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AN ASYMPTOTIC EXPANSION OF THE NONNULL DISTRIBUTION OF WILKS CRITERION FOR TESTING THE MULTIVARIATE LINEAR HYPOTHESIS

BY R. W. KULP AND B. N. NAGARSENKER

Air Force Institute of Technology

An asymptotic expansion of the nonnull distribution of the Wilks statistic for testing the linear hypothesis in multivariate analysis of variance is obtained up to the order N^{-2} where N is the sample size, for the first time in terms of noncentral beta distributions. The asymptotic distributions are better than the ones available in Anderson (1958) in the null case and in Sugiura and Fujikoshi (1969) and Posten and Bergman (1964) in the nonnull case. In fact, for certain parameters the asymptotic expansion reduces to the first term and we get the exact distribution.

1. Introduction. The usual multivariate linear model (or MANOVA) has been discussed by many authors (e.g. Anderson, 1958; Das Gupta, Anderson and Mudholkar, 1964; Roy, 1957; Seber, 1966; etc.) and the following canonical form is well known: Let each column vector of $p \times N$ matrix $\mathbf{X}$: $(\mathbf{X}_1, \cdots, \mathbf{X}_N)$ be distributed independently according to a p-variate normal distribution with common covariance matrix $\mathbf{\Sigma}$, where $\mathbf{\Sigma}(p \times p)$ is positive definite and unknown. The problem is to test the hypothesis

$$H_0\colon E(\mathbf{X}_j) = 0 \tag{1.1}$$

for $j = 1, 2, \cdots, b$ and $s + 1, \cdots, N$ with $b \le s$ against the alternative

$$\begin{aligned} K\colon E(X_j) &\neq 0 \quad \text{for some} \quad j \quad (1 \le j \le b) \\ &= 0 \quad \text{for} \quad j = s+1, \cdots, N. \end{aligned} \tag{1.2}$$

The likelihood ratio statistic for testing this hypothesis is due to Wilks (1932) and is given by

$$\Lambda_1 = W^{N/2} = (|\mathbf{S}_e|/|\mathbf{S}_e + \mathbf{S}_h|)^{N/2} \tag{1.3}$$

where $S_e = \sum_{\alpha=s+1}^{N} \mathbf{X}_\alpha \mathbf{X}_\alpha'$ and $\mathbf{S}_h = \sum_{\alpha=1}^{b} \mathbf{X}_\alpha \mathbf{X}_\alpha'$. In the context of multivariate analysis of variance, the matrix $\mathbf{S}_e$ is the sum of squares and products due to error and has the Wishart distribution $W_p(f_1, \mathbf{\Sigma})$ where $f_1 = N - s$. The matrix $\mathbf{S}_h$ is the sum of squares and products due to departure from the hypothesis and has the noncentral Wishart distribution $W_p(f_2, \mathbf{\Sigma}; \mathbf{V})$, $f_2 = b$, where the noncentrality matrix $\mathbf{V} = ½\ \mathbf{MM}'\mathbf{\Sigma}^{-1}$ with $M = E[\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_b]$ under the alternative K.

The null distribution of W was obtained by Wilks (1932) in a closed form for

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particular cases; these results with certain corrections are quoted by Anderson (1958). Various approximations to the null distribution of W were given by Bartlett (1938), Rao (1948), (1951), (1952) and Box (1949). Schatzoff (1966), Pillai and Gupta (1969) and Lee (1972) tabulated the percentage points of W. The nonnull distribution of W has been considered by Posten and Bargmann (1964), Pillai and Jayachandran (1967), Gupta (1971) and Lee (1971). Sugiura and Fujikoshi (1969) obtained the asymptotic expansion of the nonnull distribution of $P(-2\rho \log \Lambda_1 > z)$ up to the order N^{-2}, for arbitrary noncentrality matrix, in terms of noncentral chi-squared series where a correction factor ρ is determined such that under H_0, the first remainder term in the asymptotic expansion of $P(-2\rho \log \Lambda_1 > z)$ vanishes, i.e. ρ is given by

$$\rho N = f_1 + (f_2 - p - 1)/2$$

(see Anderson, 1958, page 208).

In this paper, we obtain the asymptotic expansions of the nonnull distribution of W up to the order N^{-2} in terms of $m = \rho N$ increasing instead of N as in Sugiura and Fujikoshi (1969) and as in Posten and Bargmann (1964). The asymptotic expansions are obtained in terms of noncentral beta distributions. The motivation for considering such expansions is that it is known that for $p = 1$ the exact distribution of W is a noncentral beta distribution and thus the asymptotic expansion of the nonnull distribution of W in terms of noncentral beta distributions should be better than the one in terms of noncentral chi-squared distribution available so far in Sugiura and Fujikoshi (1969). Also it is shown that for $p = 1$ and $b = 1$ the nonnull asymptotic expansion obtained in this paper reduces to the first term and we get the exact distribution. Furthermore, for $p = 2$ the asymptotic expansion in the null case reduces to the exact distribution.

2. Asymptotic distribution of W in the null case. Let $m = f_1 - 2\delta$ where δ is to be chosen later suitably. It will be seen below that $\delta = (p - f_2 + 1)/4$ so that we have $m = n - (p + f_2 + 1)/2$. In this section we shall obtain the asymptotic expansion of the null distribution of $U = W^{1/s}$ up to the order m^{-3} where s is to be determined later. Using equation (47) of Constantine (1963), the nonnull hth moment of W defined in (1.3) is given by

$$E(W^h) \quad e^{-\sigma_1} \textstyle\sum_{k=0}^{\infty} \sum_{\kappa} \dfrac{\Gamma_p(n/2, \kappa)\Gamma_p(h + f_1/2)}{\Gamma_p(f_1/2)\Gamma_p(h + n/2, \kappa)} \dfrac{C_\kappa(\mathbf{V})}{k!} \tag{2.1}$$

where $\sigma_1 = \text{tr}(\mathbf{V})$ and $n = f_1 + f_2$. $\Gamma_p(a)$ and $\Gamma_p(a, \kappa)$ are as defined in Constantine (1963). From (2.1), the hth null moments of U are given by

$$E(U^h) = K(m_1, \delta) \cdot \phi(h)$$

where

$$K(m_1, \delta) = \Gamma_p(m_1 + \delta + f_2/2)/\Gamma_p(m_1 + \delta), \quad m_1 = m/2$$

and

$$\phi(h) = \Gamma_p(h/s + m_1 + \delta)/\Gamma_p(h/s + f_2/2 + m_1 + \delta).$$

Using the inverse Mellin transform, the density of U is given by

$$f(u) = K(m_1, \delta)(2\pi i)^{-1} \int_{-i\infty}^{i\infty} u^{-h-1}\phi(h)\, dh. \tag{2.2}$$

Putting $sm_1 + h = t$ in the integrand on the right-hand side of (2.2) and proceeding as in Nagarsenker and Pillai (1973), the density of U can be put in the form

$$f(u) = K(m_1, \delta)s^{v} \textstyle\sum_{i=0}^{\infty} R_i B_u(m_1 s + a, v + i) \dfrac{\Gamma(m_1 s + a)}{\Gamma(m_1 s + a + v + i)} \tag{2.3}$$

where $v = pf_2/2$, and a is a convergence factor chosen such that $R_1 = 0$. The coefficients R_i are computed as in Nagarsenker and Pillai (1973). Using the asymptotic expansions for the logarithms of the gamma functions, in $K(m_1, \delta)$ and choosing a and s such that $R_1 = 0$ and $R_2 = 0$ respectively, we have, after some algrbra, the following theorem:

THEOREM 2.1. *The null distribution of $U = W^{1/s}$ can be expanded asymptotically as*

$$\begin{aligned} F(u) &= I_u(sm_1 + a, v) + R_3/((sm_1)^3) \\ &\quad \cdot [I_u(sm_1 + a, v + 3) - I_u(sm_1 + a, v)] + O(m_1^{-4}) \end{aligned} \tag{2.4}$$

where

$$a = \frac{1 - v}{2}, \quad s^2 = \frac{p^2 f_2^2 = 4}{(p^2 + f_2^2 - 5)}, \quad m_1 = \frac{m}{2}, \quad m = n - \frac{p + f_2 + 1}{2}, \quad R_3 = -T_3$$

and $I_u(a, b)$ is the incomplete beta function. In particular if $p = 1$, then $s = 1$, $R_3 = 0$ and the distribution of $U = W$ reduces to $I_u(f_1/2, f_2/2)$ which is the exact distribution of W. Further if $p = 2$, then $s = 2$, $R_3 = 0$ and the distribution of $U = \sqrt{w}$ reduces to $I_u(f_1 - 1, f_2)$ which is the exact distribution of $\sqrt{W}$ (see Anderson, 1958).

3. Asymptotic nonnull distribution of *W*. In this section we shall consider the asymptotic distribution of $U = W^{1/s}$ in terms of m_1 increasing where m_1 is as in Section 2. The hth moment of U from (2.1) is given by

$$E(U^h) = \textstyle\sum_{k=0}^{\infty} \sum_{\kappa} C(m_1)\phi(h)$$

where

$$C(m_1) = e^{-\sigma_1} \frac{C_\kappa(\mathbf{V})}{k!} \prod_{i=1}^{p} \left[\frac{\Gamma(m_1 + \delta + k_i + (f_2 + 1 - i)/2)}{\Gamma(m_1 + \delta + (1 - i)/2)}\right] \tag{3.1}$$

and

$$\phi(h) = \prod_{i=1}^{p} \left[\frac{\Gamma(m_1 + \delta + h/s + (1 - i)/2)}{\Gamma(m_1 + \delta + k_i + h/s + (f_2 + 1 - i)/2)}\right].$$

Using the inverse Mellin transform and proceeding as in Section 2, the density function of U can be put in the form

$$f(u) = \sum_{k=0}^{\infty} \sum_{\kappa} C(m_1)s^{v_1} \sum_{i=0}^{\infty} R_i(\kappa)B_u(m_1 s + a, v + k + i) \cdot \frac{\Gamma(m_1 s + a)}{\Gamma(m_1 s + a + v + k + i)} \tag{3.2}$$

where a, v, s are as in Section 2 and the coefficients $R_i(\kappa)$ which we need are given below:

$$R_0(\kappa) = 1, \quad R_1(\kappa) = -(t_1 k + s a_1(\kappa) - k^2)/2$$

and

$$R_2(\kappa) = k a_1 + k^2 a_2 + k^3 a_3 + k^4/8 + a_4(k + 1)a_1(\kappa) + (s/4)k^2 a_1(\kappa) + (s^2/8)a_1^2(\kappa) + (s^2/24)a_2(\kappa)$$

where

$$a_1(\kappa) = \sum_{i-1}^{p} k_i(k_i - i), \quad a_2(\kappa) = \sum_{i=1}^{p} k_i(4k_i^2 - 6ik_i + 3i^2), \quad t_1 = (sd_1 - v),$$

$$d_1 = 2\delta + f_2, \quad a_1 = (s^2 b_1 - 6t_1(v + 1) - (3v^2 - 1))/24,$$

$$b_1 = (12\delta_1^2 - 12\delta_1 + 2), \quad \delta_1 = (d_1 + 1)/2,$$

$$a_2 = (s^2 d_1^2 - 2t_1(v + 1) - v^2 + 2)/8, \quad a_3 = (1 - 3t_1)/12$$

and

$$a_4 = s(t_1 - 1)/4.$$

Using the asymptotic expansion of the logarithm of a gamma function and using some identities involving noncentral beta series (see Nagarsenker, 1979), the density of U is given by

$$f(u) = \beta_{\sigma_1}(u\colon m_1 s + a, v) + (1/m_1 s) \sum_{i=1}^{2} G_i \beta(i) + (1/(m_1 s)^2) \sum_{i=1}^{5} D_i(i) + \mathbf{O}(m_1^{-3}) \tag{3.3}$$

where

$$\sigma_i = \mathrm{tr}(\mathbf{V}^i), \quad \beta(i) = \beta_{\sigma_1}(u\colon m_1 s + a, v + i) - \beta_{\sigma_1}(u\colon m_1 s + a, v + 1 + i),$$

$$G_1 = \sigma_1(t_1 - 1)/2, \quad G_2 = (s\sigma_2 - \sigma_1^2)/2, \quad D_1 = \sigma(e_1 + E + F + 1/8),$$

$$e_1 = (3v^2 - s^2 b_1)/24, \quad E = (t_1^2 - 2v)/8, \quad F = (2 - 3t_1)/12, \quad G = st_1/4,$$

$$D_2 = \sigma_1(s^2/24 - 1/3 - a_1 - a_2 - a_3) + \sigma_1^2(E + 3F + 7/8 + s^2/8) - s^2/2\sigma_1\sigma_2 + (3/8 s^2 + 2G)\sigma_2,$$

$$D_3 = \sigma_1^3(F + 3/4) + s^2/2\sigma_3 + G\sigma_1\sigma_2 + \sigma_1^2(-a_2 - 3z_3 - 7/8 - s^2/4) + \sigma_2(s - 3a_4 - s^2/4),$$

$$D_4 = 1/8(\sigma_1^4 + s^2\sigma_2^2) - \sigma_1^3(a_3 + 3/4) - 5/12 s^2\sigma_3 + \sigma_1\sigma_2(s/4 - a_4)$$

and

$$D_5 = -\tfrac{1}{8}(\sigma_1^2 - s\sigma_2)^2.$$

This implies the following theorem:

THEOREM 3.1. *The nonnull distribution of the likelihood ratio criterion in* (1.3) *for the multivariate linear hypothesis can be approximated asymptotically up to the order* N^{-2} by

$$P(U \le u) = I_{\sigma_1}(u\colon m_1 s + a, v) + 1/m_1 s \sum_{i=1}^{2} G_i I_{\sigma_i}(i) + 1/(m_2 s)^2 \sum_{i=1}^{5} D_i I_{\sigma_1}(i) + \mathbf{O}(m_1^{-3}) \tag{3.4}$$

where $I_{\sigma_1}(u\colon a, b)$ *is the noncentral beta distribution with indices* a *and* b *and noncentrality parameter* σ_1,

$$v = pf_2/2,\ m_1 = m/2 = (f_1 - 2\delta)/2,\ \delta = (p - f_2 - 1)/4,$$

a and s are given in Section 2,

$$I_{\sigma_1}(i) = I_{\sigma_1}(u\colon m_1 s + a, v + i) - I_{\sigma_1}(u\colon m_1 s + a, v + 1 + i)$$

and the coefficients G_i and D_i are given in (3.3).

SPECIAL CASES. Putting $p = 1$, we have $s = 1$ and it can be easily checked that the coefficients G_i and D_i reduce to zero. The nonnull distribution of $U = W$ then reduces to

$$P(U \le u) = I_{\sigma_1}(u\colon f_1/2, f_2/2)$$

which is the exact nonnull distribution of U when $p = 1$ (see Bose and Roy, 1938 and Hsu, 1938). Furthermore, the exact null distributions for $p = 1$ and $p = 2$ are obtained as special cases as before.

4. Numerical results. Table I gives the comparisons of the exact values of the percentage points of W with those obtained using the first term of Box's chi-squared approximation and the first term of the beta approximation (2.4). The values obtained using the beta approximation (2.4) agree completely up to

TABLE I
Comparison of Box's and beta approximations with the exact 5% points of W

p	f_2	f_1	Exact	Box	Beta
1	1	5	.43074	.42585	.43074
		7	.55593	.55378	.55593
2	3	5	.07362	.08059	.07362
		6	.11646	.11262	.11646
3	12	12	.03379	.04128	.03382
		16	.07084	.07809	.07079
4	12	23	.08099	.08550	.08100
		33	.16434	.16771	.16434

TABLE II
Power of the test of equality of means

σ_1	$O(m^{-1})$	$O(m^{-2})$	$O(m^{-3})$	Exact	Beta
	$p = 1$, $f_2 = 8$ power using SFE neglecting term order				
0.4	.06335	.06203	.06214	.06238	.06238
4	.23696	.21572	.21724	.21753	.21753
8	.47578	.43266	.43431	.43473	.43473
	$p = 4$, $f_2 = 1$				
1	.10385	.08575	.09101	.09040	.09040
4	.31700	.22620	.24278	.24222	.24222
8	.60147	.45376	.45812	.46360	.46360

TABLE III
Comparison of the exact power of W for testing $(\upsilon_1, \upsilon_2) = (0, 0)$ *with the approximation* (3.4) *for* $p = 2$

υ_1	υ_2	$n = f_1 + f_2$ / f_2	20 / 7	46 / 13	90 / 7
0	$0^4 1$		.05000	.05000	.05000
			.05000	.05000	.05000
.01	0		.05015	.05012	.05021
			.05014	.05012	.05021
.05	.05		.05148	.05124	.05216
			.05147	.05123	.05216
0	.1		.05148	.05124	.05216
			.05148	.05124	.05216
.1	.4		.05760	.05639	.06128
			.05759	.05639	.06128
0	1		.06550	.06314	.07364
			.06551	.06313	.07364
0	2		.08256	.07786	.10162
			.08255	.07786	.10161
1	1		.08394	.07836	.10210
			.08394	.07836	.10210
0	3		.1010	.0941	.1336
			.10102	.09411	.13357

(υ_1 and υ_2 are the characteristic roots of **V**)

four significant figures with the exact values and differ only in the fifth place in some cases.

In Table II a comparison of the asymptotic nonnull distribution obtained in this paper and that obtained by Sugiura and Fujikoshi (1969) is made with the exact in terms of numerical powers. The first three columns of Table II give the power as obtained by Sugiura–Fujikoshi expansion (SFE) in Equation (1.34) of Sugiura and Fujikoshi (1969) for the test of equality of means for the cases $p = 1$ with nine groups (i.e. $f_2 = 8$) and $p = 4$ with two groups (i.e. $f_2 = 1$) with ten observations per group for both the cases.

Table III gives the comparison of the exact values obtained by Pillai and Jayachandran (1967) for $p = 2$ with the values computed using the approximation (3.4). Top values are the exact values given in Pillai and Jayachandran (1967). The second value is the power obtained using Equation (3.4) of this paper.

It is seen from Table II that even in the simplest case of $p = 1$ or $f_2 = 1$, the beta expansion seems to be better than the Sugiura–Fujikoshi expansion. The power computed from the first term of their expansion is not in close agreement with the exact power whereas the power computed from the first term in (3.4) coincides with the exact values. Also from Table III it is seen that the agreement between the exact values and the approximate values obtained using (3.4) is quite good and they differ in most cases only in the fifth significant digit.

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MATHEMATICS DEPARTMENT (AFIT/ENC)
DEPARTMENT OF THE AIR FORCE
AIR FORCE INSTITUTE OF TECHNOLOGY (ATC)
WRIGHT-PATTERSON AIR FORCE BASE, OHIO 45433