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ASYMPTOTIC NORMALITY FOR A GENERAL CLASS OF STATISTICAL FUNCTIONS AND APPLICATIONS TO MEASURES OF SPREAD[1]

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A general class of statistical functionals is introduced and the asymptotic normality of the corresponding estimators is established. Included are certain measures of spread, proposed by Bickel and Lehmann (1979), for which the asymptotic distribution theory has been an open problem. The proofs involve some new results, of independent interest, for empirical and quantile processes.

1. Introduction. Let $X_1, \cdots, X_n$ be independent random variables having common distribution function (df) F. Let h be a function from $\mathbb{R}^m$ to $\mathbb{R}$ (not necessarily symmetric) and denote by H_F the df of $h(X_1, \cdots, X_m)$. A number of parameters of interest can be expressed as $T(H_F)$, where $T(\cdot)$ is a functional of the general form

$$T(G) = \int_0^1 q(T_t(G))\, dK(t). \tag{1.1}$$

Here G is a df, q is a real-valued function of a real variable, K denotes a df on $[0, 1]$, and for each t in the support S of K, $T_t(\cdot)$ denotes a functional of the form

$$T_t(G) = \int_0^1 G^{-1}(s)\, dM_t(s), \tag{1.2}$$

where $G^{-1}(s) = \inf\{x: G(x) \ge s\}$ and M_t is a signed measure on $[0, 1]$. Whereas the $T_t(\cdot)$ are simply familiar "L-functionals", the general form (1.1) includes L-functionals, "generalized L-functionals" (Serfling, 1984), and certain special measures of spread introduced by Bickel and Lehmann (1979). These examples are discussed in detail in Section 2 and illustrate the need for study of the general functional (1.1).

The main purpose of this paper is to establish the asymptotic normality of a natural class of estimators for parameters of the form (1.1). A general result,

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Theorem 3.1, is developed in Section 3 and applied in Sections 5 and 6, respectively, to the estimators of two of the above-mentioned spread measures, estimators for which the asymptotic distribution theory has been an open problem.

The estimators we consider for the parameters $T(H_F)$ of form (1.1) are given by $T(H_n)$, where H_n is an empirical df constructed for estimation of H_F. Specifically, we define

$$H_n(y) = n_{(m)}^{-1} \sum I[h(X_{i_1}, \cdots, X_{i_m}) \le y], \qquad -\infty \le y \le \infty, \tag{1.3}$$

where the sum is taken over all $n_{(m)} = n(n-1) \cdots (n-m+1)$ m-tuples $(i_1, \cdots, i_m)$ of distinct elements from $\{1, \cdots, n\}$. In the case $h(x) = x$, H_n reduces to the usual empirical df F_n. In general, however, $H_n(y)$ is a U-statistic in structure, for each fixed y.

In verifying the conditions of Theorem 3.1 in particular applications such as treated in Sections 5 and 6, a major step is to establish, for a designated subinterval A of $[0, 1]$, that

$$\sup_A |R_n(t)| = o_p(n^{-1/2}), \tag{1.4}$$

where

$$R_n(t) = H_n^{-1}(t) - H_F^{-1}(t) - \frac{t - H_n(H_F^{-1}(t))}{h_F(H_F^{-1}(t))} \tag{1.5}$$

and h_F is the density of H_F. In this connection, we prove in Section 4 several preliminary lemmas on the empirical and quantile processes corresponding to $H_n(y)$ as defined in (1.3). These results are of independent interest and may be read independently from the rest of the paper. Even in the case that H_n reduces to the usual empirical df F_n, the results are new. In particular, Lemma 4.2 establishes an in-probability analogue of the strong approximation result of Csörgő and Révész (1978) on the distance between the quantile process and the uniform quantile process, under weaker regularity conditions on F.

The results on asymptotic normality may be applied, for example, in connection with the computation of asymptotic relative efficiencies of the spread estimators considered here, in comparison with each other and with other spread estimators. A general numerical study of a collection of spread estimators is in development by the present authors.

2. Examples

EXAMPLE 1. *L-functionals.* With $h(x) = x$, so that $H_F = F$, and with $q(x) = x$ and $M_t(\cdot) = M(\cdot)$ (independent of t), the form (1.1) reduces to $T(F) = \int_0^1 F^{-1}(s)\, dM(s)$, the familiar "$L$-functional". The asymptotic normality of the corresponding estimators has been studied by many authors (see, e.g., Serfling (1980), Huber (1981) and Helmers (1982) for general discussion).

EXAMPLE 2. *Generalized L-functionals.* With arbitrary $h(\cdot)$ and otherwise as in Example 1, we obtain the functional $T(H_F) = \int_0^1 H_F^{-1}(s)\, dM(s)$ introduced by Serfling (1984), where also the asymptotic normality is proved (see also Silverman, 1983).

EXAMPLE 3. *The spread functional* T_β. Take $h(x) = x$, so that H_F reduces to F and H_n to F_n, and take $q(x) = |x|^\gamma$. Choose M_t in (1.2) such that $T_t(F) = F^{-1}(1-t) - F^{-1}(t)$ for $t \le 1/2$ and $= F^{-1}(t) - F^{-1}(1-t)$ for $t > 1/2$. Then (1.1) becomes

$$T(F) = \int_0^1 |F^{-1}(t) - F^{-1}(1-t)|^\gamma \, dK(t),$$

a functional proposed by Bickel and Lehmann (1979) as a measure of spread of a nonsymmetric df F. In particular, they suggest the case $\gamma = 2$ and $K(\cdot)$ uniform on $(\beta, 1-\beta)$, $0 < \beta < 1/2$, giving

$$T_\beta(F) = (1/2 - \beta)^{-1} \int_{1/2}^{1-\beta} [F^{-1}(t) - F^{-1}(1-t)]^2 \, dt. \tag{2.1}$$

In Section 5 we will show that the corresponding estimator $T_\beta(F_n)$ is asymptotically normal under regularity conditions on the density f. This also provides a proof of asymptotic normality for the asymptotically equivalent estimator (proposed by Bickel and Lehmann, 1979)

$$n^{-1}(1/2 - \beta)^{-1} \sum_{k=[n/2]}^{[n(1-\beta)]} [X_{k:n} - X_{n-k+1:n}]^2,$$

where $X_{k:n}$ denote the order statistics of $X_1, \cdots, X_n$.

EXAMPLE 4. *The spread functional* $\tau^2(X - X'; \alpha, \beta)$. With $h(x_1, x_2) = x_1 - x_2$, H_F denotes the symmetric df of $X - X'$, where X and X' are independent random variables with df F. Bickel and Lehmann (1979) introduce as a spread measure the functional $\tau^2(X - X'; \alpha, \beta)$, where $\tau^2(Z; \alpha, \beta)$ is defined by (3.1) in Bickel and Lehmann (1976) and by (2.3) below. In our notation, $\tau^2(X - X'; \alpha, \beta)$ may be expressed as

$$T(H_F) = (1 - \alpha - \beta)^{-1} \int_\alpha^{1-\beta} \left[H_F^{-1}\left(\frac{t+1}{2}\right)\right]^2 dt. \tag{2.2}$$

Asymptotic normality of the natural estimator $T(H_n)$ is established in Section 6, and this result also provides a central limit theorem for the estimator (b) of Bickel and Lehmann (1979), page 39 (where in their definition of the pseudo-sample "$i < j$" should be replaced by "$i \ne j$").

EXAMPLE 5. *A measure of dispersion.* We note that the functional $\tau^2(Z; \alpha, \beta)$ mentioned in the preceding example is in fact of the form (1.1), since it can be rewritten as

$$T(F) = (1 - \alpha - \beta)^{-1} \int_\alpha^{1-\beta} \left[F^{-1}\left(\frac{t+1}{2}\right)\right]^2 dt, \tag{2.3}$$

with F the df of Z. For further discussion of this functional, see Bickel and Lehmann (1976).

OTHERS. Further interesting variations, involving Winsorizing instead of trimming, for example, could be formulated but will be left implicit.

3. A general result on asymptotic normality. Following the standard scheme of the differentiable statistical function approach (see, e.g., Serfling, 1980, Chapter 6), we approximate $T(H_n) - T(H_F)$ by

$$T(H_F; H_n - H_F) = (d/d\lambda)T(H_F + \lambda(H_n - H_F))|_{\lambda=0+}. \tag{3.1}$$

Assuming that q is differentiable, we obtain from (1.1) the expression

$$T(H_F; H_n - H_F) = \int_0^1 q'(T_t(H_F))T_t(H_F; H_n - H_F)\, dK(t),$$

where $T_t(H_F; H_n - H_F)$ is defined analogously to (3.1). Indeed, following standard treatments of L-functionals (Boos, 1979; Serfling, 1980), we obtain

$$T_t(H_F; H_n - H_F) = n_{(m)}^{-1} \sum A_t(X_{i_1}, \cdots, X_{i_m}),$$

where

$$A_t(x_1, \cdots, x_m) = \int_0^1 \frac{s - I[h(x_1, \cdots, x_m) \le H_F^{-1}(s)]}{h_F(H_F^{-1}(s))}\, dM_t(s),$$

provided that H_F has a positive density on the support of M_t. However, this proviso can be relaxed somewhat if we are more specific about the form of M_t. Namely, consider the case that

$$dM_t(s) = J_t(s)\, ds + \textstyle\sum_{j=1}^{d_t} a_{tj} I(s = p_{tj}),$$

which corresponds to smooth weighting of quantiles $G^{-1}(s)$ in $T_t(G)$ by the first term and discrete weighting of specified quantiles by the second term. We then obtain (see Serfling, 1980, pages 265 and 290, Problem 8.P.5)

$$\begin{aligned} A_t(x_1, \cdots, x_m) = &- \int_{-\infty}^{\infty} \{I[h(x_1, \cdots, x_m) \le y] - H_F(y)\} J_t(H_F(y))\, dy \\ &+ \textstyle\sum_{j=1}^{d_t} a_{tj} \dfrac{p_{jt} - I[h(x_1, \cdots, x_m) \le H_F^{-1}(p_{tj})]}{h_F(H_F^{-1}(p_{tj}))}, \end{aligned} \tag{3.2}$$

provided that H_F has a positive density at the points p_{tj}, $1 \le j \le d_t$. In the case that the second term is absent (i.e., all a_{tj}'s = 0), this proviso may be dropped. However, this term is indeed relevant in the particular examples to be treated in Sections 5 and 6.

Thus, due to the structure of $H_n(y)$ as a U-statistic, $T_t(H_F; H_n - H_F)$ is a U-statistic and hence $T(H_F; H_n - H_F)$ may be represented as a U-statistic:

$$T(H_F; H_n - H_F) = n_{(m)}^{-1} \sum G(X_{i_1}, \cdots, X_{i_m}), \tag{3.3}$$

where

$$G(x_1, \cdots, x_m) = \int_0^1 q'(T_t(H_F))A_t(x_1, \cdots, x_m)\, dK(t).$$

Consequently, noting that $E[G(X_1, \cdots, X_m)] = 0$, defining

$$G_1(x) = E[G(x, X_1, \cdots, X_{m-1}) + \cdots + G(X_1, \cdots, X_{m-1}, x)]$$

and $\sigma^2(T, F) = \text{Var}\{G_1(X)\}$, and assuming $0 < \sigma^2(T, F)$ and $E[G^2(X_1, \cdots, X_m)] < \infty$, we have by central limit theory for U-statistics:

$$n^{1/2}T(H_F; H_n - H_F) \to_d N(0, \sigma^2(T, F)). \tag{3.4}$$

Therefore, a similar limit theorem holds for $n^{1/2}[T(H_n) - T(H_F)]$ if the remainder term

$$\Delta_n = T(H_n) - T(H_F) - T(H_F; H_n - H_F)$$

can be shown to satisfy

$$\Delta_n = o_p(n^{-1/2}). \tag{3.5}$$

Now, assuming that q is twice differentiable, we may write $\Delta_n = \Delta_{n1} + \Delta_{n2}$, where

$$\Delta_{n1} = \int_0^1 q'(T_t(H_F))[T_t(H_n) - T_t(H_F) - T_t(H_F; H_n - H_F)]\, dK(t)$$

and

$$\Delta_{n2} = ½ \int_0^1 q''(\theta_t(H_F, H_n))[T_t(H_n) - T_t(H_F)]^2\, dK(t),$$

with $\theta_t(H_F, H_n)$ a random variable between $T_t(H_F)$ and $T_t(H_n)$. Assuming that q'' is uniformly continuous, it follows immediately that a set of conditions sufficient for (3.5) is given by (A)–(D) below:

$$\eta_1 = \int_0^1 |q'(T_t(H_F))|\, dK(t) < \infty; \tag{A}$$

$$\eta_2 = \int_0^1 q''(T_t(H_F))|\, dK(t) < \infty; \tag{B}$$

$$\sup_{t\in S} | T_t(H_n) - T_t(H_F) - T_t(H_F; H_n - H_F)| = o_p(n^{-1/2}); \tag{C}$$

$$\sup_{t\in S}[T_t(H_n) - T_t(H_F)]^2 = o_p(n^{-1/2}), \tag{D}$$

where S denotes the support of K.

We summarize the previous development in the following general result.

THEOREM 3.1 *Let $T(H_F)$ be given by* (1.1) *and let q have uniformly continuous second derivative. Assume that H_F has positive density h_F at appropriate points and that $0 < \sigma^2(T, F)$ and $E[G^2(X_1, \cdots, X_m)] < \infty$. Then, under conditons* (A)–(D),

$$n^{1/2}[T(H_n) - T(H_F)] \to_d N(0, \sigma^2(T, F)). \tag{3.6}$$

REMARKS. (i) In cases where $q'' = 0$, we have $\eta_2 = 0$ and condition (D) may be deleted.

(ii) In fact, the density h_F need be positive only on the union of the supports of M_t for $t \in S$. In Theorems 5.1 and 6.1 below, this set is contained in the interval given in condition (iii) of these theorems. With specific information on the form of M_t, the assumption of a density can be stated more sharply, as discussed above in connection with (3.2), and in some of these cases the assumption can be dropped completely. Also, from consideration of selected L-functionals for example, we see that the density assumption is superfluous in still other cases. However, the methods of the present paper do not yield a complete resolution of this issue, which we leave as an open matter.

(iii) For the case of (generalized) L-statistics, we have $\eta_1 = 1$ and $\eta_2 = 0$, leaving only (C) to be dealt with. In this case, (C) takes the form

$$T(H_n) - T(H_F) - T(H_F; H_n - H_F) = o_p(n^{-1/2}). \tag{3.7}$$

Thus the present approach reduces to the usual treatments of L-statistics (see Boos, 1979; and Serfling, 1980) and of generalized L-statistics (see Serfling, 1984), where the verification of (3.7) has been carried out in various situations.

4. Empirical and quantile processes of *U*-statistic structure. In Sections 5 and 6, a crucial step is to show that for appropriate t_1 and t_2,

$$\sup_{t_1 \le t \le t_2} |R_n(t)| = o_p(n^{-1/2}), \tag{4.1}$$

where $R_n(t)$ is given by (1.5). Introducing the notation $\xi_t = H_F^{-1}(t)$ and $\xi_{tn} = H_n^{-1}(t)$, and assuming that there exists a constant $M > 0$ such that $h_F(s) > M$ for $s \in [\xi_{t_1}, \xi_{t_1}]$, we have

$$\sup_{t_1 \le t \le t_2} |R_n(t)| \le M^{-1}[R_{1n} + R_{2n} + \sup_{t_1 \le t \le t_2} |H_n(\xi_{tn}) - t|],$$

where

$$R_{1n} = \sup_{t_1 \le t \le t_2} |[H_n(\xi_{tn}) - H_n(\xi_t)] - [H_F(\xi_{tn}) - H_F(\xi_t)]|$$

and

$$R_{2n} = \sup_{t_1 \le t \le t_2} |h_F(\xi_t)[\xi_{tn} - \xi_t] - [H_F(\xi_{tn}) - H_F(\xi_t)]|.$$

Since $|H_n(\xi_{tn}) - t| \le n_{(m)}^{-1}$, it is immediate that $\sup_{t_1 \le t \le t_2} |H_n(\xi_{tn}) - t| = o_p(n^{-1/2})$. Hence for (4.1) it suffices to show that

$$R_{in} = o_p(n^{-1/2}), \quad i = 1, 2. \tag{4.2}$$

Define $u(x_1, \cdots, x_m) = H_F(h(x_1, \cdots, x_m))$ and let U_n be the empirical df of the $n_{(m)}$ uniformly distributed, but dependent, random variables $u(X_{i_1}, \cdots, X_{i_m})$. Introduce the associated empirical and quantile processes,

$$\alpha_n(t) = n^{1/2}[U_n(t) - t], \quad 0 \le t \le 1,$$

and

$$u_n(t) = n^{1/2}[U_n^{-1}(t) - t], \quad 0 < t < 1,$$

respectively. Then $H_n = U_n \circ H_F$ and it follows readily that

$$R_{1n} \le n^{-1/2}\omega(\alpha_n;\ n^{-1/2}\|u_n\|_\infty), \tag{4.3}$$

where $\omega(g, \delta) = \sup_{|s-t|<\delta}|g(s) - g(t)|$, the modulus of continuity function, and $\|g\|_\infty = \sup_t|g(t)|$, the supremum norm. Also, it is readily seen that

$$R_{2n} \le n^{-1/2}\ \sup_{t_1\le t\le t_2}|v_n(t)|\ \sup_{t_1\le t\le t_2}|h_F(\xi_t) - h_F(\tilde{\xi}_{tn})|, \tag{4.4}$$

where $v_n(t) = n^{1/2}[H_n^{-1}(t) - H_F^{-1}(t)]$, $0 \le t \le 1$, and $\tilde{\xi}_{tn}$ lies between ξ_t and ξ_{tn}.

From (4.3) and (4.4) it is clear that we need appropriate results for $\|u_n\|_\infty$ and $\sup_{t_1\le t\le t_2}|v_n(t)|$. These are presented in the next two lemmas.

LEMMA 4.1 $\|u_n\|_\infty = O_p(1)$.

PROOF. From the weak convergence of $\alpha_n(\cdot)$ established by Silverman (1983), it follows that $\|\alpha_n\|_\infty$ has a limit distribution and thus $\|\alpha_n\|_\infty = O_p(1)$. Now it is readily checked that a.s.

$$u_n(t) = -\alpha_n(U_n^{-1}(t)) + O(n^{-1/2}).$$

Thus $\|u_n\|_\infty \le \|\alpha_n\|_\infty + O_p(n^{-1/2})$. □

LEMMA 4.2 *Suppose that for some* $\varepsilon > 0$, h_F *is bounded away from* 0 *and is Lipschitz continuous on* $A^\varepsilon = [\xi_{t_1} - \varepsilon, \xi_{t_2} + \varepsilon]$. *Then*

$$\sup_{t_1\le t\le t_2}|h_F(\xi_t)v_n(t) - u_n(t)| = O_p(n^{-1/2}) \tag{a}$$

and

$$\sup_{t_1\le t\le t_2}|v_n(t)| = O_p(1). \tag{b}$$

PROOF. Let $h_F(t) > M > 0$, $t \in A^\varepsilon$. Then for $t \in A^\varepsilon$,

$$\begin{aligned}|v_n(t)| &\le M^{-1}|h_F(\xi_t)v_n(t)|\\ &\le M^{-1}|u_n(t)| + M^{-1}|h_F(\xi_t)v_n(t) - u_n(t)|.\end{aligned}$$

Hence, by Lemma 4.1, (b) follows from (a). To establish (a), we use the relation $H_n^{-1} = H_F^{-1} \circ U_n^{-1}$ and write

$$\begin{aligned}h_F(\xi_t)v_n(t) &= n^{1/2}h_F(\xi_t)[H_F^{-1}(U_n^{-1}(t)) - H_F^{-1}(t)]\\ &= h_F(\xi_t)u_n(t)/h_F(H_F^{-1}(\theta_{nt})),\end{aligned}$$

where θ_{nt} lies between t and $U_n^{-1}(t)$. Thus

$$h_F(\xi_t)v_n(t) - u_n(t) = \frac{u_n(t)[h_F(\xi_t) - h_F(H_F^{-1}(\theta_{nt}))]}{h_F(H_F^{-1}(\theta_{nt}))},$$

and so by Lemma 4.1 it suffices to show that uniformly in $t_1 \le t \le t_2$

$$\frac{h_F(\xi_t) - h_F(H_F^{-1}(\theta_{nt}))}{h_F(H_F^{-1}(\theta_{nt}))} = O_p(n^{-1/2}).$$

Since $h_F(s) > M$ on A^c, $1/h_F(H_F^{-1}(\theta_{nt})) < M^{-1}$ for $t \in (t_1, t_2)$. Finally, $h_F(\xi_t) - h_F(H_F^{-1}(\theta_{nt})) = O_p(n^{-1/2})$ uniformly in $t_1 \le t \le t_2$ by the Lipschitz condition and a further application of Lemma 4.1. □

For the special case that $H_F = F$, Lemma 4.2(a) has an almost sure counterpart, with rate $O(n^{-1/2} \log \log n)$, established by Csörgő and Révész (1978) under second-order differentiability assumptions on F.

We now are in a position to give conditions under which (4.1) holds.

LEMMA 4.3. *Under the conditions of Lemma* 4.2, (4.1) *holds.*

PROOF. We wish to establish (4.2). For R_{1n}, it suffices by (4.3) and Lemma 4.1 to show that, for arbitrary K,

$$\omega(\alpha_n; n^{-1/2}K) = o_p(1). \tag{4.5}$$

Now, Silverman (1983) proves

$$\lim_{x\to 0} \limsup_{n\to\infty} E\{\omega(\alpha_n; x)\} = 0,$$

so that $E\{\omega(\alpha_n; n^{-1/2})\} \to 0$ as $n \to \infty$. This yields (4.5).

For R_{2n}, we use (4.4) and the Lipschitz condition on h_F to obtain

$$R_{2n} = n^{-1} O(\sup_{t_1 \le t \le t_2} |v_n(t)|^2),$$

which yields the desired conclusion by Lemma 4.2(b). □

5. The spread functional T_β. We return to the trimmed variance functional given by (2.1) and note that since $q(x) = x^2$ we immediately have condition (B) of Theorem 3.1. Also, since $|T_t(H_F)| = |F^{-1}(t) - F^{-1}(1-t)|$, we have regarding (A) that

$$\eta_1 = 2(1-2\beta)^{-1} \int_\beta^{1-\beta} |F^{-1}(t) - F^{-1}(1-t)|\, dt$$
$$\le 4(1-2\beta)^{-1} \int_\beta^{1-\beta} |F^{-1}(t)|\, dt.$$

As to (C), we note that in this example $H_n = F_n$ and

$$|T_t(F_n) - T_t(F) - T_t(F; F_n - F)| = |R_n(t) - R_n(1-t)|,$$

where $R_n(t)$ is the remainder term in the Bahadur representation of the tth sample quantile (see (1.5)). Hence (C) will be satisfied if

$$\sup_{\beta \le t \le 1-\beta} |R_n(t)| = o_p(n^{-1/2}). \tag{5.1}$$

As to (D) we remark that

$$\sup_{\beta \le t \le 1-\beta} [T_t(F_n) - T_t(F)]^2 \le 2 \sup_{\beta \le t \le 1-\beta} [F_n^{-1}(t) - F^{-1}(t)]^2,$$

so that (D) will be satisfied if

$$\sup_{\beta \le t \le 1-\beta} [F_n^{-1}(t) - F^{-1}(t)]^2 = o_p(n^{-1/2}). \tag{5.2}$$

We are now able to prove the following result.

THEOREM 5.1 *Suppose*

(i) $\int_{\beta}^{1-\beta} |F^{-1}(t)|\, dt < \infty$;
(ii) $0 < \sigma^2(T, F) < \infty$, *with* $\sigma^2(T, F)$ *as in Section* 3;
(iii) *For some* $\varepsilon > 0$, *the density* f *of* F *is positive and Lipschitz continuous on* $[F^{-1}(\beta) - \varepsilon, F^{-1}(1-\beta) + \varepsilon]$.

Then

$$n^{1/2}[T_\beta(F_n) - T_\beta(F)] \to_d N(0, \sigma^2(T, F)).$$

Moreover (*by an easy calculation*),

$$\sigma^2(T, F) = \frac{16}{(1-2\beta)^2} \int_\beta^{1-\beta} \int_\beta^{1-\beta} [F^{-1}(t) - F^{-1}(1-t)] \cdot [F^{-1}(s) - F^{-1}(1-s)] \Psi(t, s)\, dt\, ds,$$

where $\Psi(t, s) = [\min(t, s) - ts]/f(F^{-1}(t)) f(F^{-1}(s))$.

PROOF. From (i) and (ii) the asymptotic normality of the leading term $T_\beta(F; F_n - F)$ is immediate, i.e., (3.4) is fulfilled. (See also Remark (ii) following Theorem 3.1.) Moreover, from Lemma 4.2 (b) it follows that (5.2) and hence (D) hold. Since (C) reduces to (5.1), its validity follows from Lemma 4.3. Finally, (B) is immediate and (i) takes care of (A). □

It should be noted that in this example we could alternatively obtain (5.1) and (5.2) by applying results for the usual empirical process, namely results on Bahadur representation and quantile processes (see, e.g., Serfling, 1980, page 101, and Bickel, 1967).

We conjecture that the density assumption in this example can be relaxed.

6. The spread functional $\tau^2(X - X'; \alpha, \beta)$. Here we return to the spread functional given by (2.2) and without loss of generality we assume $\alpha < 1 - \beta$. As in Section 5, condition (B) holds trivially since $q(x) = x^2$; and it is easily seen that a sufficient condition for (A) is given by condition (i) of Theorem 6.1 below. As for (C), we note that in this example

$$T_t(H_n) - T_t(H_F) - T_t(H_F; H_n - H_F) = R_n\left(\frac{t+1}{2}\right),$$

where $R_n(t)$ is the remainder term in the Bahadur representation of the tth quantile of the empirical df H_n of U-statistic structure, so that (C) will be satisfied if

$$\sup_{\alpha \le t \le 1-\beta} \left| R_n\left(\frac{t+1}{2}\right) \right| = \sup_{(1+\alpha)/2 \le t \le 1-\beta/2} |R_n(t)| = o_p(n^{-1/2}).$$

Condition (D) in this example reduces to

$$\sup_{(1+\alpha)/2\le t\le 1-\beta/2}[H_n^{-1}(t) - H_F^{-1}(t)]^2 = o_p(n^{-1/2}).$$

Therefore, by a proof similar to that of Theorem 5.1, we have the following result.

THEOREM 6.1 *Suppose*

(i) $\int_{(1+\alpha)/2}^{1-\beta/2} |H_F^{-1}(t)|\, dt < \infty$;
(ii) $0 < \sigma^2(T, F) < \infty$, *with* $\sigma^2(T, F)$ *as in Section* 3, *and* $E[G^2(X_1, X_2)] < \infty$;
(iii) *For some* $\varepsilon < 0$, *the density* h_F *of* H_F *is positive and Lipschitz continuous on* $[H_F^{-1}((1 + \alpha)/2) - \varepsilon, H_F^{-1}(1 - \beta/2) + \varepsilon]$.

Then

$$n^{1/2}[T(H_n) - T(H_F)] \to_d N(0, \sigma^2(T, F)).$$

Moreover

$$\sigma^2(T, F) = \frac{4}{(1-\beta-\alpha)^2}\int_\alpha^{1-\beta}\int_\alpha^{1-\beta} H_F^{-1}\left(\frac{t+1}{2}\right)H_F^{-1}\left(\frac{s+1}{2}\right)\Psi\left(\frac{t+1}{2}, \frac{s+1}{2}\right) dt\, ds,$$

where

$$\Psi(t, s) = \frac{\sum P\{X_{i_1} - X_{i_2} \le H_F^{-1}(t), X_{j_1} - X_{j_2} \le H_F^{-1}(s)\} - 4ts}{h_F(H_F^{-1}(t))h_F(H_F^{-1}(s))},$$

with $\sum$ *denoting summation over* $1 \le i_1 \ne i_2 \le 2, j_1 = 1, 3, j_2 = 1, 3, j_1 \ne j_2$.

We conjecture that the density assumption in this example can be relaxed.

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