, dx < \infty.$$

With w the density of the $\mathcal{N}(0, \sigma^2)$ distribution (a proof similar to the proof of) Theorem 1.1 yields the differentiability of G and hence G_0 with derivative g_0 satisfying

$$(2.5)\qquad g_0\big(G_0^{-1}(s)\big) \le \int_s^1 H_\sigma^{-1}(t)\, dt, \qquad 0 < s < 1,$$

where H_σ is the distribution function of S from (1.3). Note that S has the same distribution as $a^{-1}\{\sum_{i=1}^n - v_i f_i(X_0)/f(X_0) - \vartheta/\sigma^2\}$, where X_0 and ϑ are independent and where X_0 has density f.

With

$$(2.6)\qquad H(z) = P\left(a^{-1} \sum_{i=1}^{n} - v_i f_i(X_0)/f(X_0) \le z\right), \qquad z \in \mathbb{R},$$

we see that the weak convergence $H_\sigma \to_w H$ as $\sigma \to \infty$ holds, which is equivalent to $H_\sigma^{-1}(u) \to H^{-1}(u)$ as $\sigma \to \infty$ for all continuity points $u \in (0,1)$ of H^{-1} [cf. (2.12)]. In view of

$$(2.7)\qquad \limsup_{\sigma \to \infty} \int_0^1 |H_\sigma^{-1}(t)|\, dt \le \int_0^1 |H^{-1}(t)|\, dt,$$

this convergence and (2.5) together yield

$$(2.8)\qquad g_0\big(G_0^{-1}(s)\big) \le \int_s^1 H^{-1}(t)\, dt, \qquad 0 < s < 1.$$

We have proved

THEOREM 2.1. *Let in the situation sketched above T be an equivariant estimator of θ. If* (2.4) *holds, then* (2.8) *is valid for G_0 defined by* (2.3) *and H by* (2.6). *This implies $G_0 \ge_1 K$, where K is defined as in* (1.4).

This result is a slight generalization of Theorem 1.1 of Klaassen (1984a), which considers $X = (X_1, \dots, X_n)'$, $X_1, \dots, X_n$ i.i.d. on $(\mathbb{R}, \mathscr{B})$. Consequently, Theorem 1.1 of the present article can be viewed as a generalization of the main result of that article.

Let H and K be defined as in (1.2) and (1.4) and denote by k and k' the density of K and its derivative, respectively. If X is a random variable with density $k(\cdot - \theta)$ with respect to Lebesgue measure on $(\mathbb{R}, \mathscr{B})$, then Theorem 2.1 shows that every translation equivariant estimator of θ, based on X, has a distribution which is more spread out than K [cf. (1.4) and (1.5)] and that X is an optimal estimator of θ. Consequently, a loose formulation of the assertion of Theorem 1.1 runs as follows. In the situation of Theorem 1.1, *estimation of θ with weight function w is at least as difficult as equivariant estimation of the location parameter θ of one observation from the distribution $K(\cdot - \theta)$*, with K given by (1.4). In this way the original estimation problem can be compared with a simpler location estimation problem.

The range of applicability of Theorems 1.1 and 2.1 is broadened considerably by the following simple remark.

LEMMA 2.1. *Let ψ and ψ_ε be measurable functions from $[0,1]$ into $[0,\infty]$ and let the possibly defective distribution function K be defined by*

$$K^{-1}(u) = \int_{1/2}^{u} \frac{1}{\psi(s)} ds, \qquad 0 < u \le 1, \tag{2.9}$$

and K_ε similarly. Let G and G_ε be distribution functions such that $G_\varepsilon \to_w G$ as $\varepsilon \to 0$. If $G_\varepsilon \ge_1 K_\varepsilon$ holds for all ε and if

$$\limsup_{\varepsilon \to 0} \psi_\varepsilon(s) \le \psi(s), \qquad 0 < s < 1, \tag{2.10}$$

holds, then

$$G \ge_1 K \tag{2.11}$$

is valid.

PROOF. Let F_n and F be distribution functions and let $F_n^{-1} \to_w F^{-1}$ mean that $\lim_{n\to\infty} F_n^{-1}(u) = F^{-1}(u)$ for all continuity points $u \in (0,1)$ of F^{-1}. We have

$$F_n \to_w F \Leftrightarrow F_n^{-1} \to_w F^{-1} \tag{2.12}$$

[see Satz 2.11 of Witting and Nölle (1970) for the implication $\Rightarrow$ and consider $F_n^{-1}(U)$ and $F^{-1}(U)$ with U uniform on $(0,1)$ for the other one].

If u and v with $0 < u < v < 1$ are continuity points of G^{-1}, then (2.12), $G_\varepsilon \to_w G$, $G_\varepsilon \ge_1 K_\varepsilon$, Fatou's lemma and (2.10) yield

$$G^{-1}(v) - G^{-1}(u) \ge \liminf_{\varepsilon \to 0} \int_u^v \frac{1}{\psi_\varepsilon(s)} ds \ge K^{-1}(v) - K^{-1}(u). \tag{2.13}$$

Since K^{-1} is continuous and G^{-1} is left-continuous on $(0,1)$, this implies (2.11). □

As an application of Lemma 2.1 we mention

COROLLARY 2.1. *Let $X_1, \dots, X_n$ be i.i.d. with density*

$$e^{-(x-\theta)}1_{[0,\infty)}(x-\theta) \tag{2.14}$$

on $\mathbb{R}$, $\theta \in \Theta = \mathbb{R}$. If T_n is a translation equivariant estimator of the location parameter θ, then, for every $\theta \in \mathbb{R}$, the distribution of $n(T_n - \theta)$ under θ is more spread out than the exponential distribution with density $e^{-x}1_{[0,\infty)}(x)$ on $\mathbb{R}$. Consequently, $T_n = X_{(1)} + c$, $c \in \mathbb{R}$, are optimal estimators.

PROOF. If $X_1, \dots, X_n$ are i.i.d. with density

$$\begin{aligned}(1-\varepsilon)e^{(1-\varepsilon)(x-\theta)/\varepsilon}1_{(-\infty,0)}(x-\theta)\\ +(1-\varepsilon)e^{-(x-\theta)}1_{[0,\infty)}(x-\theta), \qquad 0<\varepsilon<1,\end{aligned} \tag{2.15}$$

then (2.4) is satisfied and Theorem 2.1 yields $G_\varepsilon \geq_1 K_\varepsilon$, where G_ε is the distribution of $n(T_n - \theta)$ under θ and K_ε is defined by (1.4) with H replaced by H_ε, the distribution under $\theta = 0$ of

$$\frac{1}{n}\sum_{i=1}^{n}\left\{-(1-\varepsilon)/\varepsilon 1_{(-\infty,0)}(X_i) + 1_{[0,\infty)}(X_i)\right\}. \tag{2.16}$$

Clearly, $H_\varepsilon \to_w H$ for $\varepsilon \downarrow 0$ with H corresponding to point mass at 1 and more importantly,

$$\limsup_{\varepsilon\downarrow 0}\int_s^1 H_\varepsilon^{-1}(t)\,dt \leq \int_s^1 H^{-1}(t)\,dt = 1-s, \qquad 0<s<1. \tag{2.17}$$

Since

$$K^{-1}(u) - K^{-1}(0) = \int_0^u \frac{1}{\int_s^1 H^{-1}(t)\,dt}\,ds = -\log(1-u)$$

holds and since $G_\varepsilon \to_w G$ as $\varepsilon \downarrow 0$ with G the distribution of $n(T_n - \theta)$ under (2.14), a simple application of Lemma 2.1 yields the result. □

If $X_1, \dots, X_n$ are i.i.d. with a uniform distribution on $(0, \theta)$ and if one is interested in scale equivariant estimation of $\theta \in (0, \infty)$, then Corollary 2.1 implies that the distribution of $-n$ times the logarithm of any such estimator is more spread out than the exponential distribution of Corollary 2.1 and that $cX_{(n)}$, $c \in \mathbb{R}$, are optimal estimators in this sense [cf. Rao (1981)].

3. Some asymptotic consequences. In Example 1.1 and Corollary 2.1 we have seen that equality in (1.10) can be attained by suitable estimators in the "normal" and "exponential" location estimation problem. In order to judge the tightness of the spread inequality, one might also study its asymptotic implications. To that end we shall restrict attention here to the standard estimation problem of $X_1, \dots, X_n$ i.i.d. random variables with a distribution with finite Fisher information $I(\theta)$, $\theta \in \Theta$, an open interval.

Let T_n be an estimator of θ, based on $X_1, \dots, X_n$, and w an absolutely continuous density on $\mathbb{R}$ with derivative w' satisfying

$$\int |w'| < \infty. \tag{3.1}$$

Fix $\theta_0 \in \Theta$ and define

$$G_{n\sigma}(y) = \int P_\theta\left(\sqrt{n}\,(T_n - \theta) \le y\right)\sigma^{-1}\sqrt{n}\,w\left(\sigma^{-1}\sqrt{n}\,(\theta - \theta_0)\right)d\theta. \tag{3.2}$$

If w has bounded support whenever $\Theta \neq \mathbb{R}$, then, for every $\sigma > 0$, $G_{n\sigma}$ is well defined provided n is large enough. A local asymptotic minimax inequality may be obtained from the following consequence of Theorem 1.1.

THEOREM 3.1. *Let in the above situation the density f_θ of X_1 with respect to a σ-finite measure μ be $\mu \times$ Lebesgue-measurable and absolutely continuous in θ with derivative $\dot{f}_\theta$ and $I(\theta) = \int (\dot{f}_\theta f_\theta^{-1/2})^2 d\mu$. If*

$$\lim_{\theta \to \theta_0} \int \left(\dot{f}_\theta f_\theta^{-1/2} - \dot{f}_{\theta_0} f_{\theta_0}^{-1/2}\right)^2 d\mu = 0 \tag{3.3}$$

holds, then all limit points of $G_{n\sigma}$, as $n \to \infty$ and subsequently $\sigma \to \infty$, are more spread out than $\mathcal{N}(0, 1/I(\theta_0))$.

By considering the limit behavior of

$$G_n(y) = \int P_\theta\left(\sqrt{n}\,(T_n - \theta) \le y\right)w(\theta)\,d\theta \tag{3.4}$$

for various choices of w the following global asymptotic consequence of Theorem 1.1 may be obtained.

THEOREM 3.2. *Assume that in the above i.i.d. situation, $I(\cdot)$ is continuous on Θ. If there exist distributions G_θ with*

$$L\left(\sqrt{n}\,(T_n - \theta) \mid \theta\right) \to_w G_\theta, \quad \text{as } n \to \infty, \quad \theta \in \Theta, \tag{3.5}$$

then

$$G_\theta \ge_1 \mathcal{N}(0, 1/I(\theta)) \tag{3.6}$$

holds for Lebesgue almost all $\theta \in \Theta$.

Note that (3.6) implies

$$\operatorname{var}_{G_\theta} Y \ge 1/I(\theta). \tag{3.7}$$

Consequently, Theorem 3.2 extends the classical results of Le Cam (1952, 1953) and Bahadur (1964) on superefficiency.

These results show that, although for finite n the spread inequality (1.10) is not sharp for every estimation problem, it still is powerful enough to imply both local and global (sharp) asymptotic inequalities. We will not present proofs of

Theorem 3.1 and 3.2 here, but generalizations of these, including proofs, will be published elsewhere [Klaassen (1988)].

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