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ON THE SINUSOIDAL LIMIT OF STATIONARY TIME SERIES[1]

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We show that the Sinusoidal Limit Theorem of Slutsky in the Gaussian case is a consequence of the equality of certain higher order crossings.

1. Introduction. The purpose of this note is to elaborate on a limit law of E. Slutsky and tie it to higher order crossings of Gaussian time series. In the Gaussian case, as we shall see, the Sinusoidal Limit Theorem (SLT) of Slutsky is a natural consequence of relations between conspicuous visual features depicted by time series graphs: when the expected number of axis crossings of a Gaussian stationary sequence with mean zero approaches the corresponding expected number of peaks and troughs, the sequence approaches in a certain sense a sinusoid whose period depends linearly on the expected number of axis crossings. Experience shows that this may also hold for nonGaussian processes but the proof of that is still an open problem.

In his quest for an explanation to the periodic nature of time series, Slutsky (1927) discovered the Sinusoidal Limit Theorem. This result, not as well known as other probability limit theorems, is sometimes confused with the so-called "Slutsky effect" while in fact the latter is only a special case of the SLT. Originally the Slutsky effect meant the periodicity "introduced" into the data by sequential summation followed by sequential differencing in the sense that certain frequencies in the data are enhanced as a result of this repeated filtering and become dominant. This effect, albeit a special case of the SLT, may well serve as a motivation to this limit theorem and historically this could very well be the case. In any event, this discovery had a major impact on the understanding of "periodic phenomena" as it pointed out that the source for the apparent cyclical behavior of economic series may be due to linear operations and not to any strictly periodic mechanism. Of course a simpler explanation which did not escape Slutsky is the mere notion of stationarity; after all, a stationary process evolves and fluctuates around a fixed level.

The original work of Slutsky in 1927 had been followed by Romanovsky (1932, 1933) who showed that the SLT holds even when the summation filter is lengthened and also obtained a necessary and sufficient condition for the limit to be a sum of several sinusoids. He later relaxed the condition that the original series, to which the sequential filtering was applied, be completely random. Moran (1949) provided a short proof of the original Slutsky effect assuming the series is completely random and in 1950 he went on to show that repeated linear

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filtering applied to random series may result in a sum of several sinusoids whose number and periods depend on the filter coefficients. This result, closely related to Theorem D of Romanovsky (1932), extends significantly the original notion of the Slutsky effect.

More recently, Kedem and Slud (1982) have dealt with axis-crossings of processes derived by repeated differencing of stationary processes. It will be shown that these axis-crossings, referred to as higher order crossings, give rise to the SLT in the Gaussian case.

The organization of the paper is as follows: first we give a historical account by supplying a modern proof of the Slutsky effect. In our opinion this effect probably led Slutsky to the formulation of his SLT and not vice versa as recorded in his original paper. In Section 3 we introduce the higher order crossings and formulate the SLT in Theorem 3 in terms of these quantities. This is our main result. In Section 4 we show how one can go about applying our main result.

2. The Slutsky effect leads to the SLT. Let $\{Z_t\}_{t=-\infty}^{\infty}$ be a zero mean stationary process with spectral representation

$$Z_t = \int_{-\pi}^{\pi} e^{it\lambda}\, d\xi(\lambda),$$

where for convenience we assume the existence of a spectral density f and where $E\,|\,d\xi(\lambda)\,|^2 = f(\lambda)\,d\lambda$.

Let B be the backwards shift, $(BZ)_t = Z_{t-1}$. As a polynomial in B is well defined, we shall be interested in the operator

$$(1) \qquad (1-B)^m(1+B)^n,$$

applied to $\{Z_t\}$ for $m, n = 0, 1, 2, \cdots$, such that $m/n = c$ constant. Define

$$(2) \qquad Y_t^{(n)} = (1-B)^{cn}(1+B)^n\{Z_t/\sigma_n\}$$

where $\sigma_n > 0$ is defined so that $\mathrm{Var}(Y_t^{(n)}) = 1$. Strictly speaking $\{Y_t^{(n)}\}$ depends on c but this is suppressed for the sake of simplified notation. Let the transfer function of $(1-B)^{cn}(1+B)^n$ be denoted by $H_n(\lambda)$, again suppressing c. Then

$$H_n(\lambda) = (1-e^{-i\lambda})^{cn}(1+e^{-i\lambda})^n,$$

and it follows that the spectral density of $\{Y_t^{(n)}\}$ is given by

$$f_n(\lambda) = \frac{|\,H_n(\lambda)\,|^2 f(\lambda)}{\int_{-\pi}^{\pi} |\,H_n(\omega)\,|^2 f(\omega)\, d\omega}, \quad -\pi \le \lambda \le \pi.$$

We note that the squared gain $|\,H_n(\lambda)\,|^2$ is symmetric and

(i) $|\,H_n(\lambda)\,|^2 = 2^{cn+n}(1-\cos\lambda)^{cn}(1+\cos\lambda)^n$.

(ii) For $0 \le \lambda \le \pi$, the squared gain is unimodal with a peak occurring at λ_c, say. We have

$$\max_{0\le\lambda\le\pi} |\,H_n(\lambda)\,|^2 = |\,H_n(\lambda_c)\,|^2$$

where $\lambda_c = \cos^{-1}((1-c)/(1+c))$.

(iii) For sufficiently small $\varepsilon > 0$

$$\frac{|H_n(\lambda_c - \varepsilon)|^2}{|H_n(\lambda_c - \varepsilon/2)|^2} \to 0, \quad n \to \infty,$$

and the same holds if a plus replaces the minus sign.

(iv) $|H_n(\lambda_c)|^2 < |H_{n+1}(\lambda_c)|^2$.

Associated with f_n is a spectral measure $\nu_n(\cdot)$ defined on $(-\pi, \pi]$ by

$$\nu_n(\Lambda) = \int_\Lambda f_n(\lambda)\, d\lambda, \quad \Lambda \in (-\pi, \pi],$$

Λ being a Borel set.

THEOREM 1. (Slutsky Effect). *The sequence* $\{Y_1^{(n)}, \cdots, Y_N^{(n)}\}$ *defined by* (2) *converges in probability to a sinusoid given by*

(3) $$Y_t = U \cos \lambda_c t + V \sin \lambda_c t, \quad t = 1, \cdots, N.$$

where U, V *are uncorrelated random variables, provided* $f(\lambda_c) > 0$.

PROOF. The proof is fairly standard and we shall give a short account of it. First note that ν_n is symmetric on $(-\pi, \pi]$, and, following the technique of Kedem and Slud (1982), we obtain for $\varepsilon > 0$

$$\nu_n[0, \lambda_c - \varepsilon),\ \nu_n(\lambda_c + \varepsilon, \pi] \to 0, \quad n \to \infty.$$

It follows that

$$\nu_n([-\lambda_c - \varepsilon, -\lambda_c + \varepsilon] \cup [\lambda_c - \varepsilon, \lambda_c + \varepsilon]) \to 1, \quad n \to \infty,$$

and so we have the weak convergence

(4) $$\nu_n \Rightarrow \tfrac{1}{2}\delta_{-\lambda_c} + \tfrac{1}{2}\delta_{\lambda_c}, \quad n \to \infty,$$

where δ_u is the unit point mass at u. If $dF_y^{(n)}$ is the spectral distribution function of $\{Y_t^{(n)}\}$ then (4) means that

$$dF_y^{(n)}(\lambda) \to \begin{cases} 1/2, & \lambda = \pm\lambda_c \\ 0, & \text{otherwise} \end{cases}, \quad n \to \infty.$$

The proof now follows from the spectral representation of $\{Z_t\}$ (see also Grenander and Rosenblatt (1957), Section 3.3). □

It should be observed that (3) can be expressed as $A \cos(\lambda_c t + \theta)$. Thus we see that n summations, $(1 + B)^n$, followed by $cn = m$ differencings, $(1 - B)^{cn}$, produce for large n a sinusoid with period $2\pi/\lambda_c$, $\lambda_c = \cos^{-1}((1 - c)/(1 + c))$, provided $f(\lambda_c) > 0$. Of course our proof was made easy due to the spectral representation which was not available in the twenties. The method used by Slutsky made use of difference equations.

Let us compute the correlation function of $\{Y_t^{(n)}\}$. Since the complex exponential is bounded, we have from Theorem 1:

$$\mathrm{Corr}(Y_t^{(n)}, Y_{t+k}^{(n)}) = \int_{-\pi}^{\pi} e^{ik\lambda}\, \nu_n(d\lambda) \to \frac{1}{2}\,(e^{-ik\lambda_c} + e^{ik\lambda_c}) = \cos(k\lambda_c)$$

as $n \to \infty$, and here k is an integer. The basic observation which originated with Slutsky and was shared by Romanovsky and Moran is that for the limit (3)

$$(5) \quad E(Y_t - 2\cos(\lambda_c)Y_{t-1} + Y_{t-2})^2/\mathrm{Var}(Y_t) = 2(1 - 2\cos^2\lambda_c + \cos 2\lambda_c) = 0,$$

and it follows that the series Y_t satisfies a difference equation whose solution is a sinusoid with frequency λ_c.

Observe that the first order correlation of $Y_t^{(n)}$ converges to $\cos(\lambda_c)$. This and (5) suggest an immediate generalization. Any stationary sequence $\{Y_1, Y_2, \cdots\}$ with correlation function ρ_k for which

$$(6) \qquad E(Y_t - 2\rho_1 Y_{t-1} + Y_{t-2})^2/\mathrm{Var}(Y_t) = 2(1 - 2\rho_1^2 + \rho_2) = 0$$

is, with probability one, a sinusoid whose period depends on ρ_1, provided $\rho_1 < 1$. Slutsky actually used a slightly different condition. Let

$$\eta_1 = \mathrm{Corr}(\nabla^2 Y_t, Y_{t-1}), \quad \nabla = 1 - B.$$

Then,

$$(7) \qquad (1 - \eta_1^2)(6 - 8\rho_1 + 2\rho_2) = 2(1 - 2\rho_1^2 + \rho_2).$$

But if $\rho_1 < 1$ and $\eta_1 = -1$, as required by Slutsky, then $1 - 2\rho_1^2 + \rho_2 = 0$ and the sequence $\{Y_t\}$ satisfies with probability 1 the homogeneous difference equation

$$(8) \qquad Y_t - 2\rho_1 Y_{t-1} + Y_{t-2} = 0$$

whose solution is a sinusoid and this also includes (Slutsky required $|\rho_1| < 1$!) the degenerate case $\rho_1 = -1$ of extreme oscillation. We have given the essentials of the proof of Slutsky's Sinusoidal Limit Theorem which states

THEOREM 2 (SLT). *Let $Z_1, Z_2, \cdots$, be a weakly stationary sequence fulfilling the conditions*

$$EZ_t = 0, \quad EZ_t^2 = \sigma^2 = h(n), \quad \frac{EZ_t Z_{t+k}}{EZ_t^2} = \rho_k = \phi(k, n)$$

where n is a parameter specifying the series as a whole, and $h(n)$ and $\phi(k, n)$ are independent of t. If

$$(9) \qquad |\rho_1| \le \textit{constant} < 1, \quad \textit{as} \quad n \to \infty$$

and the correlation coefficient between $\nabla^2 Z_t$ and Z_{t-1}, η_1, say, is such that

$$(10) \qquad \eta_1 \to -1, \quad \textit{as} \quad n \to \infty,$$

then

(I) $(Z_t, Z_{t+1}, \cdots, Z_{t+s})$ *converges in probability to a certain sinusoid.*

(II) *The period, L, of this sinusoid is determined from the equation* $\rho_1 = \cos(2\pi/L)$.
(III) *The number of periods in the interval* $(t, t + s)$ *can be made arbitrarily large provided s, n are sufficiently large.*

Owing to (6) and (7), Romanovsky (1932) suggested replacing (9), (10) by the more direct condition that as $n \to \infty$,

$$\rho_1 < 1, \quad \text{and} \quad 1 - 2\rho_1^2 + \rho_2 \to 0. \tag{11}$$

This again, as argued above, leads to the conclusion of Theorem 2. Our observation is that condition (11) can be further modified and tied in with higher order crossings to produce the SLT. It should be remarked that in the SLT above it is more appropriate to speak of the convergence of the normalized series Z_t/σ but Slutsky did not explicitly make this assumption and we shall follow this manner.

3. Graphical considerations. Slutsky observed that almost all economic phenomena as well as social, meteorological and others occur in a sequence of rising and falling movements and that a human observer can readily detect long as well as short term fluctuations in these time series, so that the idea of some form of harmonic analysis is inescapable. Undoubtedly this has been the observation of many a scientist who watched the graph of a time series; its rising and falling movements can be quickly detected by a casual eye examination. *The problem we address here is to what extent can such an eye examination determine the deviation of a time series from a sinusoid.* By adopting a somewhat parochial view, it is possible to show that Slutsky's Limit Law is essentially a graphical law.

In this section let $\{Z_t\}$ be as above and also Gaussian. Let $\mathscr{L}$ be the clipping operator defined by

$$\mathscr{L} Y_t \equiv \begin{cases} 1, & Y_t \geq 0 \\ 0, & Y_t < 0 \end{cases}, \quad t = 0, \pm 1, \cdots.$$

Consider a sequence of binary processes $\{X_t^{(k)}\}$ defined by

$$X_t^{(k)} = \mathscr{L}(1 - B)^{k-1} Z_t, \; t = 0, \pm 1, \cdots, \quad k = 1, 2, \cdots.$$

Define

$$D_{k,N} \equiv \textstyle\sum_{t=1}^{N-1} I[X_{t+1}^{(k)} \neq X_t^{(k)}].$$

That is, $D_{k,N}$ counts the number of symbol changes in $X_1^{(k)}, \cdots, X_N^{(k)}$. But this is also the number of axis-crossings by $(1 - B)^{k-1} Z_t$, $t = 1, \cdots, N$. $D_{k,N}$ is referred to as the (number of) higher order crossings of order k. A closer examination of the $D_{k,N}$ reveals that $D_{1,N}$ counts the number of axis-crossings by $\{Z_t, t = 1, \cdots, N\}$, and except for end effects, $D_{2,N}$ is the number of peaks and troughs in this series, $D_{3,N}$ is the number of inflection points in the series, etc. Thus the $D_{k,N}$ pertain to visual features detected in time series graphs. Now these counts exhibit a monotonicity property first proved under suitable conditions in Kedem and

Slud (1982) where it was shown that

$$\limsup_{N\to\infty} N^{-1}D_{k,N} \quad \text{is almost surely increasing to 1 as} \quad k \to \infty.$$

For a Guassian process we can prove most conveniently the related result

LEMMA 1. *If $\{Z_t\}$ $t = 0, \pm 1, \cdots$ is a zero mean stationary Gaussian process with correlation function $\{\rho_k\}$ such that $\rho_1 < 1$ then we have the inequality*

$$ED_{k,N} \geq ED_{k-1,N}, \tag{12}$$

$k = 2, 3, \cdots$, uniformly in $N > k$.

PROOF. To prove (12) we shall see that it is sufficient to consider the case $k = 2$. Let $\rho_\nabla(k)$ be the correlation function of the differenced process $\{\nabla Z\}$. By stationarity

$$1 - 2\rho_1^2 + \rho_2 = \begin{vmatrix} 1 & \rho_1 & \rho_2 \\ \rho_1 & 1 & \rho_1 \\ \rho_2 & \rho_1 & 1 \end{vmatrix} \Big/ (1 - \rho_2) \geq 0$$

and so

$$\rho_1 - \rho_\nabla(1) = \frac{1 - 2\rho_1^2 + \rho_2}{2(1 - \rho_1)} \geq 0$$

or

$$\rho_\nabla(1) \leq \rho_1. \tag{13}$$

But this in conjunction with the Guassian assumption implies (Kedem, 1980)

$$ED_{1,N} = (N-1)\left(\frac{1}{2} - \frac{1}{\pi}\sin^{-1}(\rho_1)\right) \leq (N-1)\left(\frac{1}{2} - \frac{1}{\pi}\sin^{-1}(\rho_\nabla(1))\right) = ED_{2,N}.$$

Since differences of a Guassian process are again Gaussian and the requirement that their first serial correlation be strictly less than unity is automatically satisfied due to (13), we have in general

$$ED_{k,N} \geq ED_{k-1,N}, \quad k = 2, 3, \cdots$$

uniformly in $N > k$. □

It should be noted that in general the inequality (12) is strict. For example for white noise (12) is strict since

$$ED_{k+1,N} = (N-1)\left[\frac{1}{2} - \frac{1}{\pi}\sin^{-1}\left(\frac{-k}{k+1}\right)\right], \quad k \geq 0,$$

and this gives a good approximation to the behavior of $ED_{k,N}$ when k is large for many stationary sequences for which the spectral density is positive at π. Now when equality in (12) holds so does the SLT. It is convenient to normalize the

$ED_{k,N}$ by $N-1$ as $ED_{k,N}/(N-1)$ is independent of N. Unlike Slutsky, in order to prevent any confusion, we shall now state our main result without indexing the process $\{Z_t\}$.

THEOREM 3. *Let $\{Z_t\}$ be a zero mean Gaussian stationary process. Assume*

(a) $$0 < \frac{ED_{1,N}}{N-1}$$

(b) $$\frac{ED_{1,N}}{N-1} = \frac{ED_{2,N}}{N-1}.$$

Then the vector $(Z_t, \cdots, Z_{t+s})$ lies with probability one on a sinusoid with period

$$L = \frac{2(N-1)}{ED_{1,N}}.$$

PROOF. Since the process $\{Z_t\}$ is Gaussian, we have from Kedem (1980)

$$\rho_1 = \cos\left(\frac{\pi ED_{1,N}}{N-1}\right)$$

and

$$\rho_\nabla(1) = \cos\left(\frac{\pi ED_{2,N}}{N-1}\right)$$

where $\rho_\nabla(k)$ is the correlation function of $\{\nabla Z_t\}$. As before

$$\frac{E[Z_t - 2\rho_1 Z_{t-1} + Z_{t-2}]^2}{\operatorname{Var}(Z_t)}$$
$$= 2 - 4\rho_1^2 + 2\rho_2 = 4(1-\rho_1)(\rho_1 - \rho_\nabla(1))$$
$$= 4\left[1 - \cos\left(\frac{\pi ED_{1,N}}{N-1}\right)\right]\left[\cos\left(\frac{\pi ED_{1,N}}{N-1}\right) - \cos\left(\frac{\pi ED_{2,N}}{N-1}\right)\right] = 0,$$

by (a), (b). This implies

$$Z_t - 2\cos\left(\frac{\pi ED_{1,N}}{N-1}\right)Z_{t-1} + Z_{t-2} = 0, \quad \text{w.p. } 1,$$

and therefore $(Z_t, \cdots, Z_{t+s})$ for fixed s follows with probability one a difference equation of the form $Z_t - 2\rho_1 Z_{t-1} + Z_{t-2} = 0$ whose solution is the sinusoid

(14) $$Z_t = A\cos\left(\frac{\pi ED_{1,N}}{N-1}t + \theta\right). \quad \square$$

By invoking the lemma we have:

COROLLARY 1. *Assume* (a) *holds and consider the statement*

(b′) $$\frac{ED_{1,N}}{N-1} = \frac{ED_{k,N}}{N-1}, \quad all \quad k \geq 2$$

then (b) *holds if and only if* (b′) *holds.*

Thus the interpretation of the SLT in the Gaussian case is that of equality of the expected number of axis crossings to the expected number of peaks and troughs or to the expected number of changes in concavity, etc. in a time series of length N.

From Theorem 3 it also follows that

COROLLARY 2. *If for some k, $0 < ED_{k,N}/(N-1)$ and*

(b″) $$\frac{ED_{k,N}}{N-1} = \frac{ED_{k+1,N}}{N-1}$$

then the SLT *holds for the* $(k-1)$th *difference* $\{\nabla^{k-1}Z_t\}$ *and we have with probability one*

(15) $$\nabla^{k-1}Z_t = A_k\cos\left(\frac{\pi ED_{k,N}}{N-1}\,t + \theta_k\right).$$

From (15) we see that if as $k \to \infty$, $ED_{k,N}/(N-1)$ approaches 1, $\nabla^k Z_t$ approaches a degenerate state of extreme oscillation. This limiting state is guaranteed under some conditions by the Higher Order Crossings Theorem proved in Kedem and Slud (1982). Specifically, this theorem in summarized form says that, under the Gaussian assumption

THEOREM 4. (HOCT). *Under the Gaussian assumption, suppose $f(\pi) > 0$. Then*

$$\{X_t^{(k)}\} \Rightarrow \begin{array}{ll} \cdots 010101 \cdots & \textit{with probability } 1/2 \\ \cdots 101010 \cdots & \textit{with probability } 1/2 \end{array}, \quad k \to \infty,$$

and

$$\lim_{k\to\infty}\lim_{N\to\infty} N^{-1}D_{k,N} = 1 \quad \textit{with probability } 1.$$

Observe that the type of convergence spelled by the HOCT is weak convergence and pertains to the sequence $\{X_t^{(k)}\}$ as a whole while the SLT speaks about convergence of finitely long sequences.

COROLLARY 3. *Under the Gaussian assumption, if $f(\pi) > 0$ then the inequality* (12) *is strict.*

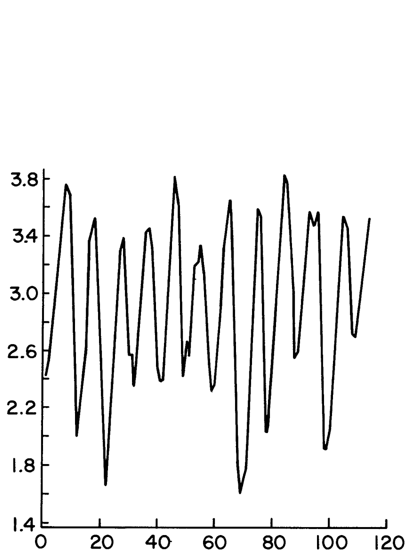

FIG. 1. *Logarithm of the Canadian Lynx series. (Source: Priestley,* 1981).

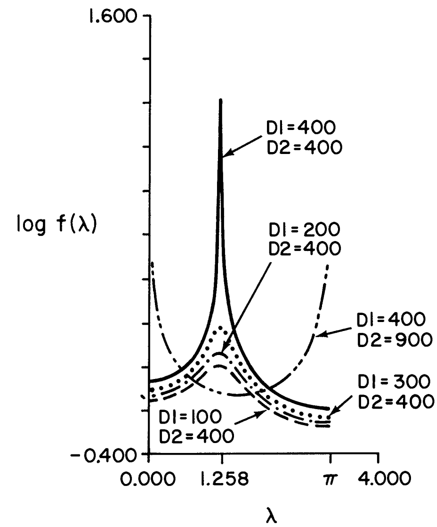

FIG. 2. *Normalized densities of 2nd order autoregressive processes (logarithmic scale) as functions of higher order crossings.*

PROOF. Suppose the conclusion is incorrect. Then by Corollary 2, for some k, $\nabla^{k-1}Z_t$ is a sinusoid whose continued iterations of ∇ will not increase the $D_{k,N}$. But this contradicts the Higher Order Crossings Theorem. □

A consequence of this corollary is that if $f(\lambda)$ is continuous and positive then for sufficiently large N the $D_{k,N}$ display strict monotinicity so that the SLT can hold only in a degenerate sense of extreme oscillation. This has actually been observed in numerous time series. The generalization of Theorem 3 to non-Gaussian sequences is still an open problem as indicated in the introduction. However, for a *bounded* stationary sequence a similar result can be obtained in terms of the crossings of a random curve. Unfortunately the conditions for the SLT to hold in that case are not as compact as in Theorem 3.

4. The Canadian lynx. The practical utility of the SLT as given in Theorem 3 is the indication of rapid oscillation in a time series when $D_{1,N}/(N-1)$ and $D_{2,N}/(N-1)$ are fairly close. *In this case a dominant frequency is present in the data* and it can be quickly estimated by $D_{1,N}$. This amounts to a form of spectral analysis based on higher order crossings, an account of which has been advanced in Kedem (1983). A concrete example is furnished by the celebrated Canadian lynx series of annual trappings of Canadian lynx from 1821 to 1934 ($N = 114$). The graph of the logarithm to base 10 of the series is given in Figure 1.

First it should be observed that the general *form* or graphical appearance of the series is not affected by the logarithmic transformation at least as far as D_1 and D_2 are concerned. Now, taking the mean as the axis, we count 23 crossings and 26 peaks and troughs and it is seen that $(D_{1,114}/113) = 0.203$ and $(D_{2,114}/113) = 0.230$ are fairly close so that the series (original!) under appropriate conditions oscillates roughly as a sinusoid with estimated period

$$2\pi \bigg/ \frac{\pi D_1}{113} = \frac{226}{23} = 9.83 \text{ years}$$

(all other analyses give similar results; see Priestley, 1981, Chapters 5, 6).

5. An illustration. The normalized spectral density of a normal stationary second order autoregressive process is completely determined by $ED_{1,N}$ and $ED_{2,N}$ and this fact provides a mean for illustrating Theorem 3. Figure 2 depicts the graph of the logarithm of this spectral density for various cases of $(ED_{1,N}, ED_{2,N})$. It is seen that the closer these two quantities are the sharper is the peak in the spectrum. In the figure, $D1$ and $D2$ stand for $ED_{1,1000}$, $ED_{2,1000}$ respectively.

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