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POWER COMPARISONS FOR INVARIANT VARIANCE RATIO TESTS IN MIXED ANOVA MODELS

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A class of invariant hypothesis tests is considered for the purpose of testing a variance ratio arising in mixed models. Members of the class are most powerful for specific alternatives and limiting members of the class are Wald's test and the locally most powerful test. It is demonstrated that the locally most powerful test has the highest and Wald's test has the lowest asymptotic power when an asymptotically unbalanced sequence of ANOVA designs is considered under Pitman alternatives.

1. Introduction. In mixed linear models it is frequently of interest to test hypotheses concerning components of variance; for example, one may be concerned that a variance "between" items (σ_B^2) is too large relative to a variance "within" items (σ^2). Such a problem may be formulated as a test of hypothesis for the ratio $\rho = \sigma_B^2/\sigma^2$ of the between to within variances.

For unbalanced experimental designs it is not generally possible to find a uniformly most powerful test under the usual invariance and/or unbiasedness constraints. Spjøtvoll (1967) derived the most powerful invariant (MPI) test for ρ in case of a simple null and alternative (H_0: $\rho = \rho_0$ versus H_1: $\rho = \rho_1$). Since the resulting test has the undesirable feature of depending on the value ρ_1, he derived a corresponding test which is independent of ρ_1 by letting $\rho_1 \to \infty$. He noted that the test resulting from this procedure achieves high power for distant alternatives and that the test is exactly equivalent to a test which was proposed by Wald (1947) and has been discussed more recently by Seely and El-Bassiouni (1983) and Harville and Fenech (1985).

In this article the class of tests considered is the set of MPI critical functions along with the Wald test and the locally most powerful invariant (LMPI) test (obtained by letting $\rho_1 \to \rho_0$ in the MPI test). When experimental designs become "large" under Pitman alternatives the LMPI test emerges with the largest asymptotic power. The asymptotic powers of the remaining members of the class of tests form a monotonically decreasing function as $\rho_1 \to \infty$. The limiting and smallest asymptotic power obtains for the Wald test.

2. A class of tests. The notation is similar to that of Seely and El-Bassiouni: Let $\mathbf{R}(A)$ and $\mathbf{r}(A)$ denote the range and rank of a matrix A, let S denote the orthogonal complement of a set S and let $N_s(\mu, \Sigma)$ denote the s-dimensional normal distribution with mean μ and covariance matrix Σ.

The model considered is

$$Y = X\Pi + Bb + e, \tag{2.1}$$

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where Π is a $p \times 1$ vector of unknown constants, $b \sim N_t(0, \sigma_B^2 I)$, $e \sim N_n(0, \sigma^2 I)$, b and e are independent and X and B are $n \times p$ and $n \times t$ matrices.

All tests we consider shall be invariant under a certain group of transformations. Letting K, L and F be matrices whose columns form orthonormal bases for $\mathbf{R}(X)$, $\mathbf{R}([X:B]) \cap \mathbf{R}(X)^{\perp}$ and $\mathbf{R}([X:B])^{\perp}$, respectively, consider transforming the data to the canonical variables

$$A = [K:L:F]'Y = [(K'Y)':(L'Y)':(F'Y)']' = [Y_0':U':Z']'$$

(say).

Define a group of transformations whose elements are $gA = g_3 g_2 g_1 A$, where $g_1 A = [(Y_0 + v)':U':Z']'$($v$ an arbitrary $\mathbf{r}(X)$-vector), $g_2 A = [Y_0':U':(QZ)']'$ (Q an arbitrary $f \times f$ orthogonal matrix), $g_3 A = cA$ (c an arbitrary nonzero scalar); then a maximal invariant is $T = f^{1/2}U/(Z'Z)^{1/2}$, using Theorem 2 of Lehmann [(1959), page 218]. The density of T is a k-component multivariate Student's t with f degrees of freedom, location vector zero and dispersion matrix $\Sigma(\rho) = \rho L'BB'L + I$.

Thus the MPI test for H_0: $\rho = \rho_0$ versus H_1: H_1: $\rho = \rho_1$ rejects when $(f + T'\{\Sigma(\rho_1)\}^{-1}T)/(f + T'\{\Sigma(\rho_0)\}^{-1}T) < c_\alpha$; letting $\Sigma_i = \Sigma(\rho_i)$ for $i = 0, 1$ and $T(Y; \rho_0, \rho_1) = U'\{(\Sigma_0^{-1} - \Sigma_1^{-1})/(\rho_1 - \rho_0)\}U/\{Z'Z + U'\Sigma_0^{-1}U\}$, it will be convenient to express the rejection region of the MPI test as $T(Y; \rho_0, \rho_1) > c_\alpha' = (1 - c_\alpha)/(\rho_1 - \rho_0)$. Letting $\rho_1 \to \rho_0$ yields the LMPI test: Note that $\lim_{\rho_1 \to \rho_0} T(Y; \rho_0, \rho_1) = U'\Sigma_0^{-1}L'BB'L\Sigma_0^{-1}U/\{Z'Z + U'\Sigma_0^{-1}U\} = T(Y; \rho_0, \rho_0)$ (say) for all $Y \in R^n$. Using Rao [(1973), page 454], it may be shown that the LMPI test rejects when $T(Y; \rho_0, \rho_0) >$ (constant); see also the development of the locally most powerful tests in El-Bassiouni (1977)

Letting $\rho_1 \to \infty$ leads to the Wald test: $\lim_{\rho_1 \to \infty}(\rho_1 - \rho_0)T(Y; \rho_0, \rho_1) = U'\Sigma_0^{-1}U/\{Z'Z + U'\Sigma_0^{-1}U\}$, which is a monotonic function of the ratio of $U'\Sigma_0^{-1}U/Z'Z = T(Y; \rho_0, \infty)$ (say), for all $Y \in R^n$. Since $T(Y; \rho_0, \infty)$ is a constant multiple (k/f) of the Wald statistic (Seely and El-Bassiouni), the Wald test is equivalent to the test based on $T(Y; \rho_0, \infty)$.

The set of test statistics will be denoted

$$\mathscr{T}(\rho_0) = \{T(Y; \rho_0, \rho_1); 0 \le \rho_0 < \infty, \rho_0 \le \rho_1 \le \infty\}.$$

3. Finite sample properties. Let $x_0, x_1, \ldots, x_k$ be independently distributed central chi-squared variables with degrees of freedom $f, 1, \ldots, 1$, respectively. Letting $\lambda_1, \ldots, \lambda_k$ denote the eigenvalues of $L'BB'L$, the distribution of $T(Y; \rho_0, \rho_1)$ for $\rho_0 \le \rho_1 < \infty$ and arbitrary $\rho \ge 0$ is that of

$$\sum_{i=1}^{k} \lambda_i(1 + \rho\lambda_i)(1 + \rho_0\lambda_i)^{-1}(1 + \rho_1\lambda_i)^{-1}x_i \Big/ \left\{x_0 + \sum_{i=1}^{k}(1 + \rho\lambda_i)(1 + \rho_0\lambda_i)^{-1}x_i\right\}$$

and the distribution of $T(Y; \rho_0, \infty)$ is that of

$$\sum_{i=1}^{k}(1 + \rho\lambda_i)(1 + \rho_0\lambda_i)^{-1}x_i/x_0.$$

Computing the distribution of these statistics requires special techniques in most

TABLE 1
Power functions for the Wald and LMPI tests at $\alpha = 0.05$, with gain in power for the LMPI test

	$n_2 = n_2 = 1, n_3 = n_4 = 10$				
	Variance ratio				
Test	**0.5**	**1.0**	**1.5**	**2.0**	**2.5**
Wald	0.381	0.560	0.665	0.733	0.780
LMPI	0.400	0.529	0.598	0.643	0.675
Gain of LMPI	0.019	−0.031	−0.067	−0.090	−0.105

	$n_1 = \cdots = n_{10} = 1, n_{11} = \cdots = n_{20} = 10$				
	Variance ratio				
Test	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**
Wald	0.352	0.651	0.821	0.907	0.950
LMPI	0.451	0.739	0.871	0.932	0.961
Gain of LMPI	0.099	0.088	0.050	0.025	0.011

cases. The technique of Davies (1980) is extremely useful: His algorithm computes $P(\sum_{i=1}^{k}\lambda_i x_i + \sigma z < c)$, where x_i are independently distributed (possibly noncentral) chi-squared variables with arbitrary integral degrees of freedom, z is independently distributed as standard normal, σ is an arbitrary nonnegative constant and the λ_i and c are arbitrary real constants.

This algorithm was used to compare power functions in the context of the one-way random effects model,

$$y_{ij} = \mu + \beta_i + \varepsilon_{ij}; \qquad i = 1, \dots, t;\ j = 1, \dots, n_i,$$

where μ is a fixed unknown parameter, the β_i are i.i.d. $N(0, \sigma_B^2)$, the ε_{ij} are i.i.d. $N(0, \sigma^2)$ and the β_i are independent of the ε_{ij}. Consider a small design with $t = 4$, $n_1 = n_2 = 1$, $n_3 = n_4 = 10$ and a large design with $t = 20$, $n_1 = \cdots n_{10} = 1$, $n_{11} = \cdots = n_{20} = 10$. The power functions for $\alpha = 0.05$ level tests of H_0: $\rho = 0$ versus H_1: $\rho > 0$ are evaluated in Table 1.

While the LMPI test is more powerful for close alternatives in the small design, the gain in power is minor compared to the more serious loss of power for more distant alternatives. In this case one might prefer the Wald test. However, in the large design, the LMPI test is superior for the entire range of alternatives of interest, which is an indication that the LMPI test might be preferred in designs where the number of groups is large.

4. Asymptotic results.

4.1. *Asymptotic design sequence.* Consider a sequence of models (2.1) with $t \to \infty$. For the asymptotic theory we make the following assumptions:

(A1) $B = \text{diag}(1_{n(i)})$, where 1_n denotes the n-dimensional column vector of 1's.

Thus there is exactly one 1 in each row of B and the remaining elements 0's. To simplify notation, let $1_i = 1_{n(i)}$ and $n_i = n(i)$.

(A2) The number of columns of X is bounded uniformly in t.

(A3) The elements n_i always take values from a finite set $S = \{m_1, \dots, m_s\}$.

(A4) Define $a_t(m_j) = [\#$ of n_i in $\{n_1, \dots, n_t\}$ which equal $m_j]$ and assume $t^{-1}a_t(m_j) = p_j + o(t^{-1/2})$. Further if $m_j = 1$ is in S, assume $p_j < 1$. (Note that if $p_j = 1$ when $m_j = 1$, then the β-effects are asymptotically confounded with the ε-effects.)

(A5) The variance ratio $\rho = \sigma_B^2/\sigma^2$ is a function of t: $\rho = \rho_t = \rho_0 + \Delta t^{-1/2} + o(t^{-1/2})$, for $\Delta \geq 0$. Thus the alternatives are of the Pitman form, and the null case is included when $\Delta = 0$.

4.2. *Asymptotic distributions.*

THEOREM 1. *Define*

$$m_{abc}(\rho_1) = \sum_{j=1}^{s} p_j m_j^a (1 + \rho_0 m_j)^{-b} (1 + \rho_1 m_j)^{-c},$$

and let

$$m = m_{100}(\rho_1) = \sum_{j=1}^{s} p_j m_j$$

and

$$m_{11} = m_{110}(\rho_1) = \sum_{j=1}^{s} p_j m_j (1 + \rho_0 m_j)^{-1}.$$

Consider model (2.1) *with* $t \to \infty$ *and* (A1)–(A5). *Then* $t^{1/2}\{T(Y; \rho_0, \rho_1) - m_{101}(\rho_1)/m\}$ *has an asymptotically normal distribution with mean* $\Delta\{m m_{211}(\rho_1) - m_{11} m_{101}(\rho_1)\}/m^2$ *and variance* $2(m m_{202}(\rho_1) - m_{101}^2(\rho_1))\}/m^3$ *for* $0 \leq \rho_0 \leq \rho_1 < \infty$ *and* $t^{1/2}\{T(Y; \rho_0, \infty) - (m-1)^{-1}\}$ *has an asymptotically normal distribution with mean* $\Delta m_{11}/(m-1)$ *and variance* $2m/(m-1)^3$.

The proof is carried out in Section 5.

4.3. *Inefficiency of the Wald test.* Let $\mu(\rho_1)$ and $\sigma^2(\rho_1)$ denote the asymptotic means and variances of the statistics $T(Y; \rho_0, \rho_1)$, for $\rho_0 < \infty$ and $\rho_0 \leq \rho_1 \leq \infty$. Using Theorem 1, define the efficiency of the Wald test relative to the LMPI test as $E(W, L) = \{\mu(\infty)/\sigma(\infty)\}^2/\{\mu(\rho_0)/\sigma(\rho_0)\}^2$. For example, in the case H_0: $\rho = 0$ and $s = 2$ we have

$$E(W, L) = (p_1 m_1 + p_2 m_2)(p_1 m_1 + p_2 m_2 - 1) \div \{p_1 m_1 (m_1 - 1) + p_2 m_2 (m_2 - 1)\}.$$

When $p_2 < \frac{1}{2}$ we have $p_2 < E(W, L) \leq 1$, with $E(W, L) \to p_2$ as $m_1/m_2 \to 0$. Thus the efficiency of the Wald test may be made arbitrary small in design sequences where there is a small proportion of relatively large group sizes.

4.4. *Asymptotic power comparisons.*

THEOREM 2. *Consider a sequence of models (2.1) along with $t \to \infty$ and (A1)–(A5). For fixed $\alpha \in (0,1)$ and $0 \le \rho_0 < \infty$, let $\beta_\alpha(\rho_1) = 1 - \Phi(Z_\alpha - \mu(\rho_1)/\sigma(\rho_1))$, which denotes the asymptotic power of the α-level test using $T(Y;\rho_0,\rho_1)$, $\rho_0 \le \rho_1 \le \infty$. Then for fixed $\Delta > 0$*

(i) *$\beta_\alpha(\rho_1)$ is a continuous function of ρ_1 with $\lim_{\rho_1\to\infty}\beta_\alpha(\rho_1) = \beta_\alpha(\infty)$ and*

(ii) *if the design is asymptotically unbalanced, i.e., if there exist m_i and m_j in S with $m_i \ne m_j$, $p_i \ne 0$ and $p_j \ne 0$, then $\beta_\alpha(\rho_1)$ is a monotonically decreasing function of ρ_1.*

PROOF. To prove part (i), note that $\lim_{\rho_1\to\infty}\rho_1\mu(\rho_1) = \Delta m_{11}(m-1)/m^2$ and $\lim_{\rho_1\to\infty}\rho_1^2\sigma^2(\rho_1) = 2(m-1)/m^3$, so that $\lim_{\rho_1\to\infty}\mu(\rho_1)/\sigma(\rho_1) = \mu(\infty)/\sigma(\infty)$.

Suppose without loss of generality that the set S consists of elements $\{m_1,\dots,m_s\}$, where $m_i < m_j$ for $i<j$ and $p_i > 0$ for all $1 \le i \le s$. Since $p_j < 1$ when $m_j = 1$, we have $\mu(\rho) > 0$ for all $\rho > 0$; hence to show (ii) it will suffice that the derivative of $\mu^2(\rho)/\sigma^2(\rho)$ is negative for $\rho > \rho_0$ when $s > 1$.

Note that $\mu^2(\rho)/\sigma^2(\rho)$ is a positive multiple of

$$f(\rho) = \{mm_{211}(\rho) - m_{11}m_{101}(\rho)\}^2/\{mm_{202}(\rho) - m_{101}^2(\rho)\}$$

and that $f'(\rho)$ has the same sign as

$$\begin{aligned} g(\rho) = {}& \{mm_{202}(\rho) - m_{101}^2(\rho)\} \\ & \times\{mm_{211}(\rho) - m_{11}m_{101}(\rho)\}\{m_{11}m_{202}(\rho) - mm_{312}(\rho)\} \\ & -\{mm_{211}(\rho) - m_{11}m_{101}(\rho)\}^2\{m_{101}(\rho)m_{202}(\rho) - mm_{303}(\rho)\}. \end{aligned}$$

It will suffice to consider

$$\begin{aligned} g_1(\rho) &= g(\rho)/\{m^2\mu(\rho)/\Delta\} \\ &= \sum_i\sum_j\sum_k\sum_l p_ip_jp_kp_lm_im_jm_km_l^2(1+\rho m_j)^{-1}(1+\rho m_l)^{-2} \\ &\quad\times\Big[\{m_j(1+\rho_0m_j)^{-1} - (1+\rho_0m_i)^{-1}\} \\ &\quad\quad\times\{m_l(1+\rho m_l)^{-1} - (1+\rho m_k)^{-1}\} \\ &\quad -\{m_j(1+\rho m_j)^{-1} - (1+\rho m_i)^{-1}\}\{m_l(1+\rho_0m_l)^{-1} - (1+\rho_0m_k)^{-1}\}\Big]. \end{aligned}$$

Note that the $[\cdot]$ term in $g_1(\rho)$ is expressible as

$$\begin{aligned} &(\rho_0-\rho)m_jm_l(m_l-m_j)(1+\rho_0m_j)^{-1}(1+\rho m_j)^{-1}(1+\rho_0m_l)^{-1}(1+\rho m_l)^{-1} \\ &\quad-(\rho_0-\rho)m_j(m_k-m_j)(1+\rho_0m_j)^{-1}(1+\rho m_j)^{-1} \\ &\quad\times(1+\rho_0m_k)^{-1}(1+\rho m_k)^{-1} \\ &\quad-(\rho_0-\rho)m_l(m_l-m_i)(1+\rho_0m_i)^{-1}(1+\rho m_i)^{-1}(1+\rho_0m_l)^{-1}(1+\rho m_l)^{-1} \\ &\quad+(\rho_0-\rho)(m_k-m_i)(1+\rho_0m_i)^{-1}(1+\rho m_i)^{-1}(1+\rho_0m_k)^{-1}(1+\rho m_k)^{-1}. \end{aligned}$$

Factoring out $(\rho_0 - \rho)$, writing $g_1(\rho)$ as four separate sums, one for each of the terms above, using straightforward algebraic manipulations and noting that the last sum becomes 0 yields

$$
\begin{aligned}
g_2(\rho) &= g_1(\rho)/(\rho_0 - \rho) \\
&= \sum_i \sum_k p_i p_k m_i m_k \sum_{l>j} p_j p_l (m_l - m_j)^2 m_j^2 m_l^2 \\
&\qquad \times (1 + \rho_0 m_j)^{-1} (1 + \rho_0 m_l)^{-1} (1 + \rho m_j)^{-3} (1 + \rho m_l)^{-3} \\
&\quad + \sum_i \sum_l p_i p_l m_i m_l^2 (1 + \rho m_l)^{-2} \sum_{k>j} p_j p_k (m_k - m_j)^2 \\
&\quad \times m_j m_k (1 + \rho m_j)(1 + \rho m_k) \\
&\quad \times (1 + \rho_0 m_j)^{-1} (1 + \rho_0 m_k)^{-1} (1 + \rho m_j)^{-3} (1 + \rho m_k)^{-3} \\
&\quad - \sum_j \sum_k p_j p_k m_j m_k (1 + \rho m_j)^{-1} \sum_{l>i} p_i p_l (m_l - m_i)^2 \\
&\quad \times m_i m_l (m_i + m_l + 2\rho m_i m_l) \\
&\quad \times (1 + \rho_0 m_i)^{-1} (1 + \rho_0 m_l)^{-1} (1 + \rho m_l)^{-3} (1 + \rho m_i)^{-3}.
\end{aligned}
$$

Letting

$$
\begin{aligned}
a_{kl} &= p_k p_l (m_k - m_l)^2 m_k m_l (1 + \rho_0 m_l)^{-1} \\
&\qquad \times (1 + \rho_0 m_k)^{-1} (1 + \rho m_k)^{-3} (1 + \rho m_l)^{-3}, \\
b_{ij} &= p_i p_j m_i m_j
\end{aligned}
$$

and relabeling indices, we have

$$
\begin{aligned}
g_2(\rho) &= \sum_i \sum_j b_{ij} \sum_{k>l} a_{kl} m_k m_l \\
&\quad + \sum_i \sum_j b_{ij} m_i (1 + \rho m_i)^{-2} \sum_{k>l} a_{kl} (1 + \rho m_k + \rho m_l + \rho^2 m_k m_l) \\
&\quad - \sum_i \sum_j b_{ij} (1 + \rho m_i)^{-1} \sum_{k>l} a_{kl} (m_k + m_l + 2\rho m_k m_l) \\
&= \sum_{k>l} a_{kl} \sum_i \sum_j b_{ij} \{ (m_k - 1)(m_l - 1) + (m_i - 1) \\
&\qquad + 2\rho m_k m_l (m_i - 1) + \rho^2 m_i m_k m_l (m_i - 1) \} / (1 + \rho m_i)^2 > 0. \quad \square
\end{aligned}
$$

5. Proof of Theorem 1. First, we note some computing formulas for the statistics in $\mathscr{T}(\rho_0)$.

For any real matrix A, define $P_A = A(A'A)^- A'$ for any choice of a generalized inverse and let $P_A^c = I - P_A$. Letting $S_e = Y' P_{[X:B]}^c Y$ we have for $0 \le \rho_0 < \rho_1 <$

∞ that

(5.1) $$T(Y; \rho_0, \rho_1) = Y'\{(L\Sigma_0^{-1}L' - L\Sigma_1^{-1}L')/(\rho_1 - \rho_0)\}Y/\{S_e + Y'L\Sigma_0^{-1}L'Y\}.$$

Letting $C_i = I + \rho_i B'P_X^c B$, $i = 0, 1, t$, we have $\Sigma_i^{-1} = I - \rho_i L'BC_i^{-1}B'L$; further if $Y_1 = B'P_X^c Y$ and $S_X = Y'P_X^c Y$ we have

$$T(Y; \rho_0, \rho_1) = Y_1'\{(\rho_1 C_1^{-1} - \rho_0 C_0^{-1})/(\rho_1 - \rho_0)\}Y_1/\{S_X - \rho_0 Y_1' C_0^{-1} Y_1\}.$$

Since $(\rho_1 C_1^{-1} - \rho_0 C_0^{-1})/(\rho_1 - \rho_0) = C_0^{-1}C_1^{-1}$, we have

(5.2) $$T(Y; \rho_0, \rho_1) = Y_1' C_1^{-1} C_0^{-1} Y_1/\{S_X - \rho_0 Y_1' C_0^{-1} Y_1\}$$

which holds for $0 \le \rho_0 \le \rho_1 < \infty$. (By continuity, the result holds for $\rho_1 = \rho_0$, which defines the LMPI statistic.)

The Wald statistic is similarly computed as

(5.3) $$T(Y; \rho_0, \infty) = \{Y'(P_{[X:B]} - P_X)Y - \rho_0 Y_1' C_0^{-1} Y_1\}/S_e.$$

A substantial portion of the proof of Theorem 1 is devoted to showing that the effect of making the statistics invariant to $X\Pi$ is asymptotically negligible. To this end, we suppose without loss of generality that $X\Pi = 0$ and consider which invariant statistics obtain when it is known in advance that $X\Pi = 0$. Letting L_1 denote a matrix whose columns form an orthonormal basis for $\mathbf{R}(B)$ and $\Sigma_{i\infty} = I + \rho_i L_1' BB' L_1$, $i = 0, 1, t$, these statistics are

$$T_\infty(Y; \rho_0, \rho_1) = Y'\{(L_1\Sigma_{0\infty}^{-1}L_1' - L_1\Sigma_{1\infty}^{-1}L_1')/(\rho_1 - \rho_0)\}Y \div \{Y'P_B^c Y + Y'L_1\Sigma_{0\infty}^{-1}L_1'Y\} \quad \text{for } 0 \le \rho_0 < \rho_1 < \infty$$

and

$$T_\infty(Y; \rho_0, \infty) = Y'L_1\Sigma_{0\infty}^{-1}L_1'Y/Y'P_B^c Y.$$

Using algebraic manipulation and letting $D_i = I + \rho_i B'B$, $i = 0, 1, t$, we have

(5.4) $$T_\infty(Y; \rho_0, \rho_1) = Y'BD_1^{-1}D_0^{-1}B'Y/\{Y'Y - \rho_0 Y'BD_0^{-1}B'Y\},$$

which holds for $0 \le \rho_0 \le \rho_1 < \infty$ (the case $\rho_1 = \rho_0$ obtains by continuity) and

(5.5) $$T_\infty(Y; \rho_0, \infty) = \{Y'P_B Y - \rho_0 Y'BD_0^{-1}B'Y\}/Y'P_B^c Y.$$

The equivalence of statistics (5.2) and (5.3) with (5.4) and (5.5), respectively, is established in

LEMMA 1. *As* $t \to \infty$ *under* (A1)–(A5), $T(Y; \rho_0, \rho_1) - T_\infty(Y; \rho_0, \rho_1) = o_p(t^{-1/2})$ *for* $\rho_0 < \infty$ *and* $0 \le \rho_0 \le \rho_1 \le \infty$.

PROOF. Consider, using (5.2) and (5.4), the difference $T(Y; \rho_0, \rho_1) - T_\infty(Y; \rho_0, \rho_1) = X_t/(U_t - V_t) - X_t^\infty/(U_t^\infty - V_t^\infty)$ (say). To prove the result for these statistics it will suffice to prove that

(i) $(U_t - V_t)/t$ and $(U_t^\infty - V_t^\infty)/t$ are bounded in probability away from 0,
(ii) $X_t^\infty/t = O_p(1)$ and
(iii) $U_t - U_t^\infty$, $V_t - V_t^\infty$ and $X_t - X_t^\infty$ all are $O_p(1)$.

To prove (i), note from (5.1) that $(U_t - V_t)/t \geq S_e/t \to_p m - 1 > 0$. Similarly, $(U_t^\infty - V_t^\infty)/t \geq Y'P_B^cY/t \to_p m - 1$, so that (i) is valid.

To verify (ii) and (iii) it will suffice that the expectations and variances of the terms in question are $O(1)$. The expectation of an arbitrary real quadratic form $Y'AY$ is $O(1)$ if $\mathrm{tr}(B'AB)$ and $\mathrm{tr}(A)$ are $O(1)$. Using the inequality $\|AB\| \leq |A|\,\|B\|$, where $|A| = \lambda_{\max}^{1/2}(A'A)$ and $\|B\| = \mathrm{tr}^{1/2}(B'B)$, one may use a symmetrized version of A to verify that $\mathrm{Var}(Y'AY) = O(1)$ if $\|A\|^2 = O(1)$.

Since $X_t^\infty/t = Y'\{BD_1^{-1}D_0^{-1}B'/t\}Y$, we have $X_t^\infty/t = O_p(1)$ since $\mathrm{tr}(BD_1^{-1}D_0^{-1}B'/t)$, $\mathrm{tr}(B'BD_1^{-1}D_0^{-1}B'B/t)$ and $\|BD_1^{-1}D_0^{-1}B'/t\|^2$ are all $O(1)$, and (ii) is verified.

To verify (iii), consider first $U_t - U_t^\infty = Y'\{-P_X\}Y$; note that

$$\mathrm{tr}(P_X) = \mathbf{r}(X), \qquad \mathrm{tr}(B'P_XB) = \|B'P_X\|^2 \leq |B|^2\|P_X\|^2 = m_s\mathbf{r}(X)$$

are all $O(1)$, so that $U_t - U_t^\infty = O_p(1)$.

Now consider for $\rho_0 \neq 0$,

$$(V_t - V_t^\infty)/\rho_0 = \left[Y'P_X^cB\{C_0^{-1} - D_0^{-1}\}B'P_X^cY\right] + \left[Y'P_XBD_0^{-1}B'P_XY - 2Y'P_XBD_0^{-1}B'Y\right].$$

The second bracketed term is seen to be $O_p(1)$ easily. [The inequality $|\mathrm{tr}(AB)| \leq \|A\|\,\|B\|$ is useful.] To take care of the first bracketed term, use $C_0^{-1} - D_0^{-1} = \rho_0D_0^{-1}B'K(I - \rho_0K'BD_0^{-1}B'K)^{-1}K'BD_0^{-1}$: The first bracketed term is $O_p(1)$ since $|(I - \rho_0K'BD_0^{-1}B'K)^{-1}| \leq 1 + \rho_0m_s = O(1)$ and $\|K\|^2 = \mathbf{r}(X) = O(1)$.

Finally,

$$X_t - X_t^\infty = Y'P_X^cBC_1^{-1}\{C_0^{-1} - D_0^{-1}\}B'P_X^cY + Y'P_X^cBD_0^{-1}\{C_1^{-1} - D_1^{-1}\}B'P_X^cY + Y'P_XBD_1^{-1}D_0^{-1}B'P_X^cY - 2Y'P_XBD_1^{-1}D_0^{-1}B'Y$$

is seen to be $O_p(1)$ using the same essential techniques used for $(V_t - V_t^\infty)/\rho_0$.

To demonstrate the lemma for the Wald statistic, write (5.3) and (5.5) as X_t/U_t and X_t^∞/U_t^∞, respectively. It will still suffice to demonstrate conditions (i)–(iii), as for the MPI tests, but now the terms $\{V_t, V_t^\infty\}$ are identically 0. Conditions (i) and (ii) are satisfied in virtually the same manner as for the LMP tests.

Consider $U_t - U_t^\infty = Y'(P_B - P_{[X:B]})Y$, so that $E(U_t - U_t^\infty) = O(1)$ and $\mathrm{Var}(U_t - U_t^\infty) = O(1)$.

Now,

$$X_t - X_t^\infty = \left[Y'(P_{[X:B]} - P_X - P_B)Y\right] - \rho_0\left[Y'P_X^cBC_0^{-1}B'P_X^cY - Y'BD_0^{-1}B'Y\right].$$

We have already seen that the second bracketed term is $O_p(1)$; this term is $V_t - V_t^\infty$ in the discussion of the LMP statistics. Noting that $E\{Y'(P_{[X:B]} - P_X - P_B)Y\} = O(1)$ and $\mathrm{Var}\{Y'(P_{[X:B]} - P_X - P_B)Y\} = O(1)$, the lemma is proven. □

Proof of Theorem 1. Let $x_{0t}, x_{1t}, \ldots, x_{st}$ be independently distributed central chi-squared variables, where x_{0t} has $n - t$ and x_{it} has $a_t(i)$ degrees of freedom. Then the distribution of $T_\infty(Y;\rho_0,\rho_1)$ is that of

$$\sum_{i=1}^{s} m_i(1+\rho_t m_i)(1+\rho_0 m_i)^{-1}(1+\rho_1 m_i)^{-1}x_{it}$$
$$\div\left\{x_{0t}+\sum_{i=1}^{s}(1+\rho_t m_i)(1+\rho_0 m_i)^{-1}x_{it}\right\}$$
$$=\sum_{i=1}^{s} m_i(1+\rho_1 m_i)^{-1}x_{it}\Big/\left\{x_{0t}+\sum_{i=1}^{s}x_{it}\right\}$$
$$+t^{-1/2}\Delta\{mm_{211}(\rho_1)-m_{11}m_{101}(\rho_1)\}/m^2+o_p(t^{-1/2}).$$

The result of the theorem follows for the $T_\infty(Y;\rho_0,\rho_1)$ statistics by using the asymptotic distribution of the vector $t^{1/2}(x_{0t}/t-(m-1), x_{1t}/t-p_1,\dots, x_{st}/t-p_s)$, a standard delta-method argument and Lemma 1.

Consider now $T_\infty(Y;\rho_0,\infty)$, which is distributed as

$$\sum_{i=1}^{s}(1+\rho_t m_i)(1+\rho_0 m_i)^{-1}x_{it}/x_{0t}$$
$$=\sum_{i=1}^{s}x_{it}/x_{0t}+t^{-1/2}\Delta m_{11}/(m-1)+o_p(t^{-1/2}).$$

The result follows from a standard delta-method argument and Lemma 1. □

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