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SEQUENTIAL TESTS FOR THE DRIFT OF A WIENER PROCESS WITH A SMOOTH PRIOR, AND THE HEAT EQUATION[1]

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Methods are described which permit one to work with continuous-time optimal stopping problems, using the heat equation, even when the prior placed on the drift parameter of a Wiener process is not normal. The details of the method are worked out for Chernoff's problem of testing the sign of the drift parameter when the prior is "smooth."

1. Introduction. Chernoff (1961, 1972) and others have shown how the heat equation and associated free boundary problems arise naturally as mathematical tools for approximating Bayes sequential procedures. Typical applications are concerned with normally distributed observations which depend on an unknown mean θ and a normal prior for θ. The present paper has two objectives: (i) to show, very generally, that the heat equation is still relevant when θ is not normally distributed and (ii) to investigate the problem of testing the sign of the normal mean for smooth priors. A smooth nonnormal prior can be viewed as a perturbed normal prior and it is possible to very precisely describe how the perturbation affects the associated free boundary.

Since the subject of free boundary problems is fraught with technical details, a conscious decision has been made here to emphasize exposition rather than rigor.

2. Bayes problems and continuous-time approximations. Suppose $X_1, X_2, \dots$ are potential observations which are independent and $N(\theta, \sigma^2)$, where σ^2 is positive and known, and θ is unknown. Set $S_n = X_1 + \cdots + X_n$, $n \geq 0$, and, for definiteness, $\sigma^2 = 1$. We shall assume that we are dealing with a concrete statistical problem which is expressed within a decision-theoretic framework with a suitable loss structure and that θ has a prior density g. The Bayes sequential procedure is sought.

Let $d(z, n)$ denote the posterior Bayes risk associated with stopping at time n, with $S_n = z$, and making an optimal terminal decision. The primary task, in a

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Bayes sequential problem, is to find the stopping time N, within $\{0,1,2,\dots\}$, which minimizes the expectation $E\{d(S_N, N)\}$.

We are only concerned here with the commonly used approximation based on the replacement of $\{S_n,\ n \geq 0\}$ by a Wiener process $X = \{X(t),\ t \geq 0\}$ with drift rate θ: Given θ, the differential $dX(t)$ has mean $\theta \cdot dt$ and variance dt. The task is to find the stopping time τ in $[0, \infty)$ which minimizes $E\{d(X(\tau), \tau)\}$. Such problems can be viewed as Markovian, with states (x, t), and the search for optimality can be viewed as a search for an "optimal continuation set" C in the (x, t) plane: The optimal stopping time $\tau = \tau(x, t)$ is the first time that X reaches the boundary of C if (x, t) is in C and it is simply t otherwise.

Let $b(x, t) := E\{d(X(\tau), \tau)\}$, represent the minimal Bayes risk that can be achieved by stopping optimally starting from state (x, t). The same argument as that used by Chernoff (1972), page 92 can be used to show that it satisfies a diffusion equation

$$b_{xx} + 2\varphi(x, t) \cdot b_x + 2b_t = 0, \qquad (x, t) \in C, \tag{1}$$

where

$$\varphi(x, t) := E\{\theta | X(t) = x\} = \frac{\int_{-\infty}^{\infty} \theta \exp\{x\theta - t\theta^2/2\} g(\theta)\, d\theta}{\int_{-\infty}^{\infty} \exp\{x\theta - t\theta^2/2\} g(\theta)\, d\theta}. \tag{2}$$

With the possible exception of certain points of singularity, the boundary conditions

$$b = d, \qquad b_x = d_x, \qquad b_t = d_t \tag{3}$$

hold on the boundary of C. Frequently, (1) and the first two conditions in (3) jointly determine the function b and the set C. But these conditions do not always guarantee a unique solution.

We note in passing that the ratio in (2) is meaningful for $t > 0$ even when the prior g lacks a finite first moment.

3. Chernoff's negative s scale. It is a standard practice to assume that the prior g is normal with some known mean μ_0 and variance σ_0^2. This makes the marginal distribution of X Gaussian but something other than a Wiener process with drift. It does not make (1) particularly attractive to work with. For this reason, Chernoff (1972) and others have found it more convenient to work with the posterior mean process $Y = \{Y(s),\ 0 < s \leq \sigma_0^2\}$, defined by

$$Y(s) = E\{\theta | X(t)\} = \left(X(t) + \mu_0/\sigma_0^2\right)/\left(t + \sigma_0^{-2}\right), \tag{4}$$

where $s = 1/(t + \sigma_0^{-2})$ is the posterior variance at time t. Then Y is a Brownian motion evolving backward in time, as measured in s (hence the name "negative s scale") and $b(x, t)$, when described as a function of (y, s), satisfies the much more familiar heat equation $u_{yy} = 2u_s$ and boundary conditions comparable to those in (3). A frequent additional advantage is that $Y(s)$ is statistically more relevant than $X(t)$, i.e., $d(x, t)$ is more naturally described as a function of (y, s).

If the prior g is non-Gaussian, the marginal distribution of X is non-Gaussian. Moreover, the posterior mean is non-Gaussian and it no longer assumes the simple linear form shown in (4).

The loss of the linear form is unfortunate, for it is the reason Y is a Brownian motion. It will be seen in the next section that it is possible to continue to work with a variant of this linear form. We will be content here with pointing out that the posterior mean is approximately equal to a linear form when t is large and the prior g is sufficiently smooth: It can be shown that $(t + c)E\{\theta | X(t)\} - (X(t) + a)$ is a convergent martingale, for arbitrary constants a and c, whenever g is continuously differentiable, in which case the limit equals $g'(\theta)/g(\theta) - a + c\theta$ a.s. A linear form will be used, as a mathematical convenience, when we consider the problem of testing the sign of the normal mean. The results so obtained will need to be restated in the coordinate system of the posterior mean before meaningful comparisons can be made with Chernoff's results for normal priors.

4. Transformations. We are concerned here with transformations of b which satisfy the heat equation $u_{xx} = 2u_t$. We start with a transformation which satisfies the backward heat equation $u_{xx} = -2u_t$.

THEOREM 1. *The function*

$$b^*(x, t) := b(x, t) \cdot \psi(x, t) \tag{5}$$

satisfies the backward heat equation in C, *where*

$$\psi(x, t) := \int_{-\infty}^{\infty} \exp\{x\theta - t\theta^2/2\} g(\theta)\, d\theta. \tag{6}$$

It is easily seen that

$$b^*_{xx} + 2b^*_t = (b_{xx} + 2\varphi b_x + 2b_t)\psi$$

at any point (x, t) at which b_{xx} and b_t exist. The theorem follows immediately from (1).

Now set

$$d^*(x, t) := d(x, t) \cdot \psi(x, t).$$

The functions b^* and d^* can be viewed as surrogates for b and d, respectively: Instead of trying to solve the free boundary problem represented by (1) with boundary conditions (3), one may solve an equivalent problem using the surrogates b^* and d^*.

What we have done can be understood from a probabilistic perspective as well: The transformation corresponds to a change of probability measures with $\psi(X(t), t)$ equal to the Radon–Nikodym derivative dP/dP^* for the (current) sigma-field $\sigma\{X(u): u \le t\}$. Under the new probability measure P^*, X is a Brownian motion, d^* and b^* are the proper analogues of d and b, respectively, and the proper analogue of (1) is the backward heat equation.

To get from the backward heat equation to the heat equation, let $y = (x + a)/(t + c)$ and $s = 1/(t + c)$ $(t > -c)$, and let C' be the image of C

in the (y, s) plane, where, for now, a and c are arbitrary constants. Thus, $x = y/s - a$ and $t = 1/s - c$. Further, let ϕ be the standard normal density and $n(y, s) = s^{-1/2}\phi(y/s^{1/2})$.

THEOREM 2. *The function*

$$b^{**}(y, s) := b^{*}(x, t) \cdot n(y, s) \tag{7}$$

satisfies the heat equation in the variables (y, s) *within* C'.

Again the proof follows by direct calculations and again a probabilistic interpretation can be given: Assume $X = \{X(t),\ t \geq 0\}$ is standard Brownian motion under the probability measure P^* and $c > 0$. Then the process $Y = \{Y(s) = (X(t) + a)/(t + c), 0 < s \leq 1/c\}$ is Brownian motion in reverse time s, under a probability measure P^{**}, if for each s, the Radon–Nikodym derivative dP^*/dP^{**} for the (current) sigma-field $\sigma\{Y(u)\colon s \leq u \leq 1/c\}$ is proportional to $n(Y(s), s)$. (The coefficient of proportionality equals $[c \cdot n(a, c)]^{-1}$.)

The transformation shown in (7) is not new with us and not even recent: Van Moerbeke (1974) credits it to Appell (1892).

Theorems 1 and 2 together yield the formula

$$b^{**}(y, s) = b(x, t) \cdot \int_{-\infty}^{\infty} n(y - \theta, s) \cdot e^{-a\theta + c\theta^2/2} g(\theta)\, d\theta. \tag{8}$$

Again, each solution of the original free boundary problem can be described, equivalently, in terms of surrogates b^{**} and

$$d^{**}(y, s) = d(x, t) \cdot \int_{-\infty}^{\infty} n(y - \theta, s) \cdot e^{-a\theta + c\theta^2/2} g(\theta)\, d\theta. \tag{9}$$

While a particular solution b, so found, may not be the solution sought for the sequential Bayes problem, one of the obtainable solutions in the (y, s) plane will, necessarily, correspond to the sequential Bayes problem; there is a simple one-to-one correspondence between solutions in the two spaces.

Notice that the original problem has been converted into an equivalent problem in the context of Chernoff's negative s scale described in Section 3. This has several advantages. An obvious advantage is that the heat equation has been widely studied by probabilists and statisticians. Many solutions are known, and these can be used to obtain asymptotic expansions and other analytic approximations to the solution of the free boundary problem. Another convenience is that Chernoff and Petkau (1984, 1986) have developed numerical algorithms for the negative s scale which find b^{**} and C', for quite general functions d^{**}. A final potential advantage is that one is able to place a large family of optimal stopping problems within a common context; each prior g contributes a member. This invites new types of comparisons. An example of this is given in the next section.

It is now a quick matter to completely recover Chernoff's framework when the prior g is normal $N(\mu_0, \sigma_0^2)$: Simply set $a = \mu_0/\sigma_0^2$ and $c = 1/\sigma_0^2$. Then (8) and (9) become $b^{**}(y, s) = n(\mu_0, \sigma_0^2) \cdot b(x, t)$ and $d^{**}(y, s) = n(\mu_0, \sigma_0^2) \cdot d(x, t)$, respectively. The positive constant $n(\mu_0, \sigma_0^2)$ contributes nothing of significance to the problem and can simply be dropped. So, effectively, we are concerned with the transformation $d^{**}(y,s) = d(x,t)$, where $y = (x + \mu_0/\sigma_0^2)/(t + \sigma_0^{-2})$ and $s = 1/(t + \sigma_0^{-2})$, and we seek a suitable solution $b^{**}(y,s)$ of the heat equation. Chernoff, by focusing attention on the posterior mean process (4), reaches the same context.

More generally, suppose, for now, the prior g is strictly positive and smooth in the sense that $\log g(\theta)$ is expressible in a Taylor series expansion,

$$\log g(\theta) = a_0 + a_1\theta + a_2\theta^2/2 + a_3\theta^3/6 + \cdots . \tag{10}$$

It is convenient to think of a_1 and $-a_2$ as prior parameters with all the other subscripted a's held fixed except a_0, which must be chosen to make the integral of $g = 1$. Then the posterior density of g given $X(t) = x$ remains in the family and it has posterior parameters $x + a_1$ and $t - a_2$. This is a generalization of the normal family, which corresponds to $a_1 = \mu_0/\sigma_0^2$ and $a_2 = -1/\sigma_0^2$, with $a_k = 0$ for $k \geq 3$. In general, the integrals in (8) and (9) take the form

$$\exp(a_0) \cdot \int_{-\infty}^{\infty} n(y-\theta, s) \cdot \exp\{a_3\theta^3/6 + a_4\theta^4/24 + \cdots\}\, d\theta \tag{11}$$

when one sets $a = a_1$ and $c = -a_2$. This makes $y = (x + a_1)/(t - a_2)$.

5. Testing the sign of the drift.

5.1. *The function $d^{**}(y,s)$.* Assume the cost of sampling is 1 per unit time and the cost due to a wrong assessment of the sign of the drift rate θ is $2k|\theta|$, where $k > 0$. This leads to

$$d(x,t) = t + 2k \cdot \min\left\{ \frac{\int_0^{\infty}\theta\, e^{x\theta - t\theta^2/2} g(\theta)\, d\theta}{\int_{-\infty}^{\infty} e^{x\theta - t\theta^2/2} g(\theta)\, d\theta}, \frac{\int_{-\infty}^{0} |\theta| e^{x\theta - t\theta^2/2} g(\theta)\, d\theta}{\int_{-\infty}^{\infty} e^{x\theta - t\theta^2/2} g(\theta)\, d\theta} \right\}$$

and

$$\begin{aligned} d^{**}(y,s) = {} & (s^{-1} + a_2)\int_{-\infty}^{\infty} n(y-\theta, s) e^{-a_1\theta - a_2\theta^2/2} g(\theta)\, d\theta \\ & + 2k \cdot \min\left\{ \int_0^{\infty} \theta \cdot n(y-\theta, s) e^{-a_1\theta - a_2\theta^2/2} g(\theta)\, d\theta, \right. \\ & \left. - \int_{-\infty}^{0} \theta \cdot n(y-\theta, s) e^{-a_1\theta - a_2\theta^2/2} g(\theta)\, d\theta \right\}. \end{aligned}$$

Before proceeding, we shall find it convenient to subtract away a solution of the heat equation $\int_{-\infty}^{\infty}(a_2 + k|\theta|) n(y-\theta, s) e^{-a_1\theta - a_2\theta^2/2} g(\theta)\, d\theta$ and then to divide by the positive constant e^{a_0}, which will not affect the boundary of C'. (This particular solution is defined for $0 < s < \infty$ if $a_2 \geq 0$ and for $0 < s < -a_2^{-1}$ if $a_2 < 0$.) *Keeping the same notation*, as a convenience, we obtain

$$\begin{aligned} d^{**}(y,s) = {} & s^{-1}\int_{-\infty}^{\infty} n(y-\theta, s) e^{-a_0 - a_1\theta - a_2\theta^2/2} g(\theta)\, d\theta \\ & - k \cdot \left| \int_{-\infty}^{\infty} \theta \cdot n(y-\theta, s) e^{-a_0 - a_1\theta - a_2\theta^2/2} g(\theta)\, d\theta \right| . \end{aligned} \tag{12}$$

Using the fact that the function $\psi_x(x, t) = \int_{-\infty}^{\infty}\theta \cdot \exp\{x\theta - t\theta^2/2\}g(\theta)\,d\theta$ is monotone in x, one can show that the second integral in (12) vanishes along a line $(y_0(s), s)$. It is positive above the line and negative below. [Note, for each t, that $\psi_x(\cdot, t)$ assumes values on both sides of the origin, since, according to (10), g has its support on both sides of $\theta = 0$.] It can be argued, along the lines of Sobel (1953), that the continuation region C' assumes the form $\{(y, s)\colon y^-(s) < y < y^+(s),\ s > 0\}$ with $(y_0(s), s)$ inside C'. One of our main objectives is to describe the asymptotic form of the boundaries $y^{\pm}(s)$ as s goes to zero. It can be shown that

$$\alpha_0(\rho) = -a_3\rho^3/2 - a_5\rho^5/8 - (a_7 + 20a_3a_4)\rho^7/48 - O(\rho^9) \quad \text{as } s \to 0,$$

where $\alpha_0(\rho) := y_0(s)s^{-1/2}$ and $\rho := s^{1/2}$. This function is identically zero when $e^{-a_1\theta}g(\theta)$ is an even function and, hence, for all normal priors.

5.2. *Expansions of* d^{**}. Let $\alpha := ys^{-1/2}$ and let $r!\mu_r(\alpha)$ be the rth moment of a standard normal random variable about $-\alpha$, i.e.,

$$\mu_r(\alpha) = \int_{-\infty}^{\infty}\phi(u)(u + \alpha)^r\,du/r!.$$

Thus $\mu_0(\alpha) = 1$, $\mu_1(\alpha) = \alpha$ and $\mu_r(\alpha) = (\mu'_{r-1}(\alpha) + \alpha \cdot \mu_{r-1}(\alpha))/r$ for $r = 1, 2, \dots$. It is useful to know that $\mu_r(\alpha)\rho^r$ satisfies the heat equation in the variables (y, s).

A preliminary step is to expand the function d^{**} in terms of the $\mu_r(\alpha)$'s and powers of ρ: For bounded α, as ρ goes to zero,

$$\begin{aligned}
&\int_{-\infty}^{\infty} n(y - \theta, s) \cdot e^{-a_0 - a_1\theta - a_2\theta^2/2} \cdot g(\theta)\,d\theta \\
&\quad\sim \int_{|\theta| \le \rho^{1/2}} n(y - \theta, s) \cdot e^{-a_0 - a_1\theta - a_2\theta^2/2} \cdot g(\theta)\,d\theta \\
&\quad= \int_{|\theta| \le \rho^{1/2}} n(y - \theta, s) \cdot e^{a_3\theta^3/6 + a_4\theta^4/24 + \cdots}\,d\theta \\
&\quad= \int_{|\theta| \le \rho^{1/2}} n(y - \theta, s) \cdot \left\{1 + a_3\theta^3/6 + a_4\theta^4/24 + \cdots\right\}d\theta \\
&\quad\sim \int_{-\infty}^{\infty} n(y - \theta, s) \cdot \left\{1 + a_3\theta^3/6 + a_4\theta^4/24 + \cdots\right\}d\theta \\
&\quad= c_0\mu_0(\alpha) + c_1\mu_1(\alpha)\rho + c_2\mu_2(\alpha)\rho^2 + c_3\mu_3(\alpha)\rho^3 + \cdots,
\end{aligned}$$

where

$$c_0 = 1, \quad c_1 = c_2 = 0, \quad c_3 = a_3, \quad c_4 = a_4, \quad c_5 = a_5,$$
$$c_6 = a_6 + 10a_3^2, \quad c_7 = a_7 + 35a_3a_4, \quad c_8 = a_8 + 35a_4^2 + 56a_3a_5,$$
$$c_9 = a_9 + 84a_3a_6 + 126a_4a_5 + 280a_3^3,$$
$$c_{10} = a_{10} + 120a_3a_7 + 126a_5^2 + 210a_4a_6 + 2100a_3^2a_4.$$

The deletion of the set $\{|\theta| > \rho^{1/2}\}$ from the range of integration and its subsequent reinsertion, contribute negligible terms of order $e^{-1/(4\rho)}$. Likewise, one obtains

$$\int_{-\infty}^{\infty} \theta \cdot n(y-\theta, s) \cdot e^{-a_0-a_1\theta-a_2\theta^2/2} \cdot g(\theta)\, d\theta$$
$$\sim c_0\mu_1(\alpha)\rho + 2c_1\mu_2(\alpha)\rho^2 + 3c_2\mu_3(\alpha)\rho^3 + \cdots$$

and, hence,

$$d^{**}(y,s) \sim \mu_0(\alpha)\rho^{-2} + \sum_{r\geq 1} \left[c_{r+2}\mu_{r+2}(\alpha) \mp rkc_{r-1}\mu_r(\alpha)\right]\rho^r, \tag{13}$$

where the sum includes a finite number of consecutive terms. The upper and lower signs in $\mp$ are applicable above and below the line $\alpha_0(\rho)$, respectively.

Note that the even and odd functions of α always go with the even and odd powers of ρ, respectively. The same must be true for the expansion of b^{**}. The presence in (13) of a negative second power of ρ forces us to say something more about solutions of the heat equation.

5.3. *Special solutions of the heat equation.* For any integer r, $V_r(\alpha)\rho^r$ is a separable solution of the heat equation (in y and s) if V_r satisfies

$$V_r''(\alpha) + \alpha \cdot V_r'(\alpha) = r \cdot V_r(\alpha). \tag{14}$$

The even and odd solutions of (14) are, respectively,

$$G_r(\alpha) = 1 + \frac{r}{2}\cdot\alpha^2 + \frac{r(r-2)}{24}\cdot\alpha^4 + \cdots + \frac{r(r-2)\cdots(r-2n+2)}{(2n)!}\cdot\alpha^{2n} + \cdots$$

and

$$H_r(\alpha) = \alpha + \frac{r-1}{6}\cdot\alpha^3 + \frac{(r-1)(r-3)}{120}\cdot\alpha^5 + \cdots + \frac{(r-1)(r-3)\cdots(r-2n+1)}{(2n+1)!}\cdot\alpha^{2n+1} + \cdots.$$

The function $\mu_r(\alpha)$ is a multiple of $G_r(\alpha)$ when r is an even integer and a multiple of $H_r(\alpha)$ when r is odd ($r = 0, 1, 2, \ldots$). For notational convenience, we will let $\mu_{-2}(\alpha) := G_{-2}(\alpha) = 1 - 2\alpha^2/2 + 2\cdot 4\alpha^4/24 - \cdots$ and $\mu_{-1}(\alpha) := H_{-1}(\alpha) = \alpha - 2\alpha^3/6 + 2\cdot 4\alpha^5/120 - \cdots$.

We shall seek an asymptotic solution for b^{**} of the general form

$$b^{**}(y,s) \sim \sum_{r\geq -2} u_r\mu_r(\alpha)\rho^r, \tag{15}$$

where, again, the sum includes a finite number of consecutive terms and the u_r's are constants which are to be determined from the boundary conditions.

5.4. *Four boundary conditions.* Two boundary conditions are described by equating the expansion of d^{**} with the presumed form of the expansion of b^{**}.

This yields

$$(16)\qquad \begin{aligned}&(u_{-2}\mu_{-2}(\alpha)-1)\rho^{-2}+u_{-1}\mu_{-1}(\alpha)\rho^{-1}+u_0\mu_0(\alpha)\\ &\quad+\sum_{r\geq 1}\left[(u_r\pm rkc_{r-1})\mu_r(\alpha)-c_{r+2}\mu_{r+2}(\alpha)\right]\rho^r=0,\end{aligned}$$

where $\pm$ refers to the upper and lower boundaries $y^{\pm}(s)$, respectively. This is to hold when $\alpha=\alpha^{\pm}(s):=y^{\pm}(s)s^{-1/2}$. The further requirement that d_y^{**} and b_y^{**} should be equal on the boundaries can be incorporated by differentiating (16) with respect to α:

$$\begin{aligned}&u_{-2}\mu'_{-2}(\alpha)\rho^{-2}+u_{-1}\mu'_{-1}(\alpha)\rho^{-1}\\ &\quad+\sum_{r\geq 1}\left[(u_r\pm rkc_{r-1})\mu'_r(\alpha)-c_{r+2}\mu'_{r+2}(\alpha)\right]\rho^r=0.\end{aligned}$$

Since $\mu'_r=\mu_{r-1}$ $(r=-1,1,2,3,\dots)$, this becomes

$$(17)\qquad \begin{aligned}&u_{-2}\mu_{-3}(\alpha)\rho^{-2}+u_{-1}\mu_{-2}(\alpha)\rho^{-1}\\ &\quad+\sum_{r\geq 1}\left[(u_r\pm rkc_{r-1})\mu_{r-1}(\alpha)-c_{r+2}\mu_{r+1}(\alpha)\right]\rho^r=0,\end{aligned}$$

where $\mu_{-3}(\alpha):=\mu'_{-2}(\alpha)=-2\alpha+2\cdot 4\alpha^3/6-2\cdot 4\cdot 6\alpha^5/120+\cdots$.

5.5. *The expansions of* $\alpha^{\pm}$. Following Chernoff (1972), page 99, we anticipate that the boundary functions $\alpha^{\pm}(s)$ have a general asymptotic form

$$(18)\qquad \alpha^{\pm}(s)\sim b_3^{\pm}\rho^3+b_4^{\pm}\rho^4+b_5^{\pm}\rho^5+\cdots.$$

The constants $b_j^{\pm}$ and the coefficients u_r are determined by the boundary conditions. The details are messy but the idea is quite simple: One replaces α everywhere in (16) and (17) by the presumed form of the asymptotic expansions of $\alpha^{\pm}(s)$ given in (18). This yields four series in ρ which can be viewed as being identically equal to zero. By setting the coefficients of the powers of ρ equal to zero, one obtains a system of equations which can be solved to yield the coefficients u_r, $r=-2,-1,0,\dots$, and $b_j^{\pm}$, $j=3,4,\dots$. One finds

$$u_{-2}=1,\qquad u_{-1}=u_0=0,\qquad u_1=-a_3/2,\qquad u_2=a_4/4,$$

$$u_3=-a_5/4,\qquad u_4=\left(a_6-2a_3^2-12k^2\right)/6,$$

$$u_5=-(a_7+11a_3a_4)/6,\qquad u_6=\left(a_8+8a_3a_5+35a_4^2\right)/8,$$

$$u_7=-\left(a_9+36a_3a_6+72a_3^3+78a_4a_5\right)/8-38k^2a_3,$$

$$u_8=\left(a_{10}+40a_3a_7+66a_5^2+210a_4a_6+380a_3^2a_4\right)/10-108k^2a_4,$$

$$b_4^{\pm}=b_6^{\pm}=b_8^{\pm}=\cdots=0,\qquad b_3^{\pm}=-a_3/2\pm k/2,$$

$$b_5^{\pm}=-a_5/8,\qquad b_7^{\pm}=-(a_7+20a_3a_4)/48\pm ka_4/4,$$

$$\begin{aligned}b_9^{\pm}=&-\left(a_9+56a_3a_6+96a_4a_5+96a_3^3\right)/384\\ &\pm k\left(a_6+4a_3^2\right)/16+k^2a_3/24\mp k^3/6.\end{aligned}$$

REMARKS. It is not immediately clear why the above system admits a solution. However, with some effort (and patience and care), one can show that a matching of coefficients indeed takes place. We have verified the accuracy of these results and of those in the next subsection by using a symbolic-math computer program (Macsyma).

5.6. *Expansions in the coordinate system of the posterior mean.* In order to clearly interpret (and simplify) the results described in the previous subsection, it is necessary to change from the coordinate system based on the mapping $y = (x + a_1)/(t - a_2)$ to the coordinate system based on $\hat{y} = E(\theta|X(t) = x) = \psi_x(x, t)/\psi(x, t)$. It is easily checked that $\hat{y}$ is a continuous and strictly increasing function of x when t is held constant. So, there is a well-defined one-to-one mapping from (y, s) to $(\hat{y}, s)$. One obvious simplification is that the function $y_0(s)$ gets mapped into the s axis of the $(\hat{y}, s)$ coordinate system. The most interesting question is "How are $y^{\pm}(s)$ mapped?" This is easily answered using the information at hand: Let $\hat{y}^{\pm}(s)$ denote the mappings of $y^{\pm}(s)$. The asymptotic expansions of $\hat{y}^{\pm}(s)s^{-1/2}$ as ρ goes to zero take the form $\hat{b}_3^{\pm}\rho^3 + \hat{b}_5^{\pm}\rho^5 + \hat{b}_7^{\pm}\rho^7 + \hat{b}_9^{\pm}\rho^9 + \cdots$, where

$$\hat{b}_3^{\pm} = \pm k/2, \qquad \hat{b}_5^{\pm} = 0, \qquad \hat{b}_7^{\pm} = \pm k a_4/2,$$
$$\hat{b}_9^{\pm} = \pm k\left(a_6 + 4a_3^2\right)/8 + k^2 a_3/6 \mp k^3/6.$$

Several observations can be made:

1. Chernoff's expansion for a normal prior has the same first term [also obtained by Bather (1962)]. Our second nonzero term does not show up in his expansion since normal priors have $a_4 = 0$. His second nonzero term is $\hat{b}_9^{\pm} = \mp k^3/6$. (Note that our k is one-half of Chernoff's k.)
2. The evidence for a nonsymmetric optimal stopping boundary, in the $(\hat{y}, s)$ coordinate system, first appears in the formula for $\hat{b}_9^{\pm}$, and then only if $a_3 \neq 0$. Its effect is to shift the continuation region upward when positive and in the opposite direction when negative.
3. The coefficients a_4 and a_6 produce similar effects: When positive, each causes the continuation region to be enlarged.
4. It seems somewhat surprising that a_4 shows up earlier than a_3 in the expansions of $\hat{y}^{\pm}$ and a_6 earlier than a_5 (which presumably appears in $\hat{b}_{11}^{\pm}$). It seems that departures from normality more readily affect the size of the continuation region than the location of its midline, at least when s is small (large t).

We are uncertain how to intuit observations (1)–(4).

It should be pointed out that we have assumed much more about the prior density g than is necessary. Clearly, for the results we have described, one only needs g to be positive and to have a suitable number of derivatives within a neighborhood of zero. Presumably, one can work out precise theorems with "stingy assumptions," but this is not our intent here. It might be interesting to investigate more severe departures from normality than we have considered. For instance, one should expect a double exponential prior $[g(\theta) = e^{-|\theta|}/2]$ to behave quite differently. The optimal boundaries for one severe departure from normality are known: It is easily seen that two point prior distributions give rise to sequential probability ratio tests.

It is tempting to speculate whether Chernoff's (1965) theory for small t (large s) can be extended from the setting of a normal prior to the more general setting of a smooth prior. We do not know whether this is feasible.

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