The Annals of Statistics
1989, Vol. 17, No. 2, 801–823

IRREVERSIBLE ADAPTIVE ALLOCATION RULES[1]

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Motivated by a scheduling problem arising from serial sacrifice experiments, the asymptotic efficiency of irreversible adaptive allocation rules is studied. The asymptotic lower bound for the regret of an adaptive allocation rule is characterized by the minimum of a linear program. Based on a class of one-sided sequential tests, asymptotically efficient rules which achieve the lower bound are constructed. The conditions necessary for this construction are verified in the serial sacrifice scheduling problem.

1. Introduction. Let Π_i, $i = 1, \dots, k$, denote statistical populations specified, respectively, by univariate densities $f_i(x|\theta)$ with respect to some measure ν, where $f_i(\cdot|\cdot)$ is known and θ is an unknown parameter belonging to some set Θ. Let $g_i(x, \theta)$ be the reward when population i is sampled and x is observed. An adaptive allocation rule is defined to be a sequence of random variables $\phi = \{\phi_n\}$ taking value in the set $\{1, \dots, k\}$ such that the event $\{\phi_n = i\}$ ("take nth sample from Π_i") belongs to the σ-field $\mathscr{F}_{n-1} = \sigma(\phi_1, X_1, \dots, \phi_{n-1}, X_{n-1})$, where X_j denotes the jth sample. Let N be the sample size. In the following we shall study the problem of designing an adaptive allocation rule which achieves the greatest possible expected reward

$$J_N(\theta) = \sum_{n=1}^{N} E_\theta\left[g_{\phi_n}(X_n, \theta)\right] \tag{1.1}$$

under the constraint

$$\phi_n \le \phi_{n+1} \quad \text{for } 1 \le n \le N-1. \tag{1.2}$$

Constraint (1.2) indicates that once a sample has been taken from Π_i, no further sampling is allowed from $\Pi_1, \dots, \Pi_{i-1}$. An adaptive allocation rule which satisfies (1.2) is said to be irreversible. When the irreversibility constraint is removed and Π_i is specified by $f(x|\theta_i)$, instead of $f_i(x|\theta)$, our allocation problem becomes the celebrated multi-armed bandit problem. Starting with Robbins (1952), there has been a considerable amount of literature on this subject. A substantial contribution has been made recently by Lai and Robbins (1984, 1985) and Lai (1987).

Our formulation is motivated by the following experimental design problem [Bergman and Turnbull (1983)]. In rodent bioassay experiments where N rodents are simultaneously put on test, it is desired to estimate the onset time distribution G_θ of a tumor. Due to the nature of the tumor, its presence

Received June 1987; revised March 1988.
[1]Supported by National Science Foundation Grant DMS-86-03084.
AMS 1980 *subject classifications*. Primary 62L10; secondary 62L05, 62P10.
Key words and phrases. Adaptive allocation rules, sequential design, sequential tests, linear programming, Kullback–Leibler information.

($\{X = 1\}$) or absence ($\{X = 0\}$) can only be detected through sacrifices. Consider a sequence of fixed times or "stages" $t_1 < \cdots < t_k$, at which sacrifices of one or more rodents could be made. Then at stage t_i, instead of observing the onset time, a binary random variable X is observed after each sacrifice. X can be viewed as a random sample from the population Π_i which is specified by

$$f_i(x|\theta) = [1 - G_\theta(t_i)]^{1-x}[G_\theta(t_i)]^x, \qquad x = 0 \text{ or } 1, \tag{1.3}$$

and ν is the counting measure on $\{0, 1\}$. For estimation purposes, we would like to allocate as many as possible sacrifices at time $t^*(\theta)$ at which the Fisher information ("the expected reward")

$$E_\theta\left[\frac{\partial}{\partial\theta}(\ln f_i(X|\theta))\right]^2 = E_\theta[g_i(X, \theta)]$$

is maximized over the set $\{1, \dots, k\}$. Without a priori knowledge of θ, the information of $t^*(\theta)$ can only come from the observed data. Thus an adaptive rule, which utilizes previous observations, is desirable. Furthermore, since a sacrifice can only be made at a time greater than or equal to the current age of the rodent, the rule must be irreversible.

Now, let us go back to the general formulation (1.1) and (1.2). Assume that $\int |g_i(x, \theta)| f_i(x|\theta)\, d\nu < \infty$ and let

$$h_i(\theta) = \int g_i(x, \theta) f_i(x|\theta)\, d\nu \tag{1.4}$$

be the expected reward if a sample is taken from Π_i. Also, let

$$T_N(i) = \sum_{n=1}^{N} 1_{\{\phi_n = i\}} \tag{1.5}$$

be the number of all samples taken from Π_i. Since

$$\begin{aligned} J_N(\theta) &= \sum_{n=1}^{N}\sum_{i=1}^{k} E_\theta\{E_\theta[g_i(X_n, \theta)1_{\{\phi_n = i\}}|\mathscr{F}_{n-1}]\} \\ &= \sum_{i=1}^{k} h_i(\theta) E_\theta T_N(i), \end{aligned} \tag{1.6}$$

the problem of maximizing $J_N(\theta)$ is therefore equivalent to that of minimizing the regret

$$R_N(\theta) = \sum_{i=1}^{k} [h^*(\theta) - h_i(\theta)] E_\theta T_N(i), \tag{1.7}$$

where

$$h^*(\theta) = \max\{h_i(\theta): 1 \le i \le k\}.$$

Let $\Theta_l = \{\theta \in \Theta: h_l(\theta) = h^*(\theta), h_l(\theta) > h_i(\theta), i = 1, \dots, l-1\}$ ("Π_l is the first best"), $\Theta_l^* = \{\theta \in \Theta: h_l(\theta) > h_i(\theta), i \ne l\}$ ("Π_l is the unique best"). Denote

Kullback–Leibler numbers by

$$I_i(\theta, \lambda) = \int \log(f_i(x|\theta)/f_i(x|\lambda)) f_i(x|\theta)\, d\nu.$$

In this article we shall always assume that for all i,

$$(1.8) \qquad 0 < I_i(\theta, \lambda) < \infty \quad \text{for } \theta \neq \lambda.$$

Furthermore, in later sections we shall also assume that $\Theta = (L, U)$, an open interval and $\{\Theta_i\}$ has a monotone structure, that is, there exist θ_i, $0 \leq i \leq k$, such that either

$$(1.9) \qquad \begin{gathered} L = \theta_k < \theta_{k-1} < \cdots < \theta_1 < \theta_0 = U, \\ \Theta_i^* = (\theta_i, \theta_{i-1}), \qquad \Theta_i = [\theta_i, \theta_{i-1}), \quad \text{for } 1 \leq i \leq k-1, \\ \Theta_k^* = \Theta_k = (L, \theta_{k-1}) \end{gathered}$$

or

$$(1.10) \qquad \begin{gathered} L = \theta_0 < \theta_1 < \cdots < \theta_{k-1} < \theta_k = U, \\ \Theta_i^* = (\theta_{i-1}, \theta_i), \qquad \Theta_i = (\theta_{i-1}, \theta_i], \quad \text{for } 1 \leq i \leq k-1, \\ \Theta_k^* = \Theta_k = (\theta_{k-1}, U). \end{gathered}$$

Since (1.10) can be transformed into (1.9) by the reparametrization $\theta \to -\theta$, we shall restrict our discussion below to the assumption (1.9) only. We shall also assume that for $\theta \in \Theta_{l+1}$ and $1 \leq j \leq l$,

(1.11) $I_j(\theta, \lambda)$ is a continuous and increasing function in $\lambda \in [\theta_l, U)$.

In Sections 3 and 4 we construct a sequence of adaptive allocation rules ϕ_N such that for $\theta \in \Theta_m$,

$$(1.12) \qquad \sum_{j=m+1}^{k} E_\theta T_N(j) = O(1), \quad \text{if } m < k,$$

$$(1.13a) \qquad R_N(\theta) \sim r(\theta, m-1) \log N, \quad \text{if } m > 1,$$

where $r(\theta, l)$ is the minimum of the following linear programming problem.

Problem A. Minimize $\sum_{i=1}^{l}(h^*(\theta) - h_i(\theta))z_i$ subject to conditions

$$(1.14) \qquad \begin{cases} I_1(\theta, \theta_1)z_1 \geq 1, \\ I_1(\theta, \theta_2)z_1 + I_2(\theta, \theta_2)z_2 \geq 1, \\ \vdots \\ I_1(\theta, \theta_l)z_1 + \cdots + I_l(\theta, \theta_l)z_l \geq 1 \end{cases}$$

and

$$(1.15) \qquad z_i \geq 0 \quad \text{for } i = 1, 2, \ldots, l.$$

The result (1.12) implies that

$$(1.13b) \qquad R_n(\theta) = O(1) \quad \text{if } \theta \in \Theta_1.$$

This specifies the order of the regret when the best population is the first one while (1.13a) gives the order when it is in a later stage. The result (1.12) together with (1.13a) also imply that for all $\theta \in \Theta_j$,

$$E_\theta(N - T_N(j)) = O(\log N).$$

Therefore, if more than one population gives the greatest reward (this is the case when $\theta = \theta_j$ for some j), our rules would tend to choose the first best one. This is a desirable property for our experimental design problem since these rules would terminate the experiment as soon as possible. In Section 2 we shall show that these rules are optimal in the sense of the following theorem.

THEOREM 1. *Assume that* (1.8), (1.9) *and* (1.11) *hold. Let* ϕ_N *be a sequence of irreversible rules such that for all* $\theta \in \Theta$,

$$R_N(\theta) = o(N^a) \quad \text{for every } a > 0. \tag{1.16}$$

Then for every $\theta \in \Theta_{l+1}$,

$$\liminf_{N \to \infty} R_N(\theta)/\log N \geq r(\theta, l). \tag{1.17}$$

Condition (1.16) of Theorem 1 implies that for all θ,

$$\lim_{N \to \infty} N^{-1} J_N(\theta) = h^*(\theta). \tag{1.18}$$

The rules that satisfy (1.18) are said to be *consistent*. Under the assumptions of Theorem 1, the rules that satisfy (1.13a, b) are said to be *asymptotically efficient*. The lower bound $r(\theta, l)$ of asymptotically efficient rules depends on when the best population is available. This effect of "time arrow" does not appear in the multi-armed bandit problem [Lai and Robbins (1985), (1.11)], where the rules are allowed to be reversible.

Bergman and Turnbull (1983) and Louis (1984) have studied the serial sacrifice experiments mentioned at the beginning of this article. Under the assumption that $G_\theta(t) = 1 - e^{-\theta t}$ [see (1.3)], they proposed rules which are consistent. The efficiency issue (in our setting) had not been discussed. Furthermore, the consistency result of Bergman and Turnbull (1983) is obtained when the sacrifice times become dense and Louis' problem is formulated under a continuous time framework. As noted by Bergman and Turnbull (1983), in the carcinogen bioassay problem the sacrifice times could be at convenient weekly or monthly intervals. In this case, our formulation should be more realistic than the others, especially when there are only a few allowed sacrifice times. For some other related work, see Louis and Orav (1985), Turnbull and Hayter (1985) and Morris (1987).

In Section 3, based on a collection of one-sided sequential tests, a general method for constructing asymptotically efficient rules is described. In Section 4, an application to the serial sacrifice problem is discussed. In Section 5, a simulation study is reported for the finite sample case.

2. A lower bound for the expected regret. In this section, it is convenient to assume (and we shall assume) that there exist independent random variables $\{X_{in}, 1 \le i \le k, n \ge 1\}$ such that for each i, $\{X_{in}, n \ge 1\}$ is i.i.d. with common density $f_i(x|\theta)$ ("a random sample from Π_i"). For each irreversible rule ϕ, the associated $T_N(i)$ defined in (1.4) can then be viewed as an $\mathscr{F}_n(i)$-stopping time, where

$$\mathscr{F}_n(1) = \sigma\big(X_{1j}: 1 \le j \le n\big) \quad \text{and for } i > 1,$$

$$\mathscr{F}_n(i) = \sigma\big(X_{mj}, T_N(m): 1 \le m < i, 1 \le j \le T_N(m)\big) \vee \sigma\big(X_{ij}, 1 \le j \le n\big). \tag{2.1}$$

The following lemma provides a constraint for the expected sample size.

Lemma 2.1. *Let ϕ_N be a sequence of irreversible rules which satisfies* (1.16). *Then for every $\theta \in \bigcup_{m=j+1}^{k}\Theta_m$ and every $\lambda \in \Theta_j^*$,*

$$\liminf_{N\to\infty} \left[\sum_{i=1}^{j} I_i(\theta,\lambda)E_\theta(T_N(i))\right]\Big/\log N \ge 1. \tag{2.2}$$

Remark. Our proof below follows closely that of Theorem 2 of Lai and Robbins (1985).

Proof. Since $\lambda \in \Theta_j^*$, $h_j(\lambda) > h_i(\lambda)$ for $i \ne j$. By (1.16)

$$N - E_\lambda(T_N(j)) = \sum_{i\ne j} E_\lambda(T_N(i)) = o(N^a) \quad \text{for } a > 0. \tag{2.3}$$

In view of (2.3) and the Markov inequality, for any $\delta < 1$,

$$\begin{aligned} &P_\lambda\left[\sum_{i=1}^{j} I_i(\theta,\lambda)T_N(i) < (1-\delta)\log N\right] \\ &\quad \le I_j(\theta,\lambda)[N - E_\lambda(T_N(j))]/[I_j(\theta,\lambda)N - (1-\delta)\log N] \\ &\quad = o(N^{a-1}) \quad \text{for } a > 0. \end{aligned} \tag{2.4}$$

Now for $\mathbf{n} = (n_1,\dots,n_j)$ and $\theta, \lambda \in \Theta$, define

$$L(\theta,\lambda,\mathbf{n}) = \sum_{i=1}^{j}\sum_{n=1}^{n_i} \log[f_i(X_{in}|\theta)/f_i(X_{in}|\lambda)].$$

Let $\mathbf{T}_N = (T_N(1),\dots,T_N(j))$ and for $\delta > a$, set

$$A_N = \left\{\sum_{i=1}^{j} I_i(\theta,\lambda)T_N(i) < (1-\delta)\log N,\; L(\theta,\lambda,\mathbf{T}_N) \le (1-a)\log N\right\}. \tag{2.5}$$

Then by (2.4), (2.5) and Wald's likelihood ratio identity [Siegmund (1985)],

$$
\begin{aligned}
P_\theta[A_N] &= \int_{A_N} \exp[L(\theta, \lambda, \mathbf{T}_N)]\, dP_\lambda \\
&\le \int_{A_N} \exp[(1-a)\log N]\, dP_\lambda \qquad (2.6)\\
&= N^{1-a} P_\lambda(A_N) = o(1).
\end{aligned}
$$

In view of the strong law of large numbers and (1.8), as $\sum_{i=1}^{j} n_i \to \infty$,

$$
\begin{aligned}
\left| L(\theta, \lambda, \mathbf{n}) - \sum_{i=1}^{j} I_i(\theta, \lambda) n_i \right| &= o\left(\sum_{i=1}^{j} n_i \right) \\
&= o\left(\sum_{i=1}^{j} I_i(\theta, \lambda) n_i \right), \quad \text{a.s. } [P_\theta].
\end{aligned}
$$

Since $1 - a > 1 - \delta$, it follows that as $N \to \infty$,

$$
P_\theta\left\{ L(\theta, \lambda, \mathbf{n}) > (1-a)\log N \text{ for some } \mathbf{n} \text{ such that } \sum_{i=1}^{j} I_i(\theta, \lambda) n_i < (1-\delta)\log N \right\} \to 0.
$$

This in turn implies that as $N \to \infty$,

$$
P_\theta\left\{ L(\theta, \lambda, \mathbf{T}_N) > (1-a)\log N, \ \sum_{i=1}^{j} I_i(\theta, \lambda) T_N(i) < (1-\delta)\log N \right\} \to 0. \qquad (2.7)
$$

By (2.5), (2.6) and (2.7)

$$
\lim_{N\to\infty} P_\theta\left\{ \sum_{i=1}^{j} I_i(\theta, \lambda) T_N(i) < (1-\delta)\log N \right\} = 0,
$$

from which (2.2) follows. □

Applying Lemma 2.1 successively for $j = 1, \dots, l$, we obtain the following corollary.

COROLLARY 2.2. *Assume that* (1.8) *holds. Let* ϕ_N *be a sequence of irreversible rules which satisfies* (1.16). *Then for every* $\theta \in \Theta_{l+1}$ *and* $\lambda_i \in \Theta_i^*$,

$1 \le i \le l$,

$$\text{(2.8)} \qquad \begin{cases} \liminf_{N\to\infty} I_1(\theta, \lambda_1) E_\theta[T_N(1)]/\log N \ge 1, \\ \quad \vdots \\ \liminf_{N\to\infty} \sum_{i=1}^{l} I_i(\theta, \lambda_l) E_\theta[T_N(i)]/\log N \ge 1. \end{cases}$$

Since our goal is to minimize $\Sigma_{i=1}^{l}[h^*(\theta) - h_i(\theta)]E_\theta(T_N(i))$, (2.8) leads us to consider the following linear programming problem. For the background knowledge of the linear programming and the terminology used in this article, the reader is referred to Duffin, Peterson and Zener (1967).

PROBLEM B. Minimize $\Sigma_{i=1}^{l} b_i z_i$, subject to the conditions

$$\text{(2.9)} \qquad \begin{cases} a_{11}z_1 \ge 1, \\ a_{21}z_1 + a_{22}z_2 \ge 1, \\ \quad \vdots \\ a_{l1}z_1 + \cdots + a_{ll}z_l \ge 1, \end{cases}$$

and

$$\text{(2.10)} \qquad z_i \ge 0, \qquad i = 1, \dots, l.$$

LEMMA 2.3. *Assume that for* $1 \le i \le l$,

$$\text{(2.11)} \qquad b_i > 0 \quad \text{and} \quad a_{ij} > 0 \quad \text{for } 1 \le j \le i.$$

Then Problem B *has a solution.*

The proof of this lemma is easy and we omit it.

From now on we assume that $\Lambda_l = \Theta_1^* \times \cdots \times \Theta_l^*$ is nonempty. For each $\lambda = (\lambda_1, \dots, \lambda_l) \in \Lambda_l$ and $\theta \in \Theta_{l+1}$, set $b_i = h^*(\theta) - h_i(\theta)$ and $a_{ij} = I_j(\theta, \lambda_i)$ in Problem B. Assume that (1.8) holds; then (2.11) holds as well. By Lemma 2.3, Problem B has a solution. We denote its minimum by $r(\theta, l, \lambda)$.

THEOREM 2. *Assume that* (1.8) *holds and* Λ_l *is nonempty. Let* ϕ_N *be a sequence of irreversible rules which satisfies* (1.16). *Then for every* $\theta \in \Theta_{l+1}$,

$$\text{(2.12)} \qquad \liminf_{N\to\infty} R_N(\theta)/\log N \ge \sup_{\lambda \in \Lambda_l} r(\theta, l, \lambda).$$

PROOF. If $\liminf_{N\to\infty} R_N(\theta)/\log N = \infty$, then (2.12) is automatically satisfied. Assume that $\liminf_{N\to\infty} R_N(\theta)/\log N = c < \infty$. Since $h^*(\theta) - h_i(\theta) > 0$ for $1 \le i \le l$ and $R_N(\theta) \ge \Sigma_{i=1}^{l}(h^*(\theta) - h_i(\theta))E_\theta(T_N(i))$, $\liminf_{N\to\infty} E_\theta(T_N(i))/\log N < \infty$ for $1 \le i \le l$. We can choose a subsequence N_n such that

$$\lim_{n\to\infty} R_{N_n}(\theta)/\log N_n = c$$

and

$$\lim_{n\to\infty} E_\theta\big(T_{N_n}(i)\big)/\log N_n = z_i, \qquad 1 \le i \le l. \tag{2.13}$$

It is clear that

$$z_i \ge 0 \quad \text{for } 1 \le i \le l \tag{2.14}$$

and

$$c \ge \sum_{i=1}^{l} \big(h^*(\theta) - h_i(\theta)\big) z_i. \tag{2.15}$$

For each $\lambda \in \Lambda_l$, by Corollary 2.2, we have that

$$\begin{cases} I_1(\theta, \lambda_1) z_1 \ge 1, \\ \quad\vdots \\ I_1(\theta, \lambda_l) z_1 + \cdots + I_l(\theta, \lambda_l) z_l \ge 1. \end{cases} \tag{2.16}$$

By (2.14) and (2.16), $\sum_{i=1}^{l}(h^*(\theta) - h_i(\theta))z_i \ge r(\theta, l, \lambda)$. Hence

$$\sum_{i=1}^{l} \big(h^*(\theta) - h_i(\theta)\big) z_i \ge \sup_{\lambda \in \Lambda_l} r(\theta, l, \lambda).$$

This and (2.15) complete our proof. □

REMARK. In Theorem 2, the monotone structure (1.9) was not necessary and Θ could have been any set.

The following lemma provides a link between the lower bound of (2.12) and that of Theorem 1.

LEMMA 2.4. *Assume that* (1.8), (1.9) *and* (1.11) *hold. Then the minimum of Problem* A *is*

$$r(\theta, l) = \sup_{\lambda \in \Lambda_l} r(\theta, l, \lambda). \tag{2.17}$$

PROOF. By our assumptions, we have that $\Theta_{l+1} = [\theta_{l+1}, \theta_l)$, $\Lambda_l = (\theta_1, U) \times \cdots \times (\theta_l, \theta_{l-1})$ and $\theta_{l+1} < \theta_l < \cdots < \theta_1 < U$. Hence for each $\lambda \in \Lambda_l$, $\lambda_i > \theta_i$ for $1 \le i \le l$. By (1.11), for any $\theta \in \Theta_{l+1}$, we have

$$I_j(\theta, \theta_i) \le I_j(\theta, \lambda_i), \qquad 1 \le i \le l, 1 \le j \le l. \tag{2.18}$$

Now let $\mathbf{z}$ be a solution of Problem A. In view of (1.14) and (2.18), $\mathbf{z}$ also satisfies (2.16). Consequently,

$$r(\theta, l) = \sum_{i=1}^{l} \big(h^*(\theta) - h_i(\theta)\big) z_i \ge r(\theta, l, \lambda).$$

Hence

$$r(\theta, l) \ge \sup_{\lambda \in \Lambda_l} r(\theta, l, \lambda). \tag{2.19}$$

Choose $\lambda_n = (\lambda_1(n), \dots, \lambda_l(n)) \in \Lambda_l$ such that

$$\lim_{n\to\infty} \lambda_n = (\theta_1, \dots, \theta_l). \tag{2.20}$$

Fix $\theta \in \Theta_{l+1}$. Let $\mathbf{z}_n = (z_1(n), \dots, z_l(n))$ be a solution of Problem B with $b_i = h^*(\theta) - h_i(\theta)$ and $a_{ij} = I_j(\theta, \lambda_i(n))$. Set

$$c_i(n) = \max\{I_i(\theta, \lambda_i(n))/I_i(\theta, \theta_i), \dots, I_i(\theta, \lambda_l(n))/I_i(\theta, \theta_l)\}. \tag{2.21}$$

In view of (2.20) and (1.11),

$$\lim_{n\to\infty} c_i(n) = 1, \qquad 1 \le i \le l. \tag{2.22}$$

By (2.21), $(c_1(n)z_1(n), \dots, c_l(n)z_l(n))$ satisfies (1.14) and (1.15). Hence

$$\Big[\max_{1\le i\le l} c_i(n)\Big] r(\theta, l, \lambda_n) \ge \sum_{i=1}^{l} b_i c_i(n) z_i(n) \ge r(\theta, l).$$

Applying (2.22), we obtain

$$\sup_{\lambda\in\Lambda_l} r(\theta, l, \lambda) \ge r(\theta, l).$$

Now (2.17) follows from this and (2.19). □

PROOF OF THEOREM 1. Since all conditions of Theorem 2 and Lemma 2.4 are satisfied, (1.17) is a direct consequence of (2.12) and (2.17). □

To understand the lower bound (1.17) of Theorem 1, it is natural to ask if we replace all inequalities in (1.14) by identities whether the solution of the resulting equations is a solution of Problem A. To answer this question, let us consider Problem B. Let $\mathbf{0} = (0, \dots, 0)$, $\mathbf{1} = (1, \dots, 1)$ and $\mathbb{C}$ be the $l \times l$ triangular matrix with components

$$c_{ij} = \begin{cases} a_{ij}/b_j, & l \ge i \ge j, \\ 0, & \text{otherwise}. \end{cases} \tag{2.23}$$

We shall use the conventions that $\mathbf{x} \ge \mathbf{y}$ iff $x_i \ge y_i$ for $1 \le i \le l$ and $\mathbf{x} > \mathbf{y}$ iff $x_i > y_i$ for $1 \le i \le l$. We shall also use " ′ " to denote the transpose of a vector or a matrix.

LEMMA 2.5. *In Problem* B, *assume that* (2.11) *holds and the solution* $\mathbf{z}_0$ *of the equations*

$$\begin{cases} a_{11}z_1 = 1, \\ a_{21}z_1 + a_{22}z_2 = 1, \\ \quad\vdots \\ a_{l1}z_1 + \cdots + a_{ll}z_l = 1 \end{cases} \tag{2.24}$$

satisfies (2.10). *Then* $\mathbf{z}_0$ *is a solution of Problem* B *if*

$$\mathbf{1}\mathbb{C}^{-1} \ge \mathbf{0} \tag{2.25}$$

and $\mathbf{z}_0$ *is the unique solution if* (2.25) *is strengthened to be*

$$\mathbf{1}\mathbb{C}^{-1} > \mathbf{0}. \tag{2.26}$$

PROOF. By changing the variables $\alpha = (b_1 z_1, \dots, b_l z_l)$, Problem B is transformed into the problem to minimize $\sum_{i=1}^{l} \alpha_i$ subject to conditions

$$\alpha\mathbb{C}' \geq \mathbf{1} \tag{2.27}$$

and

$$\alpha \geq \mathbf{0}. \tag{2.28}$$

The solution $\mathbf{z}_0$ of (2.24) is also transformed into a vector γ which satisfies $\gamma\mathbb{C}' = \mathbf{1}$ and $\gamma \geq \mathbf{0}$. Now let α be a solution of the new problem. Clearly,

$$\sum_{i=1}^{l} \alpha_i - \sum_{i=1}^{l} \gamma_i \leq 0. \tag{2.29}$$

By (2.25) and (2.27), there exist $\mathbf{s} \geq \mathbf{0}$ and $\mathbf{y} \geq 0$ such that $\mathbf{1} = \mathbf{s}\mathbb{C}$ and $\alpha\mathbb{C}' - \mathbf{1} = \mathbf{y}$. Hence

$$\alpha\mathbf{1}' - \mathbf{y}\mathbf{s}' = \alpha\mathbb{C}'\mathbf{s}' - \mathbf{y}\mathbf{s}' = \mathbf{1}\mathbf{s}' = \gamma\mathbb{C}'\mathbf{s}' = \gamma\mathbf{1}'$$

or

$$\sum_{i=1}^{l} \alpha_i - \sum_{i=1}^{l} \gamma_i = \sum_{i=1}^{l} y_i s_i \geq 0. \tag{2.30}$$

In view of (2.29) and (2.30), γ is a solution of the new problem and

$$\sum_{i=1}^{l} y_i s_i = 0. \tag{2.31}$$

Furthermore, if (2.26) holds, then $\mathbf{s} > \mathbf{0}$. The fact $\mathbf{y} \geq \mathbf{0}$ and (2.31) imply that $\mathbf{y} = \mathbf{0}$. Consequently, $\alpha\mathbb{C}' = \mathbf{1}$. Since $\mathbb{C}$ is invertible, $\gamma = \alpha$. □

LEMMA 2.6. *In* (2.24), *assume that* $a_{ij} > 0$ *for* $l \geq i \geq j \geq 1$. *If for each* j,

$$a_{ij}\ (\textit{strictly}) \downarrow \text{ as } i \uparrow \text{ for } l \geq i \geq j, \tag{2.32}$$

then the solution $\mathbf{z} \geq \mathbf{0}$ $(\mathbf{z} > \mathbf{0})$.

PROOF. The proof follows easily from the identities

$$(a_{i,1} - a_{i+1,1})z_1 + \cdots + (a_{i,i} - a_{i+1,i})z_i = a_{i+1,i+1}z_{i+1}, \quad 1 \leq i \leq l-1. \tag{2.33}$$

Since (2.25) is equivalent to

$$\mathbf{1} = \mathbf{y}\mathbb{C} \quad \text{for some } \mathbf{y} \geq \mathbf{0}, \tag{2.34}$$

with the same proof as in Lemma 2.6 we have the following result.

LEMMA 2.7. *Assume that* $b_i > 0$ *and* $a_{ij} > 0$ *for* $l \geq i \geq j \geq 1$. *If for each* i,

$$c_{ij} \text{ (strictly) } \uparrow \text{ as } j \uparrow \text{ for } i \geq j \geq 1, \tag{2.35}$$

then (2.25) [(2.26)] *holds.*

As a summary, we state the following theorem which provides a necessary and sufficient condition for a sequence of rules to be asymptotically efficient.

THEOREM 3. *Assume that* (1.8), (1.9) *and* (1.11) *hold. Let* ϕ_N *be a sequence of irreversible rules. Suppose that for any* $\theta \in \Theta_{l+1}$, *any* $i \leq l$,

$$I_j(\theta, \theta_i)/\big(h^*(\theta) - h_j(\theta)\big) \uparrow \text{ as } j \uparrow \text{ for } 1 \leq j \leq i. \tag{2.36}$$

Then ϕ_N *is asymptotically efficient if* (1.13b) *holds and*

$$\lim_{N \to \infty} E_\theta\big(T_N(i)\big)/\log N = z_i, \qquad \forall\, \theta \in \Theta_{l+1}, \tag{2.37}$$

where z_i *solve*

$$\begin{cases} I_1(\theta, \theta_1) z_1 = 1, \\ \vdots \\ I_1(\theta, \theta_l) z_1 + \cdots + I_l(\theta, \theta_l) z_l = 1. \end{cases} \tag{2.38}$$

Furthermore, if (2.36) *is strengthened to be*

$$I_j(\theta, \theta_i)/\big(h^*(\theta) - h_j(\theta)\big) \text{ strictly } \uparrow \text{ as } j \uparrow \text{ for } 1 \leq j \leq i, \tag{2.39}$$

then (1.13b), (2.37) *and* (2.38) *are also necessary for* ϕ_N *to be asymptotically efficient.*

PROOF. Fix $\theta \in \Theta_{l+1}$. Let $a_{ij} = I_j(\theta, \theta_i)$ and $b_j = h^*(\theta) - h_j(\theta)$. Then Problem A is equivalent to Problem B. Define c_{ij} as in (2.23). Then (2.36) is equivalent to (2.34). Hence (2.25) holds by Lemma 2.7. From (2.37), (2.38) and $\mathbf{z} \geq \mathbf{0}$, (2.24) follows. By Lemma 2.5, $\mathbf{z}$ is a solution of Problem A, that is,

$$\lim_{N \to \infty} R_N(\theta)/\log N = \sum_{i=1}^{l} b_i z_i = r(\theta, l).$$

Thus (1.13a) is satisfied and ϕ_N is therefore asymptotically efficient. On the other hand, if (2.39) holds, then by Lemmas 2.7 and 2.5, Problem A has a unique solution, say, $\mathbf{z}_0$. Let ϕ_N be asymptotically efficient. Then (1.13b) holds by definition and, in view of (1.13a), any limit point $\mathbf{z}$ of

$$\big\{\big(E_\theta[T_N(1)], \ldots, E_\theta[T_N(l)]\big)/\log N \colon N \geq 2\big\}$$

satisfies (2.24). By Lemma 2.5, $\mathbf{z} = \mathbf{z}_0$. Consequently, (2.37) and (2.38) hold. □

3. Construction of efficient rules. In this section we describe a general method of constructing asymptotically efficient rules under the assumptions of Theorem 3. First, note that the monotonicity assumption (1.9) suggests that we consider a class of one-sided tests. More precisely, in order to ensure $R_n(\theta) = O(1)$ when the first population is the best one, we always start sampling from Π_1. Since $\Theta_1 = [\theta_1, U)$, we then perform a one-sided test to see whether $\theta < \theta_1$. In view of Theorem 3, if ϕ_N is asymptotically efficient and $\theta < \theta_1$, then we need about $\log N/I_1(\theta, \theta_1)$ observations from Π_1 to be reasonably confident that the best one is ahead. We then sample from Π_2 and perform the one-sided test to see whether $\theta < \theta_2$. Since the observations from Π_1 carry some information about θ, they should be incorporated into the new test. As a result, instead of taking $\log N/I_2(\theta, \theta_2)$ observations from Π_2, Theorem 3 informs us that about $[1 - I_1(\theta, \theta_2)/I_1(\theta, \theta_1)]\log N/I_2(\theta, \theta_2)$ observations would be sufficient to be reasonably sure that the best one is still ahead if $\theta < \theta_2$. This procedure goes on until all N samples have been taken.

To fix the ideas, for each $1 \le i \le k$, let $\{X_{in}\}$ be a random sample from Π_i. For any sequence of integer-valued random variables $T_N(1), \dots, T_N(k)$, define $\mathscr{F}_n(i)$ as in (2.1). From Section 2, for any irreversible rule ϕ the associated $T_N(i)$ defined in (1.5) is an $\mathscr{F}_n(i)$-stopping time. Conversely, if for each i, $T_N(i)$ is an $\mathscr{F}_n(i)$-stopping time, then $\phi = \{\phi_j\}$ is an irreversible rule where

$$\phi_j = l \quad \text{if } \sum_{i=0}^{l-1} T_N(i) < j \le \sum_{i=0}^{l} T_N(i) \text{ and } T_N(0) = 0. \tag{3.1}$$

Our goal is therefore to construct $\mathscr{F}_n(i)$-stopping times $T_N(i)$ that satisfy (1.13b), (2.37) and (2.38). To this end, for each l, let F_l be a probability distribution with support (L, θ_l). For nonnegative integers $n_1, \dots, n_l$, define

$$M_l(n_1, \dots, n_l) = \int_L^{\theta_l} \prod_{i=1}^{l} \prod_{n=1}^{n_i} f_i(X_{in}|\theta)\, dF_l(\theta) \Bigg/ \left[\prod_{i=1}^{l} \prod_{n=1}^{n_i} f_i(X_{in}|\theta_l)\right]. \tag{3.2}$$

Now define $T_N(i)$, $1 \le i \le k$, inductively by

$$\begin{aligned} T_N(0) &= 0, \\ \tau_N(l) &= \inf\{n: M_l(T_N(1), \dots, T_N(l-1), n) > N\}, \\ T_N(l) &= \min\left\{\tau_N(l), N - \sum_{i=1}^{l-1} T_N(i)\right\}, \quad \text{for } 1 \le l < k, \\ T_N(k) &= N - \sum_{i=1}^{k-1} T_N(i). \end{aligned} \tag{3.3}$$

Clearly, $T_N(i)$ is an $\mathscr{F}_n(i)$-stopping time and $\sum_{i=1}^{k} T_N(i) = N$. It is also clear from (3.3) that $\tau_N(l)$ is the sample size of a sequential one-sided test for testing H_0: $\theta < \theta_l$. The idea of using the mixture of likelihood ratios such as (3.2) in the sequential testing problem is due to Robbins (1970).

In the following we shall assume that for $1 \le i \le k$,

$$f_i(x|\theta) = \exp(\alpha_i(\theta)x - \psi_i(\theta)) \tag{3.4}$$

where α_i is a continuous and strictly increasing function. If we set $\eta = \alpha_i(\theta)$,

$\Phi_i(\eta) = \psi_i(\alpha_i^{-1}(\eta))$ and $V_i = \alpha_i((L, U))$, then V_i is an open interval and

$$\tilde{f}_i(x|\eta) = \exp(\eta x - \Phi_i(\eta)) \tag{3.5}$$

a canonical exponential family with parameter space V_i. Under this setting, the asymptotic behavior of $E_\theta(\tau_N(1))$ had been studied thoroughly by Pollak and Siegmund (1975). In fact, the proof of the following lemma follows closely that of Theorem 1 of Pollak and Siegmund (1975). Since we only have to obtain the first-order approximation, our proof is simpler.

LEMMA 3.1. *Assume that* (3.4) *holds,*

$$L < \theta_{k-1} < \cdots < \theta_1 < U \tag{3.6}$$

and

$$0 < \sigma_i^2(\theta) = \operatorname{var}_\theta(X_{i1}) < \infty \quad \text{for } 1 \le i \le k. \tag{3.7}$$

Let $\{T_N(i)\}$ *be defined as in* (3.3). *Then for* $\theta \ge \theta_m$ (1.12) *holds and for* $1 \le l \le k-1$ *and* $\theta \in [\theta_{l+1}, \theta_l)$,

$$\limsup_{N\to\infty} \left[\sum_{i=1}^{l} I_i(\theta, \theta_l) E_\theta T_N(i) \right] \Big/ \log N \le 1. \tag{3.8}$$

Before we give the proof of Lemma 3.1, we need another lemma.

LEMMA 3.2. *Assume that* (3.4) *and* (3.7) *hold. Then* (1.8) *holds,*

(3.9) $\quad$ *$I_i(\theta, \lambda)$, as a function of λ, is continuous and increasing for $\lambda \ge \theta$*

and

$$E_\beta\big(f_i(X_{i1}|\theta)/f_i(X_{i1}|\lambda)\big) \le 1 \quad \text{for } \beta \ge \lambda \ge \theta. \tag{3.10}$$

PROOF. Fix i. Let us use (3.5). By (3.7),

$$0 < \sigma^2\big(\alpha_i^{-1}(\eta)\big) = \Phi_i''(\eta) \quad \text{for all } \eta \in V_i. \tag{3.11}$$

Thus Φ_i is strictly convex and Φ_i' is strictly increasing. Now, for $\gamma, \eta \in V_i$, let $\tilde{I}_i$ be the Kullback–Leibler number of $\tilde{f}_i(x|\gamma)$ with respect to $\tilde{f}_i(x|\eta)$. Then

$$\begin{aligned} \tilde{I}_i(\gamma, \eta) &= (\gamma - \eta) E_\eta(X_{i1}) - [\Phi_i(\gamma) - \Phi_i(\eta)] \\ &= (\gamma - \eta)\Phi_i'(\gamma) - \Phi_i(\gamma) + \Phi_i(\eta). \end{aligned} \tag{3.12}$$

Hence

$$\frac{\partial \tilde{I}_i(\gamma, \eta)}{\partial \eta} = \Phi_i'(\eta) - \Phi_i'(\gamma) > 0 \quad \text{for } \eta > \gamma.$$

Since $\tilde{I}_i(\gamma, \gamma) = 0$ and $I_i(\theta, \lambda) = \tilde{I}_i(\alpha_i(\theta), \alpha_i(\lambda))$, this implies (1.8) as well as (3.9). For (3.10), let $\eta \ge \gamma \ge \zeta$. Then

$$\eta \ge \eta - \gamma + \zeta \ge \zeta. \tag{3.13}$$

Hence $\eta - \gamma + \zeta \in V_i$. This implies the finiteness of

$$(3.14)\quad \begin{aligned} E_\eta\big(\tilde{f}_i(X_{i1}|\zeta)/\tilde{f}_i(X_{i1}|\gamma)\big) &= e^{-\Phi_i(\zeta)+\Phi_i(\gamma)-\Phi_i(\eta)}\int e^{(\eta-\gamma+\zeta)x}\,d\nu(x) \\ &= e^{-\Phi_i(\zeta)+\Phi_i(\gamma)-\Phi_i(\eta)+\Phi_i(\eta-\gamma+\zeta)} \le e^0 = 1 \end{aligned}$$

by (3.13) and the convexity of Φ_i.

Now, (3.10) follows directly from (3.14). □

PROOF OF LEMMA 3.1. Given $\theta \ge \dot{\theta}_m$, let us first show that (1.12) holds. For this, it is sufficient to show that

$$(3.15)\quad P_\theta[\tau_N(m) < \infty] \le 1/N.$$

This is because (3.15) implies that

$$P_\theta[T_N(1) + \cdots + T_N(m) = N] \ge P_\theta[\tau_N(m) = \infty] \ge 1 - 1/N.$$

Consequently,

$$(3.16)\quad \sum_{j=m+1}^{k} E_\theta(T_N(j)) \le (k-m)NP_\theta\left[\sum_{j=m+1}^{k} T_N(j) > 0\right] \le k - m.$$

For (3.15), it follows from a similar argument as that given in (15) of Robbins (1970).

Now given $\theta < \theta_l$, let us show (3.8). In the remaining part of this proof, to simplify the notation, we shall use T_i instead of $T_N(i)$. Let $\mu_i(\theta) = E_\theta(X_{i1})$, $S_{T_i} = \sum_{n=1}^{T_i} X_{in}$, $T = \min\{\tau_N(l) - 1, T_l\}$, $S_T = \sum_{n=1}^{T} X_{ln}$ and

$$(3.17)\quad \begin{aligned} II(\lambda,\gamma) = &\left\{\sum_{i=1}^{l-1}[\alpha_i(\lambda) - \alpha_i(\gamma)]S_{T_i} - T_i[\psi_i(\lambda) - \psi_i(\gamma)]\right\} \\ &+ [\alpha_l(\lambda) - \alpha_l(\gamma)]S_T - T[\psi_l(\lambda) - \psi_l(\gamma)]. \end{aligned}$$

By the definition of T,

$$(3.18)\quad \begin{aligned} \log N &\ge \log M_l(T_1,\dots,T_{l-1},T) \\ &= \log\int_L^{\theta_l} \exp(II(\lambda,\theta_l))\,dF_l(\lambda) \\ &= II(\theta,\theta_l) + \log\int_L^{\theta_l}\exp(II(\lambda,\theta))\,dF_l(\lambda) \\ &\ge II(\theta,\theta_l) + \log\int_{|\lambda-\theta|<\delta}\exp(II(\lambda,\theta))\,dF_l(\lambda), \end{aligned}$$

for any δ such that $L < \theta - \delta < \theta + \delta < \theta_l$.

By Jensen's inequality,

$$(3.19)\quad \begin{aligned} &\int_{|\lambda-\theta|<\delta}\exp(II(\lambda,\theta))\,dF_l/F_l((\theta-\delta,\theta+\delta)) \\ &\quad\ge \exp\int_{|\lambda-\theta|} II(\lambda,\theta)\,dF_l/F_l((\theta-\delta,\theta+\delta)). \end{aligned}$$

In view of (3.18) and (3.19),

$$\log N \geq II(\theta, \theta_l) + \log F_l((\theta - \delta, \theta + \delta)) + \int_{|\lambda - \theta|} II(\lambda, \theta)\, dF_l / F_l((\theta - \delta, \theta + \delta)). \tag{3.20}$$

Applying the identity $I_i(\lambda, \gamma) = [\alpha_i(\lambda) - \alpha_i(\gamma)]\mu_i(\lambda) - [\psi_i(\lambda) - \psi_i(\gamma)]$ to (3.17), we obtain

$$\begin{aligned} II(\lambda, \gamma) = &\left\{ \sum_{i=1}^{l-1} [\alpha_i(\lambda) - \alpha_i(\gamma)] \left[S_{T_i} - \mu_i(\lambda) T_i \right] \right\} \\ &+ [\alpha_l(\lambda) - \alpha_l(\gamma)][S_T - \mu_l(\lambda) T] \\ &+ \sum_{i=1}^{l-1} I_i(\lambda, \gamma) T_i + I_l(\lambda, \gamma) T. \end{aligned} \tag{3.21}$$

Since $T + 1 \leq N + 1$ is a stopping time and all stopping times T_i involved in (3.21) are bounded by N, Wald's identity [Chow and Teicher (1978), page 137] implies that

$$\begin{aligned} E_\theta II(\lambda, \gamma) = &\left\{ \sum_{i=1}^{l-1} [\alpha_i(\lambda) - \alpha_i(\gamma)][\mu_i(\theta) - \mu_i(\lambda)] E_\theta(T_i) \right\} \\ &+ [\alpha_l(\lambda) - \alpha_l(\gamma)][\mu_i(\theta) - \mu_i(\lambda)] E_\theta(T + 1) \\ &+ \sum_{i=1}^{l-1} I_i(\lambda, \gamma) E_\theta T_i + I_l(\lambda, \gamma) E_\theta T \\ &- [\alpha_l(\lambda) - \alpha_l(\gamma)] E_\theta [X_{l, T+1} - \mu_l(\lambda)]. \end{aligned} \tag{3.22}$$

Hence

$$\begin{aligned} E_\theta II(\theta, \theta_l) = &\sum_{i=1}^{l-1} I_i(\theta, \theta_l) E_\theta T_i + I_l(\theta, \theta_l) E_\theta(T + 1) \\ &- [\alpha_l(\theta) - \alpha_l(\theta_l)] E_\theta [X_{l, T+1} - \mu_l(\theta)]. \end{aligned} \tag{3.23}$$

Since α_i and μ_i are continuous, given $\varepsilon > 0$, we can choose δ so small that for all $1 \leq i \leq l$, $|\lambda - \theta| < \delta$ implies

$$|\alpha_i(\lambda) - \alpha_i(\theta)|\, |\mu_i(\theta) - \mu_i(\lambda)| \leq \varepsilon. \tag{3.24}$$

Now by Wald's identity and Hölder's inequality,

$$E_\theta |X_{l, T+1} - \mu_l(\theta)| \leq E_\theta \left[\sum_{n=1}^{T+1} (X_{ln} - \mu_l(\theta))^2 \right]^{1/2} \leq \sigma_\theta [E_\theta(T + 1)]^{1/2},$$

where $\sigma_\theta^2 = \mathrm{Var}_\theta(X_{l1})$. In view of this, (3.22), (3.24) and the fact that $I_i(\lambda, \theta) \geq 0$, we have

$$E_\theta \int_{|\lambda-\theta|<\delta} II(\lambda,\theta)\,dF_l(\lambda) \tag{3.25}$$
$$\geq \left\{ -\varepsilon \left[\sum_{i=1}^{l-1} E_\theta T_i + E_\theta(T+1) \right] - K\left[E_\theta(T+1)\right]^{1/2} \right\}$$
$$\times F_l((\theta-\delta,\theta+\delta)),$$

where $K = \sigma_\theta \sup\{|\alpha_l(\theta) - \alpha_l(\lambda)|: |\theta - \lambda| \leq \delta \text{ or } \theta = \theta_l\}$. Apply (3.23) and (3.25) to (3.20). We obtain

$$\log N \geq \sum_{i=1}^{l-1} [I_i(\theta,\theta_l) - \varepsilon] E_\theta T_i + [I_l(\theta,\theta_l) - \varepsilon] E_\theta(T+1) \tag{3.26}$$
$$-2K\{E_\theta(T+1)\}^{1/2} + \log F_l((\theta-\delta,\theta+\delta)).$$

By the definition of T and T_l, $T \geq T_l - 1$. Note also that $\theta < \theta_l$, by Lemma 3.2, $I_i(\theta,\theta_l) > 0$ for all i. Using these facts and the fact that ε can be arbitrarily small, (3.8) follows from (3.26). □

THEOREM 4. *With the same assumptions as in Lemma* 3.1, *we have that* (i) *for* $\theta \geq \theta_m$, (1.12) *holds, and* (ii) *for* $1 \leq l \leq k-1$ *and* $\theta < \theta_l$, (2.37) *and* (2.38) *hold.*

PROOF. The result (i) had been shown in Lemma 3.1. We only have to show (ii). First, we claim that if $\theta < \theta_l$, then

$$\liminf_{N\to\infty} \sum_{i=1}^{l} I_i(\theta,\theta_l) E_\theta T_N(i)/\log N \geq 1. \tag{3.27}$$

For this, we shall apply Lemma 2.1. In order to apply Lemma 2.1, let $q_i(\theta)$ be a continuous function such that $q_i(\theta) > 0$ for $\theta \in (\theta_i, \theta_{i-1})$ and < 0 for $\theta \notin [\theta_i, \theta_{i-1}]$. Define the reward function $g_i(x,\theta) = q_i(\theta)$. Hence $h_i(\theta) = q_i(\theta)$ and $\Theta_i^* = (\theta_i, \theta_{i-1})$, $\Theta_i = [\theta_i, \theta_{i-1})$ for $1 \leq i < k-1$ and $\Theta_k^* = \Theta_k = (\theta_1, U)$. By Lemma 3.2, (1.8) holds. Furthermore, $\{T_N(i)\}$ satisfies (1.12) and (3.8) by Lemma 3.1. This in turn implies that (1.16) holds. Therefore by Lemma 2.1, for $\theta < \theta_l$, $\lambda \in (\theta_l, \theta_{l-1})$,

$$\liminf_{N\to\infty} \sum_{i=1}^{l} I_i(\theta,\lambda) E_\Theta(T_N(i))/\log N > 1. \tag{3.28}$$

Using the continuity of $I_i(\theta,\lambda)$, (3.27) follows from (3.28).

Now by (3.8) of Lemma 3.1 and (3.27), we have that

$$\lim_{N\to\infty} \sum_{i=1}^{l} I_i(\theta,\lambda) E_\theta(T_N(i))/\log N = 1. \tag{3.29}$$

This implies that any limit point of $(T_N(1)/\log N, \ldots, T_N(l)/\log N)$ should satisfy (2.38). But (2.38) has a unique solution. Hence (2.37) and (2.38) follow. □

As a summary, we state the following theorem.

THEOREM 5. *Assume* (3.4), (3.7), (1.9) *and* (2.36) *hold. Define* $\{T_N(i)\}$ *as in* (3.3). *Then the associated* ϕ_N *defined by* (3.1) *is asymptotically efficient.*

PROOF. By Lemma 3.2, (1.8) holds and (1.11) is a consequence of (3.9). By (i) of Theorem 4, (1.12) is satisfied. This in turn implies (1.13b). By (ii) of Theorem 4, (2.37) and (2.38) hold. Thus all conditions of Theorem 3 are met. Therefore ϕ_N is asymptotically efficient. □

4. An application. In this section we shall apply Theorem 5 to the serial sacrifice problem stated in Section 1. More precisely, for $\theta, t > 0$, let

$$f(x|t,\theta) = (1 - e^{-\theta t})^x (e^{-\theta t})^{1-x}, \qquad x = 0, 1. \tag{4.1}$$

Also let $0 < t_1 < \cdots < t_k$. Then Π_i is specified by the density $f_i(x|\theta) = f(x|t_i, \theta)$ with respect to ν, the counting measure on $\{0, 1\}$. Note that

$$f(x|t,\theta) = \exp[\alpha(t,\theta)x - \psi(t,\theta)] \tag{4.2}$$

where

$$\alpha(t,\theta) = \ln(e^{\theta t} - 1) \quad \text{and} \quad \psi(t,\theta) = \theta t.$$

It is easy to see that (3.4) and (3.7) hold with $\Theta = (0, \infty)$. By (4.10) for $\theta < \lambda$,

$$\frac{\partial I_j(\theta,\lambda)}{\partial \lambda} = t_j(e^{-t_j\theta} - e^{-t_j\lambda})/(1 - e^{-t_j\lambda}) > 0.$$

Hence (1.11) holds. Now, it only remains to verify (1.9) and (2.36) to obtain asymptotically efficient rules via Theorem 5. For this, the following lemma provides a convenient sufficient condition to prove (1.9).

LEMMA 4.1. *Assume that all* h_i *are continuous and that there exists* $\{\theta_i, 1 \le i < k\}$, *such that* $L = \theta_k < \theta_{k-1} < \cdots < \theta_1 < \theta_0 = U$,

$$\{\theta\colon h_i(\theta) > h_{i+1}(\theta)\} = (\theta_i, U) \tag{4.3}$$

and

$$\{\theta\colon h_i(\theta) < h_{i+1}(\theta)\} = (L, \theta_i) \quad \textit{for } 1 \le i < k. \tag{4.4}$$

Then (1.9) *holds.*

PROOF. By the assumptions, $(\theta_1, \theta_0) \subset (\theta_2, \theta_0) \subset \cdots \subset (\theta_{k-1}, \theta_0)$ and $(\theta_k, \theta_1) \supset (\theta_k, \theta_2) \supset \cdots \supset (\theta_k, \theta_{k-1})$. Hence if $h_i(\theta) > h_{i+1}(\theta)$, then $h_i(\theta) > h_j(\theta)$ for $j > i$. Similarly, if $h_{i-1}(\theta) < h_i(\theta)$, then $h_j(\theta) < h_i(\theta)$ for $j < i$. By the continuity of h_j, $h_i(\theta_i) = h_{i+1}(\theta_i)$. These facts clearly imply that $\Theta_1^* = (\theta_1, \theta_0)$ and $\Theta_1 = [\theta_1, \theta_0)$. Replace the roles of θ_1 and θ_0 by θ_1 and θ_2. It follows that $\Theta_2^* = (\theta_2, \theta_1)$ and $\Theta_2 = [\theta_2, \theta_1)$. Applying this argument inductively, we obtain (1.9). □

Now let us return to our special case and define

$$h(t,\theta) = t^2/(e^{t\theta} - 1). \tag{4.5}$$

Then

$$h_i(\theta) = E_\theta\left(\frac{\partial \ln f_i}{\partial \theta}\right)^2 = h(t_i, \theta).$$

LEMMA 4.2. *The function h has the following three properties*:

(4.6) *For each $\theta > 0$, there is $t(\theta) > 0$ such that h, as a function of t, is strictly increasing in $(0, t(\theta))$ and strictly decreasing in $(t(\theta), \infty)$. Consequently, $h(t(\theta), \theta) = \sup_t h(t, \theta)$.*

(4.7) *Let $t_3 > t_2 > t_1 > 0$. Then for $\theta > 0$, $h(t_1, \theta) > h(t_2, \theta) \Rightarrow h(t_2, \theta) > h(t_3, \theta)$. Similarly, for $\theta > 0$, $h(t_3, \theta) > h(t_2, \theta) \Rightarrow h(t_2, \theta) > h(t_1, \theta)$. Furthermore, there is no $\theta > 0$ such that $h(t_1, \theta) = h(t_2, \theta) = h(t_3, \theta)$.*

(4.8) *Let $t_2 > t_1 > 0$. Then there is $\infty > \lambda > 0$ such that $\{\theta: h(t_1, \theta) > h(t_2, \theta)\} = (\lambda, \infty)$ and $\{\theta: h(t_1, \theta) < h(t_2, \theta)\} = (0, \lambda)$.*

PROOF. Consider the function $d(x) = 2e^x - 2 - xe^x$. Since $d'(x) = e^x(1 - x)$, d is strictly increasing from $d(0) = 0$ up to $d(1) = e - 2$ and then strictly decreasing to $d(\infty) = -\infty$. Thus d has a unique root ρ such that $d(x) > 0$ or < 0 according to $0 < x < \rho$ or $x > \rho$. Now, for $\theta > 0$, $\partial h(t, \theta)/\partial t = td(t\theta)/(e^{t\theta} - 1)^2$. Set

$$t(\theta) = \rho/\theta. \tag{4.9}$$

Then $\partial h/\partial t > 0$ or < 0 according to $t(\theta) > t > 0$ or $t > t(\theta)$. Hence (4.6) is proved.

Since (4.7) is an immediate consequence of (4.6), let us prove (4.8). Let $t_2 > t_1 > 0$. First, note that $h(t_1, \theta) - h(t_2, \theta) > 0$, $= 0$ or < 0 iff $u(\theta) = t_1^2(e^{t_2\theta} - 1) - t_2^2(e^{t_1\theta} - 1) > 0$, $= 0$ or < 0. Note that $u''(\theta) = (t_1t_2)^2(e^{t_2\theta} - e^{t_1\theta}) > 0$. Thus $u'(\theta) = t_1t_2(t_1e^{t_2\theta} - t_2e^{t_1\theta})$ is strictly increasing. Since $u'(0) = t_1t_2(t_1 - t_2) < 0$, u is strictly decreasing from $u(0) = 0$ to $u(t_0)$ where $u'(t_0) = 0$ and then strictly increasing to $u(\infty) = \infty$. Hence there is $\lambda \in (0, \infty)$ such that $u(\lambda) = 0$, $u(\theta) < 0$ if $0 < \theta < \lambda$ and $u(\theta) > 0$ if $\theta > \lambda$. This completes our proof of (4.8). □

LEMMA 4.3. *For the serial sacrifice problem stated above*, (1.9) *holds*.

PROOF. Note that (4.8) implies that there exists $\{\theta_i, 1 \le i < k\}$ such that (4.3) and (4.4) hold with $U = \infty$ and $L = 0$. By (4.7), we know that $(\theta_i, U) \subset (\theta_{i+1}, U)$. Thus $\theta_i \ge \theta_{i+1}$. Since h_j are continuous, $h_i(\theta_i) = h_{i+1}(\theta_i)$ and $h_{i+1}(\theta_{i+1}) = h_{i+2}(\theta_{i+1})$. Hence $\theta_i = \theta_{i+1}$ is impossible for it is against (4.7). Thus $\theta_i > \theta_{i+1}$. Thus all conditions of Lemma 4.1 are satisfied and therefore (1.9) is proved. □

LEMMA 4.4. *For the serial sacrifice problem stated above,* (2.39) *holds. Consequently,* (2.36) *is also true.*

Note that $I_j(\theta, \lambda) = I(t_j, \theta, \lambda)$ where

$$(4.10)\qquad \begin{aligned} I(t, \theta, \lambda) &= E_\theta \ln(f(x|t, \theta)/f(x|t, \lambda)) \\ &= (\lambda - \theta)t + (1 - e^{-t\theta})\ln\{(e^{t\theta} - 1)/(e^{t\lambda} - 1)\}. \end{aligned}$$

PROOF OF LEMMA 4.4. Let $\theta \in [\theta_{l+1}, \theta_l)$. Then $h^*(\theta) = h_{l+1}(\theta) > h_l(\theta)$. By (4.7) of Lemma 4.2, $h_l(\theta) > h_{l-1}(\theta) > \cdots > h_1(\theta)$. Hence $h^*(\theta) - h_j(\theta)$, as a function of j, is strictly decreasing. Hence in order to show (2.39), it is sufficient to show that for any $i \le l$, $I_j(\theta, \theta_i)$, as a function of j, is increasing for $j \le i$. Since $I_j(\theta, \theta_i) = I(t_j, \theta, \theta_i)$, it is sufficient to show that

$$(4.11)\qquad I(t, \theta, \theta_i), \text{ as a function of } t, \text{ is increasing for } t \le t_i.$$

But (4.6) and (4.9) of Lemma 4.2 imply that $t_i < \rho/\theta_i$, for $h_{i+1}(\theta_i) = h_i(\theta_i) = h^*(\theta_i)$. Also note that $\theta < \theta_i$. Thus (4.11) follows from the lemma stated below.

LEMMA 4.5. *Let* $\lambda > \theta > 0$. *Then* $I(t, \theta, \lambda)$, *as a function of* t, *is increasing for* $0 < t < \rho/\lambda$, *where* $\rho > 0$ *and satisfies*

$$(4.12)\qquad 2e^\rho - 2 - \rho e^\rho = 0.$$

PROOF. Let $u = \lambda t$ and $\alpha = \theta/\lambda$. Then by (4.10), the problem is equivalent to showing that for each $0 < \alpha < 1$,

$$(4.13)\qquad \begin{aligned} &I(u) = (1 - \alpha)u + (1 - e^{-\alpha u})\ln\{(e^{\alpha u} - 1)/(e^u - 1)\} \\ &\text{is increasing for } 0 < u < \rho. \end{aligned}$$

Hence it is sufficient to show that $I'(u) \ge 0$ for $0 < u < \rho$. After calculation, we find that

$$(4.14)\qquad I'(u) = e^{-\alpha u}\{V(u) + \alpha \ln[1 - V(u)]\},$$

where

$$V(u) = (e^u - e^{\alpha u})/(e^u - 1).$$

Now,

$$V'(u) = \{(1 - \alpha)e^{(1+\alpha)u} - e^u + \alpha e^{\alpha u}\}/(e^u - 1) > 0,$$

by the strict convexity of the exponential function e^x. Hence V strictly increases from $\lim_{u \to 0} V(u) = 1 - \alpha$ to $V(\infty) = 1$. Note that on the range $1 - \alpha < s < 1$, the function $f(s) = s + \alpha \ln(1 - s)$ has a zero $s_0 > 0$ such that $f(s) > 0$ or < 0 according to $s < s_0$ or $s > s_0$. Therefore if $f(V(\rho)) \ge 0$, then $I'(u)e^{\alpha u} = f(V(u)) \ge 0$ for all $\alpha u < \rho$. Hence we only have to show that for $0 < \alpha < 1$,

$$(4.15)\qquad v(\alpha) = (e^\rho - e^{\alpha\rho})/(e^\rho - 1) + \alpha \ln[(e^{\alpha\rho} - 1)/(e^\rho - 1)] \ge 0.$$

After calculation we have

$$(4.16)\qquad \begin{aligned} v''(\alpha) &= (-\rho^2 e^{\alpha\rho})/(e^\rho - 1) + \rho/(1 - e^{-\alpha\rho}) \\ &\quad + \{(\rho + \alpha\rho^2)(1 - e^{-\alpha\rho}) - \alpha\rho^2\}/(1 - e^{-\alpha\rho})^2. \end{aligned}$$

Applying (4.12) to the first term of (4.16), we obtain

$$\begin{aligned} v''(\alpha) &= -2\rho e^{(\alpha-1)\rho} \\ &\quad + \{\rho(1-e^{-\alpha\rho}) + (\rho+\alpha\rho^2)(1-e^{-\alpha\rho}) - \alpha\rho^2\}/(1-e^{-\alpha\rho})^2 \\ &= \rho\{2[1-e^{(\alpha-1)\rho}(1-e^{-\alpha\rho})] - [\alpha\rho/(1-e^{-\alpha\rho})]e^{-\alpha\rho}\}/(1-e^{-\alpha\rho}). \end{aligned} \tag{4.17}$$

Using the fact that $x/(1-e^{-x})$ is an increasing function and $(\alpha\rho)/(1-e^{-\alpha\rho}) \le \rho/(1-e^{-\rho}) = 2$,

$$\begin{aligned} v''(\alpha) &\ge 2\rho[1+e^{-\rho} - e^{\rho(\alpha-1)} - e^{-\alpha\rho}]/(1-e^{-\alpha\rho}) \\ &\ge 0, \end{aligned} \tag{4.18}$$

by the convexity of the function e^x and the fact that $-\alpha\rho$ and $\rho(\alpha-1)$ lie between $-\rho$ and 0. Observe that

$$v'(\alpha) = (-\rho e^{\alpha\rho})/(e^{\rho}-1) + \ln\{(e^{\alpha\rho}-1)/(e^{\rho}-1)\} + \alpha\rho e^{\alpha\rho}/(e^{\alpha\rho}-1).$$

Hence

$$\lim_{\alpha\to 1} v'(\alpha) = 0. \tag{4.19}$$

In view of (4.18) and (4.19), $v'(\alpha) \le 0$ for $0 < \alpha < 1$. Since $v(1) = 0$, (4.15) is proved. □

REMARK. Lemmas 4.3 and 4.4 show that the rules constructed in Section 3 are asymptotically efficient and their asymptotic sample size satisfies (2.37) and (2.38). Furthermore, (2.39) holds by Lemma 4.4. Therefore, by Theorem 3, any asymptotically efficient rule should also satisfy (2.37) and (2.38).

5. Simulation and comparison. In this section we report some simulation results on two classes of allocation rules, which are the asymptotically efficient rules discussed in Section 3 and the ratio rules proposed by Bergman and Turnbull (1983). The simulation results are conducted under the benchmark situation studied by Bergman and Turnbull.

More precisely, we take sample size $N = 200$, population size $k = 18$, stages $\{1,\dots,18\}$ and the density of Π_i to be

$$f_i(x|\theta) = (1-e^{-i\theta})^x(e^{-i\theta})^x, \tag{5.1}$$

where $x \in \{0,1\}$ and $i \in \{1,\dots,18\}$. Recall that under these assumptions, the allocation rules constructed in Section 3 are shown to be asymptotically efficient in Section 4.

Also recall that $\{X_{in}\}$ is defined to be a random sample from the population Π_i and $S_{n_i} = \sum_{n=1}^{n_i} X_{in}$. For any $b > 0$ and $z^* \ge 0$, the ratio rule $\psi(b, z^*) = (\psi_1,\dots,\psi_k)$ is then defined inductively by

$$\begin{aligned} \psi_0 &= 0, \\ \tilde{\psi}_l &= \inf\{n_l: bS_{n_l} - (n_l - S_{n_l}) > z^*\}, \\ \psi_l &= \min\left\{\tilde{\psi}_l, 200 - \sum_{i=0}^{l-1}\psi_i\right\}, \quad \text{for } 1 \le l < k, \end{aligned}$$

and

$$\psi_k = 200 - \sum_{i=0}^{k-1} \psi_i.$$

First notice that the data from previous stages are not used in defining $\tilde{\psi}_l$. The only information used in specifying the next sample size ψ_l is the remaining sample size $200 - \sum_{i=0}^{l-1}\psi_i$. Second, in practice the constants b and z^* can be adjusted so that a reasonable rule can be achieved. When b is fixed, larger z^* would prevent the switch of populations too soon and smaller z^* would leave the inferior population earlier. Following Bergman and Turnbull, we take $b = 4$ and $z^* = 4, 8$ for our simulation study.

We can also introduce an adjustment factor into our rules (3.3). For any constant $c > 0$, redefine

$$\tau_N(l) = \inf\{n: M_l(T_N(1), \dots, T_N(l-1), n) > cN\}.$$

With minor changes, it can be shown easily that Theorem 5 still holds. In our study below we choose $c = 0.01$. We also choose $F_l(\theta)$ to be the uniform distribution over $(0, \theta_l)$. Hence for $1 \le l \le k$,

$$M_l(n_1, \dots, n_l) = \theta_l^{-1} \int_0^{\theta_l} \prod_{i=1}^{l} \left(\frac{e^{i\theta} - 1}{e^{i\theta_l} - 1} \right)^{S_{n_i}} e^{i(\theta_l - \theta)n_i}\, d\theta$$

and the asymptotically efficient rule $\phi = (T(1), \dots, T(k))$ is then defined inductively by

$$T(0) = 0,$$

$$\tau(l) = \inf\{n: M_l(T(1), \dots, T(l-1), n) > 2\},$$

$$T(l) = \min\left\{\tau(l), 200 - \sum_{i=0}^{l-1} T(i)\right\}, \qquad 1 \le l < k-1,$$

and

$$T(k) = 200 - \sum_{i=0}^{k-1} T(i).$$

Note that $\{\theta_i\}$ is determined by (4.3) and (4.5). Under the assumption (5.1), we have Table 1.

Following Bergman and Turnbull (1983), for a given rule $\mathbf{R} = (R_1, \dots, R_k)$, define its efficiency to be

$$e_N(\mathbf{R}, \theta) = E_\theta(\hat{J}_N(\theta))/J(\theta) = J_N(\theta)/J(\theta),$$

where

$$\hat{J}_N(\theta) = \sum_{i=1}^{k} R_i[i^2/(e^{i\theta} - 1)]$$

and

$$J(\theta) = N \sup_{0<t<\infty} [t^2/(e^{t\theta} - 1)].$$

TABLE 1
Values of $\{\theta_i\}$

i	0	1	2	3	4	5	6	7	8	9
θ_i	∞	1.099	0.645	0.458	0.355	0.290	0.246	0.212	0.188	0.168
i	10	11	12	13	14	15	16	17	18	
θ_i	0.152	0.139	0.127	0.118	0.116	0.103	0.00966	0.00911	0.0000	

When $\theta \in \Theta_i = (\theta_{i-1}, \theta_i)$, the ith population is the best one.

TABLE 2
Estimated efficiencies and standard deviations for $\psi(4,4)$, $\psi(4,8)$ *and* ϕ

θ (best population)	1.0(2)	0.5(3)	0.25(6)	0.167(9)	0.125(13)	0.1(16)
$\psi(4,4)$	0.888 ± 0.011	0.936 ± 0.005	0.952 ± 0.002	0.945 ± 0.001	0.937 ± 0.001	0.923 ± 0.001
$\psi(4,8)$	0.925 ± 0.008	0.950 ± 0.003	0.938 ± 0.001	0.928 ± 0.001	0.905 ± 0.001	0.882 ± 0.002
ϕ	0.924 ± 0.004	0.943 ± 0.004	0.946 ± 0.003	0.930 ± 0.004	0.931 ± 0.004	0.938 ± 0.003

[See (1.6) and (4.5) for statistical interpretations.] Given a sample of $\mathbf{R}$ and the value of θ, one can use the sample mean of $\hat{J}_N(\theta)/J(\theta)$ to estimate $e_N(\mathbf{R}, \theta)$. Based on 100 simulations, Table 2 gives the estimated efficiencies and their standard deviations for the rules $\psi(4,4)$, $\psi(4,8)$ and ϕ. For comparison, we take the same θ from Bergman and Turnbull (1983). It is interesting to note that for $\psi(4,4)$ and $\psi(4,8)$, the estimated efficiencies are very close to the efficiencies listed in their article.

Although the ratio rules $\psi(4,4)$ and $\psi(4,8)$ do not use the data from previous stages, Table 2 shows that these rules perform reasonably well for various θ. However, as expected, $\psi(4,4)$ favors smaller θ and $\psi(4,8)$ bigger θ. When the range of θ is not known a priori, it may be difficult to choose between $\psi(4,4)$ and $\psi(4,8)$. On the contrary, our rule ϕ which performs uniformly well, does not have this disadvantage. Furthermore, this uniformity over a wide range of parameter values seems to indicate that our rules may be asymptotically optimal for a broad class of prior distributions in the Bayesian setting. Further research along this line is of interest. For the related multi-armed bandit problem, see Lai (1987).

Acknowledgments. We would like to thank an Associate Editor and two reviewers for their constructive comments. The first author would also like to thank Abba Krieger for a helpful conversation concerning Lemma 2.5.

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