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SECOND ORDER EFFICIENCY IN THE SEQUENTIAL DESIGN OF EXPERIMENTS[1]

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In the sequential design of experiments an experimenter performs experiments sequentially to make an eventual inference about the true state of nature. A Bayesian formulation of this problem is considered. The parameter space is assumed finite and there are a finite number of repeatable experiments. Sufficient conditions are given for a procedure to be second order efficient as the sampling costs approach zero. The asymptotic analysis of a related Markov control problem is also presented.

1. Introduction. In the sequential design of experiments, an experimenter performs experiments sequentially to make an inference about the state of nature. Various authors have suggested procedures, i.e. rules dictating the experimenter's actions at each stage. Of particular note is procedure δ^A proposed by Chernoff (1959). In his paper, Chernoff proves that δ^A is asymptotically optimal in a large sample limit. To investigate its performance for moderate sample sizes, numerical simulations were performed by Blot and Meeter (1973), and Meeter, Pirie and Blot (1970). In these simulations, δ^A is compared with three other procedures, δ^{BH}, δ^B and δ^M, proposed by Box and Hill (1967), Blot and Meeter (1973), and Chernoff (1972). These three procedures have no compelling theoretical justification and in general are not asymptotically optimal. However, for the problems simulated they often outperformed δ^A. In one case, the average sample size for δ^A was 176, and the average sample size for δ^{BH} was 115, and it certainly appears that asymptotic optimality is not a definitive performance criterion unless the sample sizes are a fair amount larger. As the simulations performed were necessarily limited in scope, there is little reason to advocate use of δ^{BH}, δ^B or δ^M on arbitrary problems.

The goal of this paper is to refine Chernoff's asymptotic analysis of this problem. A Bayesian formulation is used with a general loss structure. The parameter space is assumed finite and there are a finite number of repeatable experiments. All costs of sampling are scaled by a constant α, and the asymptotic limit considered is as $\alpha \to 0$ which corresponds with large samples.

Theorem 2.1 gives an approximation for the (optimal) Bayes risk which has error $o(\alpha\sqrt{-\ln\alpha})$ as $\alpha \to 0$ uniformly over prior probabilities. A procedure is called second order efficient if its risk differs from the Bayes risk by an amount $o(\alpha\sqrt{-\ln\alpha})$ as $\alpha \to 0$ uniformly over prior probabilities. The same theorem gives sufficient conditions for a procedure to be second order efficient. Chernoff's

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procedure is usually not second order efficient, and the difference between its risk and the Bayes risk is $O(\alpha\sqrt{-\ln\alpha})$. Savings in expected sample size from using a second order efficient procedure will be the same magnitude as the square root of the expected sample size. It seems likely that these savings may be quite substantial in problems with moderate sample sizes. In a sequel to this paper, we plan to introduce several procedures suggested by this analysis, and compare those procedures against δ^A, δ^B, δ^{BH}, and δ^M using simulation. In a narrower class of problems, Lalley and Lorden have recent unpublished results describing procedures with risk $O(\alpha)$ greater than the Bayes risk.

The bulk of this paper deals with the asymptotic analysis of a Markov control problem intimately related to the sequential design of experiments. The problem concerns driving a discrete time vector valued process $\{S_n\}$ from an initial position $S_0 = s$ to the first quadrant. There is a finite set $\{P_1 \cdots P_k\}$ of allowed distributions for steps, and a cost c_i is incurred each time distribution P_i is used. The asymptotic limit studied is as the distance from s to the first quadrant approaches infinity. This control problem will be studied in Section 4 and in Section 3 a related nonstochastic optimization problem will be analyzed. In Section 2 the notation for sequential design will be introduced and the main theorem stated. Proofs are given in Section 5.

2. Sequential design: main theorem and notation. The formulation given here is similar to that of other authors although a slightly more general loss structure is used and second moment conditions are imposed.

θ represents the state of nature and assumes values in a finite set $\Theta = \{\theta_0, \cdots, \theta_p\}$. The experimenter chooses experiments from a finite set $\mathscr{E} = \{e_1 \cdots e_k\}$. When the experimenter performs an experiment, he observes a random variable. Conditional on θ and the choice of experiment, this random variable is independent of the past and has a known density $f(\cdot, \theta, e)$. The distributions corresponding to these densities are assumed mutually absolutely continuous. The prior probabilities will be denoted

$$\pi^{(j)} = \pi^{(j)}(0) = P(\theta = \theta_j),$$

and the posterior probabilities are

$$\pi^{(j)}(n) = P(\theta = \theta_j \mid Y_1 \cdots Y_n, E_1 \cdots E_n)$$

where E_i is the ith experiment performed and Y_i is the ith observation. Let

$$S_n^{(j)} = \ln \begin{bmatrix} \pi^{(j)}(n)/\pi^{(0)}(n) \\ \vdots \\ \pi^{(j)}(n)/\pi^{(j-1)}(n) \\ \pi^{(j)}(n)/\pi^{(j+1)}(n) \\ \vdots \\ \pi^{(j)}(n)/\pi^{(p)}(n) \end{bmatrix}.$$

By Bayes theorem,

$$\begin{aligned}\mathscr{L}&(S_{n+1}^{(j)} - S_n^{(j)} \mid \theta = \theta_j, E_{n+1} = e_i)\\ &= \mathscr{L}\left[\left.\begin{bmatrix}\ln(f(Y, \theta_j, e_i)/f(Y, \theta_0, e_i))\\ \vdots \\ \ln(f(Y, \theta_j, e_i)/f(Y, \theta_{j-1}, e_i))\\ \ln(f(Y, \theta_j, e_i)/f(Y, \theta_{j+1}, e_i))\\ \vdots \\ \ln(f(Y, \theta_j, e_i)/f(Y, \theta_p, e_i))\end{bmatrix}\right| \theta = \theta_j, E_{n+1} = e_i\right]\\ &= P_i^{(j)}.\end{aligned}$$

The mean and covariance of $P_i^{(j)}$ will be $\mu_i^{(j)}$ and $\Sigma_i^{(j)}$ respectively. The components of $\mu_i^{(j)}$ are the Kullback-Leibler information numbers for distinguishing θ_j from other states of nature using e_i when θ_j is correct. The cost for performing e_i when $\theta = \theta_j$ will be $\alpha\, c_i^{(j)}$. The limit in our asymptotic expansions will be $\alpha \to 0$ which corresponds with large samples. The set of actions the exprimenter can take after sampling is assumed finite and the loss for any action is nonnegative. Another assumption made is that for each θ_j, there is a unique action which gives 0 loss if $\theta = \theta_j$. The loss using this action when $\theta = \theta_i$ will be denoted $\ell_i^{(j)}$. Let $c_{\max} = \max\{c_i^{(j)}\}$ and $\ell_{\min} = \min\{\ell_i^{(j)}: \ell_i^{(j)} > 0\}$.

COMMENT. Although this structure for the terminal actions is reasonably general, there are interesting situations not covered. One example might be with 3 states of nature and 2 actions with the losses for the actions given by the table,

loss	0	1	2
a_1	0	0	1
a_2	0	1	0

For this loss structure, the asymptotic shape of the stopping region does not allow use of the results for the Markov control problem.

Another assumption made is that $c_i^{(j)} > 0$ and $\mu_i^{(j)} > 0$ for every i, j.

The functions $t_i^{(j)}(s)$ are defined to minimize

$$\textstyle\sum_{i=1}^{k} c_i^{(j)} t_i^{(j)}(s)$$

under the constraints

$$t_i^{(j)}(s) \ge 0 \quad \text{for} \quad 1 \le i \le k,$$

and

$$s + \textstyle\sum_{i=1}^{k} \mu_i^{(j)} t_i^{(j)}(s) \ge 0.$$

Finding these functions for fixed s is a problem in linear programming and for varying s the techniques of parametric linear programming can be applied (see

Vajda, 1961). The last assumption made is that the functions $t_i^{(j)}(\cdot)$ are unique. Section 3 contains further discussion of these functions and Theorem 3.1 gives sufficient conditions for uniqueness.

Let

$$R_d^{(j)}(s) = \sum_{i=1}^k c_i^{(j)} t_i^{(j)}(s), \quad \hat{\Sigma}^{(j)}(s) = \sum_{i=1}^k t_i^{(j)}(s)\Sigma_i^{(j)}$$

and

$$\hat{R}^{(j)}(s) = \int R_d^{(j)}(s + x)N(0, \hat{\Sigma}^{(j)}(s))\{dx\}.$$

A procedure $\mathscr{P}$ will be a rule which tells the experimenter which experiment to perform, when to stop, and which action to take ($\mathscr{P}$ must satisfy measureability conditions insuring that decisions are based on the past only). $R(\pi, \alpha, \mathscr{P})$ will be the risk for procedure $\mathscr{P}$ given the prior distribution and α, and $R^{(j)}(\pi, \alpha, \mathscr{P})$ will be the risk given $\theta = \theta_j$.

The primary concern will be with procedures where the sample size is given by

$$N = \inf\{n \geq 0: S_n^{(j)} \geq \Lambda^{(j)} - \ln\alpha\ \mathbf{1},\ 0 \leq j \leq p\}, \tag{2.1}$$

and the terminal action is the action best for θ_j if $S_N^{(j)} \geq \Lambda^{(j)} - \ln\alpha\ \mathbf{1}$. For convenience all components of the $\Lambda^{(j)}$ vectors will be either 0 or $-\infty$, so that $S_n^{(j)} \geq \Lambda^{(j)} - \ln\alpha\ \mathbf{1}$ is equivalent to

$$\frac{\pi^{(j)}}{\pi^{(i)}} \geq \frac{1}{\alpha} \quad \forall i \quad \text{s.t.} \quad \ell_i^{(j)} > 0.$$

Let

$$a^{(j)} = \{\theta_i: \ell_i^{(j)} > 0\} \quad \text{and} \quad a^{(j)} = \{i: \ell_i^{(j)} > 0\}.$$

Suppose $S_0^{(j)} \geq \Lambda^{(j)} - \ln\alpha\ \mathbf{1}$. Then $P(\theta \in a^{(j)}) \leq p\alpha$. Conditioning on $S_N^{(j)}$, if $\mathscr{P}$ satisfies (2.1) the risk due to the terminal action is less than $p\ell_{\max}\alpha$ for any prior and any α.

For a vector x let x^- be defined by $x_i^- = \max\{0, -x_i\}$ and let

$$\tau(x) = \sup_i x_i^-. \tag{2.2}$$

Let

$$R(\pi, \alpha) = \inf_{\mathscr{P}} R(\pi, \alpha, \mathscr{P}) \quad \text{and}$$
$$\hat{R}(\pi, \alpha) = \alpha \sum_{i=1}^p \pi^{(i)}\hat{R}^{(i)}(S_0^{(i)} + \ln\alpha\ \mathbf{1} - \Lambda^{(i)}).$$

THEOREM 2.1.

$$R(\pi, \alpha) = \hat{R}(\pi, \alpha) + o(\alpha\sqrt{-\ln\alpha})$$

uniformly in π *as* $\alpha \to 0$. *If* h_1 *and* h_2 *are functions satisfying*

$$\lim_{\tau\to\infty} \frac{h_1(\tau)}{\sqrt{\tau \ln \tau}} = \infty \tag{2.3}$$

and

$$\text{(2.4)} \qquad \lim_{\alpha \to 0} \frac{h_2(\alpha)}{\sqrt{-\ln\alpha}} = 0,$$

and if $\mathscr{P}$ *is a procedure that stops according to* (2.1) *and satisfies*

$$\text{(2.5)} \qquad \begin{aligned} &P(S_n^{(j)} \geq h_2(\alpha),\ t_i^{(j)}(S_n^{(j)} + \ln\alpha\ \mathbf{1} - \Lambda^{(j)}) \\ &\qquad \leq h_1(\tau(S_n^{(j)} + \ln\alpha\ \mathbf{1} - \Lambda^{(j)}),\ E_{n+1} = e_i)) = 0 \end{aligned}$$

for $0 \leq j \leq p$, $1 \leq i, \leq k$, $n \geq 0$, $\alpha > 0$, *then*

$$R(\pi, \alpha, \mathscr{P}) = \hat{R}(\pi, \alpha) + o(\alpha\sqrt{-\ln\alpha})$$

uniformly in π *as* $\alpha \to 0$.

Roughly speaking, $t_i^{(j)}(S_n^{(j)} + \ln\alpha\ \mathbf{1} - \Lambda^{(j)})$ is the number of times e_i should be performed if $\theta = \theta^{(j)}$ and if the likelihood ratios evolve as dictated by the Kullback Leibler information numbers. Condition (2.5) says that if $t_i^{(j)}$ is small and $\theta^{(j)}$ is very likely, the next experiment should not be e_i. Procedures which satisfy this condition are easy to find. Two simple examples are the procedures which select at each stage the experiment e_i which maximizes $\sum_j \pi_j(n)\ t_i^{(j)}(S_n^{(j)} + \ln\alpha\ \mathbf{1} - \Lambda^{(j)})$ or $t_i^{(\hat{J})}(S_n^{(\hat{J})} + \ln\alpha\ \mathbf{1} - \Lambda^{(\hat{J})})$ where $\hat{J}$ is the index of the most probable state of nature.

3. Nonstochastic control. Asymptotics for the Markov control problem studied later will be related to the nonstochastic problem of minimizing the cost for "driving" a continuous time vector valued process $\{S_t\}$ from its initial position $S_0 = s$ into the 1st quadrant, $\{x \in \mathbb{R}^p\colon x \geq 0\}$. There is a finite sequence $(\mu_1, \cdots, \mu_k)$ of allowed velocities for the process, and for $i = 1, \cdots, k$ we are allowed to choose a time t_i that the process spends traveling at velocity μ_i. The cost per unit time for traveling at velocity μ_i is c_i, so that the total cost for a given choice of $t_1, \cdots, t_k$ is $\sum_{i=1}^k t_i c_i$.

This control problem is easily solved by linear programming techniques. Define $t_i(s)$ as the *optimal* length of time spent at velocity μ_i when $S_0 = s$. Then for fixed s, $t_1(s), \cdots, t_k(s)$ are constants which minimize

$$\sum_{i=1}^k c_i t_i(s)$$

under the constraints

$$t_i(s) \geq 0, \quad i = 1 \cdots k,$$

$$s + \sum_{i=1}^k t_i(s)\mu_i \geq 0.$$

To insure that this minimization problem has a solution, it is assumed that

$$\text{(3.1)} \qquad c_i > 0, \quad i = 1 \cdots k,$$

and

$$\text{(3.2)} \qquad \mu_i > 0, \quad i = 1 \cdots k.$$

The function $R_d(s)$ will be the minimum cost,

$$R_d(s) = \sum_i c_i t_i(s). \tag{3.3}$$

The sequence of vectors $(\mu_1/c_1, \cdots, \mu_k/c_k)$ will be called *regular* if the minimization problem above uniquely determines $t_i(s) \forall s, \forall i$. Theorem 3.1 gives sufficient conditions for regularity.

Considering the dual linear programming problem, define $\lambda(s)$ for $s \in \mathbb{R}^p$ as a vector which minimizes

$$\lambda(s) \cdot s$$

under the constraints

$$-\lambda(s) \cdot \mu_i \leq c_i, \quad i = 1 \cdots k, \tag{3.4}$$

$$\lambda(s) \leq 0. \tag{3.5}$$

From the duality theorem of linear programming,

$$R_d(s) = \lambda(s) \cdot s.$$

LEMMA 3.1. *If* (3.1) *and* (3.2) *hold, then*

(i) *R_d is convex, piecewise linear, and homogeneous of degree* 1.
(ii) *At points where R_d is differentiable,*

$$\lambda(s) = \nabla R_d(s). \tag{3.6}$$

(iii) *$\lambda(\cdot)$ can be chosen to be a homogeneous function of degree* 0.
(iv) *If $(\mu_1/c_1, \cdots, \mu_k/c_k)$ is regular then $t_i(\cdot)$ is a homogeneous function of degree* 1.

PROOF. The proof of this lemma is elementary and will be omitted.

Checking the regularity of $(\mu_1/c_1, \cdots, \mu_k/c_k)$ is a hard task which must be accomplished to apply several later theorems. The next result gives sufficient conditions for regularity. Verification of these conditions involves computing $\sum_{j\geq 1}\binom{k}{j+1}\binom{p}{j}$ determinants. This will only be practical if k or p are small, so further results would be useful. The number of determinants evaluated can sometimes be reduced by throwing away all vectors μ_i strictly less than a convex combination of other μ's. In assessing the restrictiveness of assumptions of regularity, it is worth noting that the conditions fail on a set with Lebesgue measure zero in $\mathbb{R}^{kp}$. The result is stated, with $c_1 = c_2 = \cdots = 1$.

THEOREM 3.1. *$(\mu_1 \cdots \mu_k)$ is regular provided for any $1 \leq j \leq p \wedge (k-1)$, for any $(j+1) \times p$ matrix A whose columns are μ_i's for distinct i's, and for any $j \times p$ matrix B whose rows are distinct rows of the identity matrix,*

$$\det[B(A - \bar{A})(A - \bar{A})' B'] \neq 0$$

where each column of $\bar{A}$ is the average of the columns of A.

PROOF. Language used in this proof is explained in most texts on linear programming such as Vajda (1961). Suppose uniqueness fails when $s = s_0$. Then the simplex method will find a basic feasible solution $\hat{t}_1 \cdots \hat{t}_k$ minimizing $\sum_{i=1}^k t_i$, but some adjacent basic feasible solution $\tilde{t}_1 \cdots \tilde{t}_k$ will also attain the minimum. If j is the number of $\hat{t}$ which are basic variables, then $p - j$ of the corresponding slack (or additional) variables are basic and j components of $s_0 + \sum_{i=1}^k \hat{t}_i \mu_i$ are zero. Suppose when $\hat{t}$ becomes $\tilde{t}$ a t variable is introduced to the basic variable list. Then $\sum_{i=1}^k (\hat{t}_i - \tilde{t}_i)\mu_i$ is a contrast (since $\sum_{i=1}^k \hat{t}_i = \sum_{i=1}^k \tilde{t}_i$) of $(j+1)$ μ's and has j zero components. If B is the matrix whose rows correspond with these zero components and A the matrix whose columns are the $(j+1)$ μ's, then $B(A - \bar{A})$ has rank at most $j - 1$ and $\det[B(A - \bar{A})(A - \bar{A})'B'] = 0$. If instead a slack variable is introduced to the basic variable list when $\hat{t}$ becomes $\tilde{t}$, $\sum_{i=1}^k (\hat{t}_i - \tilde{t}_i)\mu_i$ is a contrast of j μ's with $j - 1$ zero components and again one determinant vanishes.

4. Stochastic control. This section considers control of a discrete time vector valued stochastic process $\{S_n\}_{n=0}^{\infty}$ from its initial position $S_0 = s$ into the 1st quadrant. Define steps for the process as

$$Z_i = S_i - S_{i-1}, \quad i = 1 \cdots.$$

There is a finite set $\{P_1 \cdots P_k\}$ of allowed distributions for the steps, and for each n, one may choose a distribution for the next step using the history of the process till time n.

To be more specific, define independent random vectors

$$Z(i, j), \quad i = 1 \cdots k, \quad j = 1, \cdots$$

with marginal distributions

$$\mathscr{L}(Z(i, j)) = P_i, \quad i = 1 \cdots k, \quad j = 1, \cdots.$$

The control of the process is through a stochastic process $\{\nu_n\}_{n=1}^{\infty} \in \{1, \cdots, k\}^{\infty}$. The sequence $\{S_n\}$ given by this control is defined by

$$S_n = s + \sum_{i=1}^k \sum_{j=1}^{N_i(n)} Z(i, j)$$

where

$$N_i(n) = \#\{k \le n \colon \nu_k = i\}.$$

The only restriction imposed on our control process is that for each n, ν_n is measurable with respect to $\mathscr{F}_{n-1} = \sigma(S_1, \cdots, S_{n-1}) \times \mathscr{F}_0$, where $\mathscr{F}_0$ is independent of all the $Z(i, j)$.

With these definitions, ν_n is the index of the distribution used for step Z_n and $N_i(n)$ is the number of times P_i is used in the first n steps.

Define the stopping time

$$N = \inf\{i \colon S_i \ge 0\}$$

and let $N_i = N_i(N)$ for $i = 1 \cdots k$.

Our objective in this problem is to choose a control process $\{\nu_n\}$ to minimize

$$E \sum_{i=1}^{k} c_i N_i,$$

i.e. there is a cost c_i for each step with distribution P_i.

We assume that for each i, P_i has a finite covariance $\boldsymbol{\Sigma}_i$ and mean

$$\mu_i > 0,$$

and $c_i > 0$.

We will define a procedure $\mathscr{P}$ as a function which associates with each (starting position) s, a control process $\mathscr{P}(s)$ satisfying the measurability condition above. All the random variables used, except the $Z(i, j)$ will be thought of as functions of s and $\mathscr{P}$, but the dependence on s and $\mathscr{P}$ will be suppressed for notational convenience. With this convention, define the risk for a procedure as $R(s, \mathscr{P}) = E \sum_{i=1}^{k} c_i N_i$.

The greatest lower bound for these risk functions will be defined as

$$R(s) = \inf_{\mathscr{P}} R(s, \mathscr{P}).$$

Although we give no proof, it seems likely that an optimal procedure $\mathscr{P}_0$ exists which attains this lower bound, i.e. $R(s) = R(s, \mathscr{P}_0)$. In some problems it may be possible to find $\mathscr{P}_0$ using backwards induction.

As backwards induction is impractical in many situations our goal will be to find approximations for $R(s)$ and procedures which are almost optimal. The asymptotic limit we use is as $\tau(s) \to \infty$ (see equation (2.2)). For notational convenience we will replace statements like $f(s) = O(g(\tau(s)))$ uniformly in s as $\tau(s) \to \infty$; with $f(s) = O(g(\tau))$ as $\tau \to \infty$. Theorem 4.1 shows that R_d is a lower bound for $R(s)$ and Theorem 4.2 shows that to first order this bound can be attained as $\tau \to \infty$. In Theorem 4.3 we show that a procedure is first order efficient if and only if $E \mid (N_i - t_i)/\tau \mid \to 0$. The main result is Theorem 4.4 which gives a second order approximation for $R(s)$ and gives conditions sufficient for a procedure to be second order efficient.

The following lemma is needed in later arguments.

LEMMA 4.1. *Define*

$$M = \#\{n \geq 0 \colon S_n \not\geq s + \mathbf{1}\}.$$

There exists a constant A such that for any control and for any s,

$$EM \leq A.$$

PROOF. We will assume that S_n is 1-dimensional. The result in higher dimensions is an easy consequence of the result in one dimension.

For $i = 1 \cdots k$, and $t \in \mathbb{R}$, let

$$M_i = \inf\{\textstyle\sum_{j=1}^{n} Z(i, j) \colon n = 1, 2, \cdots\} \cup \{0\},$$

$$V_i(x) = \#\{n \colon \textstyle\sum_{j=1}^{n} Z(i, j) < x\},$$

and

$$U_i(x) = EV_i(x).$$

For any s and any control, the number of n such that $S_n < s + 1$ and $\nu_n = 1$ is at most

$$V_1(1 + |M_2| + \cdots + |M_k|).$$

Consequently, $M \le \sum_{i=1}^k V_i(1 + \sum_{j=1,j\neq i}^k |M_j|)$.
Therefore

$$EM \le \sum_{i=1}^k EU_i(1 + \sum_{j=1,j\neq 1}^k |M_j|)$$

for any control and any s. The right-hand side of this expression is finite because U_i grows linearly by Theorems 1 and 2 of Stone (1965) and $E|M_j| < \infty$ by Theorem 11 of Kingman (1962).

COROLLARY 4.1. *Suppose $a \in (-1, 0]$ and f is a bounded function. If $f(s) = O(\tau^a)$ as $\tau \to \infty$ then*

$$E \sum_{i=1}^N f(S_i) = O(\tau^{1+a}) \quad \text{as} \quad \tau \to \infty,$$

and if $f = o(\tau^a)$, as $\tau \to \infty$

$$E \sum_{i=1}^N f(S_i) = o(\tau^{1+a}), \quad \text{as} \quad \tau \to \infty.$$

Also

$$E\,\#\{n: S_n \not\geq s + x\,\mathbf{1}\} = O(x)$$

uniformly in s as $x \to \infty$.

PROOF. The last expression follows immediately from Lemma 4.1. The other two expressions follow from the upper bound

$$\sum_{i=1}^N |f(S_i)| \le |f(S_N)| + \#\{n \ge 1: \tau(S_n) > \tau(s) - 1\}\sup_{\tau(x)>\tau(s)-1}|f(x)|$$
$$+ \sum_{j=1}^{[\tau(s)]} \#\{n \ge 1: \tau(S_n) \in (j-1, j]\} \sup_{\tau(x)\in(j-1,j]}|f(x)|$$

since expectations of the set cardinalities are bounded above by $1 + A$ where A is the constant in Lemma 4.1.

THEOREM 4.1. $R(s) \ge R_d(s)\ \forall s \in \mathbb{R}^n$.

PROOF. Using equation (2.4) we see that for any prodecure, $\{\sum_{i=1}^k c_iN_i(n) + \lambda(s) \cdot S_n, \mathscr{F}_n\}_{n=0}^\infty$ is a submartingale. By Corollary 4.1, EN is finite and the optional stopping theorem gives

$$\lambda(s) \cdot s \le E \sum_{i=1}^k c_iN_i + \lambda(s) \cdot ES_N \le E \sum_{i=1}^k c_iN_i.$$

The left- and right-hand sides of this inequality are $R_d(s)$ and $R(s, \mathscr{P})$ respectively proving the theorem.

THEOREM 4.2. *$R(s) = R_d(s) + O(\sqrt{\tau})$ as $\tau \to \infty$.*

PROOF. Let $[x]$ be the greatest integer $\leq x$. Define a procedure $\mathscr{P}$ in the following manner: Begin by taking $[t_1(s)] + 1$ steps with distribution P_1. Next take $[t_2(s)] + 1$ steps with distribution P_2, etc. After $M = k + \sum_{i=1}^k [t_i(s)]$ steps have been taken, continue using steps with distribution P_1. Conditioning on the value of S_M and using Corollary 4.1 we get

$$R(S, \mathscr{P}) \leq \sum_{i=1}^k c_i([t_i(s)] + 1) + c_{\max} EA(1 + \tau(S_M))$$
$$\leq \sum_{i=1}^k c_i t_i(s) + c_{\max}(A + k) + c_{\max} A \sqrt{\tau(s)} E\tau(S_M/\sqrt{\tau(s)})$$

where $c_{\max} = \max\{c_1, \cdots, c_k\}$. As $\tau \to \infty$, the covariance of $S_M/\sqrt{\tau(s)}$ is uniformly bounded. Since $ES_M \geq 0$ this implies that $E\tau(S_M/\sqrt{\tau(s)})$ is uniformly bounded at $\tau \to \infty$. Consequently

$$R(s, \mathscr{P}) \leq R_d(s) + O(\sqrt{\tau})$$

as $\tau \to \infty$, proving the theorem.

A procedure $\mathscr{P}$ will be called first order efficient if

$$\frac{R(s, \mathscr{P})}{R_d(s)} \to 1$$

as $\tau \to \infty$.

THEOREM 4.3. *Suppose $(\mu_1/c, \cdots, \mu_k/c)$ is regular. Then $\mathscr{P}$ is first order efficient if and only if*

$$E\left|\frac{N_i - t_i(s)}{\tau}\right| \to 0$$

for every i as $\tau \to \infty$.

PROOF. Let $\{s_m\}$ be a sequence for which the conclusion fails. For the sequence $\{s_m\}$, the distributions of the vectors $(N_1/\tau, \cdots, N_k/\tau)$ are uniformly tight by Corollary 4.1. Taking subsequences, we can assume $t_i/\tau \to \hat{t}_i$, $-s_m/\tau \to \hat{s}$ and $(N_1/\tau, \cdots, N_k/\tau) \to_L (\hat{N}_1, \cdots, \hat{N}_k)$. By the SLLN for the $Z(i, j)$, $\sum_{i=1}^k \sum_{i=1}^{N_i} Z(i, j)/\tau - \sum_{i=1}^k N_i\mu_i/\tau \to 0$ in probability. Since

$$-s_m + \sum_{i=1}^k \sum_{i=1}^{N_i} Z(i, j) \geq 0,$$

it follows that $\hat{s} + \sum_{i=1}^k \hat{N}_i\mu_i \geq 0$ a.s. Regularity of the set $\{\mu_1/c_1, \cdots \mu_k/c_k\}$ implies $E \sum_{i=1}^k c_i\hat{N}_i \geq R_d(\hat{s})$ with equality only if $\hat{N}_i = \hat{t}_i$ a.s. Since $\mathscr{P}$ is first order efficient this proves convergence in probability. Taking a further subsequence, we can assume

$$E\left|\frac{N_i - t_i}{\tau}\right| \to \delta_i$$

where the δ_i are nonnegative, possibly infinite and at least one $\delta_i > 0$. Then

$$\liminf (1/\tau) E \sum_{i=1}^{k} c_i N_i > E \sum_{i=1}^{k} c_i \hat{N}_i$$

and the theorem follows.

The second order approximation for $R(s)$ is

$$\hat{R}(s) = \int R_d(s + x) N(0, \hat{\Sigma}(s))\{dx\}$$

where

$$\hat{\Sigma}(s) = \sum_{i=1}^{k} t_i(s) \Sigma_i.$$

THEOREM 4.4. *If* $(\mu_1/c_1, \cdots, \mu_k/c_k)$ *is regular then*

$$R(s) = \hat{R}(s) + o(\sqrt{\tau})$$

as $\tau \to \infty$. *Let* h_2 *be an arbitrary positive function and let* h_1 *be a function satisfying*

$$\lim_{\tau \to \infty} \frac{h_1(\tau)}{\sqrt{\tau} \ln \tau} = \infty.$$

There is a constant A such that

$$R(s, \mathscr{P}) \leq \hat{R}(s) + A h_2(\tau(s)) + o(\sqrt{\tau}) \tag{4.5}$$

as $\tau \to \infty$, *whenever* $\mathscr{P}$ *is a procedure such that*

$$P(\nu_n = i, t_i(S_{n-1}) \leq h_1(\tau(S_{n-1})), \tau(s - S_{n-1}) \geq h_2(\tau(s))) = 0 \tag{4.6}$$

for all $s \in \mathbb{R}^p$, $n \geq 1$, $1 \leq i \leq k$.

PROOF. For any procedure,

$$S_N = s + \sum_{i=1}^{k} \sum_{j=1}^{N_i} (Z(i, j) - \mu_i) + \sum_{i=1}^{k} N_i \mu_i \geq 0, \quad \text{a.s.}$$

Hence

$$R(s, \mathscr{P}) \geq ER_d(s + \sum_{i=1}^{k} \sum_{j=1}^{N_i} (Z(i, j) - \mu_i))$$
$$= \sqrt{\tau}\, ER_d\left(\frac{s}{\sqrt{\tau}} + \sum_{i=1}^{k} \sum_{j=1}^{N_i} \frac{Z(i, j) - \mu_i}{\sqrt{\tau}}\right).$$

The first step in the proof will be to show that when $\mathscr{P}$ is first order efficient this last expression equals $\hat{R}(s) + o(\sqrt{\tau})$ as $\tau \to \infty$. If not, there is a sequence $\{s_m\}$ with $\tau_m = \tau(s_m) \to \infty$ such that

$$\lim_{m \to \infty} ER_d\left(\frac{s_m}{\sqrt{\tau_m}} + \sum_{i=1}^{k} Y_{im}\right) - \frac{\hat{R}(s_m)}{\sqrt{\tau_m}} \neq 0$$

where $Y_{im} = \sum_{j=1}^{N_i} ((Z(i, j) - \mu_i)/\sqrt{\tau_m})$. Taking a subsequence, we can assume that

$$\lim_{m \to \infty} \frac{t_i(s_m)}{\tau_m} = t_i^0,$$

and using Theorem 4.3 this implies

$$\lim_{m\to\infty} E \left| \frac{N_i}{\tau_m} - t_i^0 \right| = 0.$$

As in the proof of Anscombe's Theorem one can show that for each i,

$$Y_{im} - \sum_{j=1}^{[t_i^0 \tau_m]} \frac{Z(i, j) - \mu_i}{\sqrt{\tau_m}} \to 0$$

in probability. Consequently

$$\mathscr{L}(\sum_{i=1}^k Y_{im}) \to N(0, \sum_{i=1}^k t_i^0 \Sigma_i).$$

Using Theorem 2 of Chow et al. (1979) the families $\{Y_{im}, m \geq 1\}$ are uniformly integrable implying that $\{\sum_{i=1}^k Y_{im}, m \geq 1\}$ is uniformly integrable. Since the function R_d satisfies the condition

$$| R_d(x) - R_d(y) | \leq \Lambda \| x - y \|$$

for some constant Λ, the uniform integrability and convergence in law above imply that

$$\lim_{m\to\infty} E R_d\left(\frac{s_m}{\sqrt{\tau_m}} + \sum_{i=1}^k Y_{im}\right) - \int R_d\left(\frac{s_m}{\sqrt{\tau_m}} + x\right) N(0, \sum_{i=1}^k t_i^0 \Sigma_i)\{dx\} = 0$$

which proves that $R(s, \mathscr{P}) \geq \hat{R}(s) + o(\sqrt{\tau})$ as $\tau \to \infty$ since

$$\lim_{m\to\infty} \int R_d\left(\frac{s_m}{\sqrt{\tau_m}} + x\right) N(0, \sum_{i=1}^k t_i^0 \Sigma_i)\{dx\} - \frac{\hat{R}(s_m)}{\sqrt{\tau_m}} = 0.$$

The proof will be completed by establishing inequality (4.5). The basic idea behind the proof is to write $R(s, \mathscr{P}) - \hat{R}(s) = E \sum_{i=1}^N (c_{\nu_i} + \hat{R}(S_i) - \hat{R}(S_{i-1}))$ and use a Taylor series argument to show the terms in the sum are small. Unfortunately $\hat{R}$ is not smooth enough to make this work so it will be replaced with a smoother function $\tilde{R}_\epsilon$. The next three lemmas deal with differentiating quantities like $\tilde{R}_\epsilon$, and Lemmas 4.5 and 4.6 give asymptotic relations used to show the terms in the sum above are small with $\tilde{R}_\epsilon$ replacing $\hat{R}$.

LEMMA 4.2. *Suppose f: $\mathbb{R}^p \to \mathbb{R}$ grows algebraically and that μ and Ω are differentiable and that Ω is positive definite. Then*

$$\begin{aligned}\Delta_i \int f(x) N(\mu(s), \Omega(s))\{dx\} &= \int [-\tfrac{1}{2} \operatorname{tr}(\Omega^{-1}(s)(\Delta_i \Omega(s))) \\ &\quad + \tfrac{1}{2}(x - \mu(s))'\Omega^{-2}(s)(\Delta_i \Omega(s))(x - \mu(s)) \\ &\quad + (x - \mu(s))'\Omega^{-1}(s)\Delta_i \mu(s)] f(x) N(\mu(s), \Omega(s))\{dx\}.\end{aligned}$$

PROOF. The density for $N(\mu(s), \Omega(s))$ is

$$(1/(2\pi)^{p/2}\,|\,\Omega(s)\,|^{1/2})\exp\{-\tfrac{1}{2}(x-\mu(s))'\Omega^{-1}(s)(x-\mu(s))\}.$$

Now

$$\Delta_i|\,\Omega(s)\,| = |\,\Omega(s)\,|\ \mathrm{tr}(\Omega^{-1}(s)\Delta_i\Omega(s)).$$

Using this, Δ_i of the density equals the density times

$$-\tfrac{1}{2}\ \mathrm{tr}(\Omega^{-1}(s)\Delta_i\Omega(s)) + \tfrac{1}{2}\ (x-\mu(s))'\Omega^{-2}(s)\Delta_i\Omega(s)(x-\mu(s)) + (x-\mu(s))'\Omega^{-1}(s)\Delta_i\mu(s).$$

Since the difference quotients of the density are bounded in magnitude by an algebraic function times a normal density, the dominated convergence theorem allows an interchange of differentiation and integration, proving the lemma.

LEMMA 4.3. *Suppose that f is continuous and grows algebraically, that Δf exists* a.e., *that difference quotients of f are bounded by an algebraic function, and that Σ is positive definite. Then*

$$\int f(x)\Sigma^{-1}(x-s)N(s,\Sigma)\{dx\} = \int \Delta f(x)N(s,\Sigma)\{dx\}.$$

PROOF. Using Lemma 4.2,

$$\begin{aligned}\int f(x)\Sigma^{-1}(x-s)N(s,\Sigma)\{dx\} &= \Delta\int f(x)N(s,\Sigma)\{dx\} = \Delta\int f(x+s)N(0,\Sigma)\{dx\}\\ &= \int \Delta f(x+s)N(0,\Sigma)\{dx\} = \int \Delta f(x)N(s,\Sigma)\{dx\},\end{aligned}$$

where the interchange of integration and differentiation is permissible by the dominated convergence theorem.

LEMMA 4.4 *Suppose that f and Ω satisfy all the conditions in Lemmas* 4.2 *and* 4.3. *Then*

$$\Delta_i\int f(x)N(s,\Omega(s))\{dx\} = \int\left[\frac{1}{2}(x-s)'\Omega^{-1}(s)(\Delta_i\Omega(s))\Delta f(x) + \Delta_i f(x)\right]N(s,\Omega(s))\{dx\}.$$

PROOF. Using Lemma 4.3,

$$\begin{aligned}&\int f(x)[\Omega^{-1}(s)(\Delta_i\Omega(s))(x-s)]_j[\Omega^{-1}(s)(x-s)]_jN(s,\Omega(s))\{dx\}\\ &= \int \{(\Delta_j f(x))[\Omega^{-1}(s)(\Delta_i\Omega(s))(x-s)]_j + f(x)[\Omega^{-1}(s)(\Delta_i\Omega(s))]_{jj}\}N(s,\Omega(s))\{dx\}.\end{aligned}$$

Summing over $1 \le j \le p$,

$$\int f(x)(x-s)'\Omega^{-2}(s)(\Delta_i\Omega(s))(x-s)N(s,\Omega(s))\{dx\}$$
$$= \int \{(x-s)'\Omega^{-1}(s)(\Delta_i\Omega(s))\Delta f(x) + f(s)\operatorname{tr}[\Omega^{-1}(s)(\Delta_i\Omega(s))]\}N(s,\Omega(s))\{dx\}.$$

The result now follows from Lemmas 4.2 and 4.3.

Let $g(\cdot) > 1$ be a function satisfying

$$g(s) \sim \tau$$

and

$$\Delta^j g(s) = O(\tau^{1-j}), \quad j = 1, 2, 3$$

as $\tau \to \infty$. Define for $\varepsilon > 0$,

$$\Sigma_\varepsilon(s) = \varepsilon g(s)I + \int \hat{\Sigma}(s)N(s, g(s)I)\{dx\} \tag{4.7}$$

and

$$\tilde{R}_\varepsilon(s) = \int R_d(x)N(s, \Sigma_\varepsilon(s))\{dx\}.$$

$\Sigma_\varepsilon(\cdot)$ and $\tilde{R}_\varepsilon(\cdot)$ are smoothed versions of $\hat{\Sigma}(\cdot)$ and $\hat{R}(\cdot)$. Let

$$A_i(s, x) = \frac{-p\Delta_i g(s)}{2g(s)} + \frac{(x-s)^2\Delta_i g(s)}{2g^2(s)} + \frac{(x-s)_i}{g(s)}.$$

Using Lemma 4.2,

$$\Delta_i\Sigma_\varepsilon(s) = \varepsilon\Delta_i g(s)I + \int A_i(s, x)\Sigma(x)N(s, g(s)I)\{dx\},$$

$$\Delta_i\Delta_j\Sigma_\varepsilon(s) = \varepsilon\Delta_i\Delta_j g(s)I + \int \left(A_i(s, x)A_j(s, x) + \frac{\partial}{\partial s_j}A_i(s, x)\right) \cdot \hat{\Sigma}(x)N(s, g(s)I)\{dx\}, \tag{4.8}$$

and

$$\Delta_i\Delta_j\Delta_q\Sigma_\varepsilon(s) = \varepsilon\Delta_i\Delta_j\Delta_q g(s)I + \int \left\{A_i(s, x)A_j(s, x)A_q(s, x) + A_q(s, x)\frac{\partial}{\partial s_j}A_i(s, x) + \frac{\partial}{\partial s_q}\left[A_i(s, x)A_j(s, x) + \frac{\partial}{\partial s_j}A_i(s, x)\right]\right\} \hat{\Sigma}(x)N(s, g(s)I)\{dx\}.$$

These equations also hold when $\hat{\Sigma}(\cdot)$ is a constant function. In this case, $\Delta_i\Sigma_\varepsilon(s) = \varepsilon\Delta_i g(s)I$, $\Delta_i\Delta_j\Sigma_\varepsilon(s) = \varepsilon\Delta_i\Delta_j g(s)I$, and $\Delta_i\Delta_j\Delta_q\Sigma(s) = \varepsilon\Delta_i\Delta_j\Delta_q g(s)I$. Consequently the three integrals in the equations above are identically zero if $\hat{\Sigma}(x)$ is replaced by $\hat{\Sigma}(s)$, and the equations are correct if $\hat{\Sigma}(x)$ is replaced by $(\hat{\Sigma}(x) - \hat{\Sigma}(s))$. Using this substitution, it is not hard to show that for any ε,

$$\Delta_i \Delta_j \hat{\Sigma}_\varepsilon(s) = O(1/\sqrt{\tau})$$

and

$$\Delta_i \Delta_j \Delta_q \hat{\Sigma}_\varepsilon(s) = O(1/\tau)$$

as $\tau \to \infty$ (these expansions are not necessarily uniform in ε as $\hat{\Sigma}$ need not be invertable). Using Lemma 4.3 and equation (4.8),

$$\begin{aligned}\Delta_i \hat{\Sigma}_\varepsilon(s) &= \varepsilon \Delta_i g(s) I + \int \left\{ \left(\frac{-p}{2g(s)} + \frac{(x-s)^2}{2g^2(s)} \right) \right. \\ &\qquad \left. (\hat{\Sigma}(x) - \hat{\Sigma}(s)) \Delta_i g(s) + \Delta_i \hat{\Sigma}(x) \right\} N(s, g(s)I)\{dx\} \qquad (4.9) \\ &= \varepsilon \Delta_i g(s) I + \int \Delta_i \hat{\Sigma}(x) N(s, g(s)I)\{dx\} + O(1/\sqrt{\tau})\end{aligned}$$

as $\tau \to \infty$. Also

$$\begin{aligned}\hat{\Sigma}_\varepsilon(s) &= \hat{\Sigma}(s) + \varepsilon g(s) I + \int (\hat{\Sigma}(x) - \hat{\Sigma}(s)) N(s, g(s)I)\{dx\} \\ &= \hat{\Sigma}(s) + \varepsilon g(s) I + O(\sqrt{\tau})\end{aligned}$$

as $\tau \to \infty$. Using Lemma 4.4,

$$\Delta_i \tilde{R}_\varepsilon(s) = \int \left\{ \frac{1}{2} (x-s)' \hat{\Sigma}_\varepsilon^{-1}(s) (\Delta_i \hat{\Sigma}_\varepsilon(s)) \lambda(x) + \lambda_i(x) \right\} N(s, \hat{\Sigma}_\varepsilon(s))\{dx\}.$$

One implication of this equation is that $\Delta_i \tilde{R}_\varepsilon$ is a bounded function. Let

$$\begin{aligned}B_j(s, x, \varepsilon) &= -\tfrac{1}{2} \operatorname{tr}(\hat{\Sigma}_\varepsilon^{-1}(s) \Delta_j \hat{\Sigma}_\varepsilon(s)) \\ &\quad + \tfrac{1}{2}(x-s)' \hat{\Sigma}_\varepsilon^{-2}(s) (\Delta_j \hat{\Sigma}_\varepsilon(s))(x-s) + (\hat{\Sigma}_\varepsilon^{-1}(s)(x-s))_j.\end{aligned}$$

Then

$$\begin{aligned}\Delta_i \Delta_j \tilde{R}_\varepsilon(s) &= \int \left\{ B_j(s, x, \varepsilon) \left(\frac{1}{2} (x-s) \hat{\Sigma}_\varepsilon^{-1}(s) (\Delta_i \hat{\Sigma}_\varepsilon(s)) \lambda(x) + \lambda_i(x) \right) \right. \\ &\qquad \left. + \frac{\partial}{\partial s_j} \left(\frac{1}{2} (x-s)' \hat{\Sigma}_\varepsilon^{-1}(s) \Delta_i \hat{\Sigma}(s) \right) \lambda(x) \right\} N(s, \hat{\Sigma}_\varepsilon(s))\{dx\} \qquad (4.10) \\ &= \int \lambda_i(x) (\hat{\Sigma}_\varepsilon^{-1}(s)(x-s))_j N(s, \hat{\Sigma}_\varepsilon(s))\{dx\} + O(1/\tau)\end{aligned}$$

as $\tau \longrightarrow \infty$. Finally,

$$\begin{aligned}
&\Delta_i \Delta_j \Delta_q \tilde{R}_\varepsilon(s) \\
&\quad = \int \left\{ \left(B_q(s, x, \varepsilon) + \frac{\partial}{\partial s_q} \right) \left[B_j(s, x, \varepsilon) \left(\frac{1}{2} (x - s)' \Sigma_\varepsilon^{-1}(s) (\Delta_i \Sigma_\varepsilon(s)) \lambda(x) + \lambda_i(x) \right) \right. \right. \\
&\qquad \left. \left. + \frac{\partial}{\partial s_q} \left(\frac{1}{2} (x - s)' \Sigma_\varepsilon^{-1}(s) \Delta_i \Sigma_\varepsilon(s) \right) \lambda(x) \right] \right\} \qquad (4.11) \\
&\qquad \cdot N(s, \Sigma(s))\{dx\} \\
&\quad = O(1/\tau)
\end{aligned}$$

as $\tau \longrightarrow \infty$.

From equation (4.7),

$$\Sigma_\varepsilon(s) - \hat{\Sigma}(s) \sim \varepsilon \tau(s) I$$

as $\tau \longrightarrow \infty$. For τ large enough,

$$\begin{aligned}
\hat{R}_\varepsilon(s) &= \int \int R_d(s + x + y) N(s, \hat{\Sigma}(s))\{dx\} N(0, \Sigma_\varepsilon(s) - \hat{\Sigma}(s))\{dy\} \\
&= \hat{R}(s) + \int \int [R_d(s + x + y) - R_d(s + x)] N(s, \hat{\Sigma}(s))\{dx\} \\
&\qquad N(0, \Sigma_\varepsilon(s) - \hat{\Sigma}(s))\{dy\}.
\end{aligned}$$

It follows that for some constant K,

$$\limsup_{\tau \to \infty} \frac{\hat{R}_\varepsilon(s) - \hat{R}(s)}{\sqrt{\tau}} \le K\sqrt{\varepsilon} \qquad (4.12)$$

for all $\varepsilon \in (0, 1)$.

LEMMA 4.5. *Let v be a fixed vector. Then*

$$\tilde{R}_\varepsilon(s + v) - [\tilde{R}_\varepsilon(s) + v'\Delta\tilde{R}_\varepsilon(s) + \tfrac{1}{2} \textstyle\sum_{i=1}^p \sum_{j=1}^p v_i v_j \Delta_i \Delta_j \tilde{R}_\varepsilon(s)] = o(1/\sqrt{\tau})$$

as $\tau \longrightarrow \infty$.

PROOF. This follows from Taylor's Theorem given the asymptotic expression for $\Delta_i \Delta_j \Delta_q \tilde{R}_\varepsilon$ given in equation (4.11).

LEMMA 4.6. *Let Q be a distribution with mean zero and covariance Ω. Then*

$$\int \tilde{R}_\varepsilon(s + x) Q\{dx\} = \tilde{R}_\varepsilon(s) + \frac{1}{2} \textstyle\sum_{i=1}^p \sum_{j=1}^p \Omega_{ij} \Delta_i \Delta_j \tilde{R}_\varepsilon(s) + o(1\sqrt{\tau})$$

as $\tau \longrightarrow \infty$.

PROOF. Define

$$E_\varepsilon(s, x) = [\tilde{R}_\varepsilon(s + x) - \tilde{R}_\varepsilon(s) - x'\Delta\tilde{R}_\varepsilon(s) - \tfrac{1}{2} \textstyle\sum_{i=1}^{p} \sum_{j=1}^{p} x_i x_j \Delta_i \Delta_j R_\varepsilon(s)]\sqrt{\tau(s)}.$$

From (4.10),

$$\sup_{i,j,\|x\| \le \tau(s)/2} \left| \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} E_\varepsilon(s, x) \right| = O(1) \tag{4.13}$$

as $\tau \to \infty$. Hence for some constant $K_1 = K_1(\varepsilon)$,

$$| E_\varepsilon(s, x) | \le K_1 x^2$$

whenever $\| x \| \le \tau(s)/2$. Using the fact that $\tilde{R}_\varepsilon$ has a bounded gradient, there exist constants $K_2 = K_2(\varepsilon)$ and $K_3 = K_3(\varepsilon)$, such that

$$| E_\varepsilon(s, x) | \le \sqrt{\tau} \ \| x \| \ K_2 + x^2 K_3$$

for all s, x. These last two inequalities imply that for some constant $K_4 = K_4(\varepsilon)$, $| E_\varepsilon(s, x) | \le x^2 K_4$ for all s, x. The function x^2 is integrable (Q) and the lemma now follows from Lemma 4.5 and dominated covergence.

Define the functions

$$d_i(s) = \inf\{ \| x \| : t_i(s + x) = 0\}.$$

From Lemma 4.5,

$$\begin{aligned} \tilde{R}_\varepsilon(s - \mu_i) = \tilde{R}_\varepsilon(s) - \int \Big[&\frac{1}{2} (x - s)' \Sigma_\varepsilon^{-1}(s)(\mu_i' \Delta \Sigma_\varepsilon(s))\lambda(x) \\ &+ \mu_i' \lambda(x) - \frac{1}{2} (\mu_i' \lambda(x))(\mu_i' \Sigma_\varepsilon^{-1}(s)(x - s)) \Big] \\ &\cdot N(s, \Sigma_\varepsilon(s))\{dx\} + o(1/\sqrt{\tau}) \end{aligned} \tag{4.14}$$

as $\tau \to \infty$. On $\{x: t_i(x) > 0\}$, $\mu_i' \lambda(x) = -c_i$ and $\mu_i' \Delta \hat{\Sigma}(x) = -\Sigma_i$.

From equation (4.9),

$$\begin{aligned} \mu_i' \Delta \Sigma_\varepsilon(s) &= \varepsilon(\mu_i' \Delta g(s))I - \Sigma_i + O\left(\frac{1}{\sqrt{\tau}} + \int_{\| x \| > d_i(s)} N(0, g(s)I)\{dx\} \right) \\ &= \varepsilon(\mu_i' \Delta g(s))I - \Sigma_i + O\left(\frac{1}{\sqrt{\tau}} + P\left(\chi_p^2 > \frac{d_i^2(s)}{g(s)} \right) \right) \\ &= \varepsilon(\mu_i' \Delta g(s))I - \Sigma_i + O\left(\frac{1}{\sqrt{\tau}} + \left(\frac{d_i^2(s)}{\tau} \right)^{1/2p-1} \exp\left(\frac{-d_i^2(s)}{2\tau} \right) \right) \end{aligned}$$

as $\tau \longrightarrow \infty$. Using this and (4.14),

$$\begin{aligned}
\tilde{R}_\varepsilon(s - \mu_i) &= R_\varepsilon(s) + c_i + \int^{1/2} (x - s)'\Sigma_\varepsilon^{-1}(s)\,(\Sigma_i - \varepsilon(\mu_i'\Delta g(s))I)\lambda(x)N(s, \Sigma_\varepsilon(s))\{dx\} \\
&\quad + o\left(\frac{1}{\sqrt{\tau}}\right) + O\left(\frac{1}{\sqrt{\tau}}\left(\frac{d_i^2(s)}{\tau}\right)^{1/2p-1}\exp\left(\frac{d_i^2(s)}{2\tau}\right)\right) \\
&\quad + O\left(\int_{\|x\|>d_i(s)}\left(1 + \frac{\|x\|}{\sqrt{\tau}}\right)N(s, \Sigma_\varepsilon(s))\{dx\}\right) \\
&= R_\varepsilon(s) + c_i + \int^{1/2} (x - s)'\Sigma_\varepsilon^{-1}(s)(\Sigma_i - \varepsilon(\mu_i'\Delta g(s)I))\lambda(x)\, N(s, \Sigma_\varepsilon(s))\{dx\} \\
&\quad + o\left(\frac{1}{\sqrt{\tau}}\right) + O\left(\exp\left\{-K_5\frac{d_i^2(s)}{\tau}\right\}\right)
\end{aligned}$$

as $\tau \longrightarrow \infty$ for some positive constant $K_5 = K_5(\varepsilon)$. Using Lemma 4.6 and equation (4.10),

$$\begin{aligned}
\int \tilde{R}_\varepsilon(s + x - \mu_i)P_i\{dx\} \\
&= \tilde{R}_\varepsilon(s) + \frac{1}{2}\int (x - s)'\Sigma^{-1}(s)\Sigma_i\lambda(x)N(s, \Sigma_\varepsilon(s))\{dx\} + o(1/\sqrt{\tau})
\end{aligned}$$

as $\tau \longrightarrow \infty$. Combining this with the previous equation gives

$$\begin{aligned}
\tilde{R}_\varepsilon(s) - \int \tilde{R}_\varepsilon(s + x)P_i\{dx\} \\
&= c_i - \frac{\varepsilon}{2}(\mu_i'\Delta g(s))\int (x - s)'\,\Sigma_\varepsilon^{-1}(s)\lambda(x)N(s, \Sigma_\varepsilon(s))\{dx\} \\
&\quad + o(1/\sqrt{\tau}) + O\left(\exp\left(-K_5\frac{d_i^2(s)}{\tau}\right)\right)
\end{aligned} \tag{4.15}$$

as $\tau \longrightarrow \infty$. Since $d_i(s)/t_i(s)$ is bounded below by a positive constant, the last term in (4.15) can be replaced by $O(\exp(-K_6(t_i^2(s)/\tau)))$ for some $K_6 = K_6(\varepsilon) > 0$. Also there is a constant K_7 *independent* of ε such that

$$\left|\frac{\varepsilon}{2}(\mu_i'\Delta g(s))\int (x - s)'\Sigma_\varepsilon^{-1}(s)\lambda(x)N(s, \Sigma_\varepsilon(s))\{dx\}\right| \le K_7\sqrt{\varepsilon/\tau} + o(1/\sqrt{\tau})$$

as $\tau \to \infty$. Consequently

$$\left| \tilde{R}_\varepsilon(s) - c_i - \int \tilde{R}_\varepsilon(s+x) P_i\{dx\} \right| \tag{4.16}$$
$$\leq K_7\sqrt{\varepsilon/\tau} + o(1/\sqrt{\tau}) + O\left(\exp\left(-K_6 \frac{t_i^2(s)}{\tau}\right)\right)$$

uniformly as $\tau \to \infty$. To complete the proof of Theorem 4.4, note that for any ε,

$$R(s, \mathscr{P}) - \hat{R}(s) = \tilde{R}_\varepsilon(s) - \hat{R}(s) + E[\tilde{R}_\varepsilon(S_N) + \textstyle\sum_{i=1}^N (c_{\nu_i} + \tilde{R}_\varepsilon(S_i) - \tilde{R}_\varepsilon(S_{i-1}))]$$

and hence

$$\begin{aligned} |R(s, \mathscr{P}) - \hat{R}(s)| &\\ \leq |\tilde{R}_\varepsilon(s) - \hat{R}(s)| &+ E\tilde{R}_\varepsilon(S_N) \\ &+ E \textstyle\sum_{i=1}^N |c_{\nu_i} + \tilde{R}_\varepsilon(S_i) - \tilde{R}_\varepsilon(S_{i-1})| \; I\{t_{\nu_i}(S_{i-1}) \geq h_1(\tau(S_{i-1}))\} \\ &+ E \textstyle\sum_{i=1}^N |c_{\nu_i} + \tilde{R}_\varepsilon(S_i) - \tilde{R}_\varepsilon(S_{i-1})| \, I\{\tau(s - S_{i-1}) < h_2(\tau(s))\}. \end{aligned}$$

Now $E\tilde{R}_\varepsilon(S_N)$ is bounded, and from (4.12) $|\tilde{R}_\varepsilon(s) - \hat{R}(s)| \leq K_8\sqrt{\varepsilon\tau}$ for some K_8 independent of ε. Since $E(|c_{\nu_i} + \tilde{R}_\varepsilon(S_i) - \tilde{R}_\varepsilon(S_{i-1})| \mid S_i)$ is bounded, there is a constant A independent of ε such that

$$E \textstyle\sum_{i=1}^N |c_{\nu_i} + \tilde{R}_\varepsilon(S_i) - \tilde{R}_\varepsilon(S_{i-1})| \, I\{\tau(s, S_{i-1}) < h_2(\tau(s))\} \leq Ah_2(\tau(s)) + O(1)$$

as $\tau \to \infty$.

Using equations (4.6) and (4.16).

$$E[|c_{\nu_i} + \tilde{R}_\varepsilon(S_i) - \tilde{R}_\varepsilon(S_{i-1})| \, I\{t_{\nu_i}(S_{i-1}) \geq h_1(\tau(S_{i-1}))\} \mid S_{i-1} = x]$$
$$\leq o(1/\sqrt{\tau(x)}) + K_7\sqrt{\varepsilon/\tau(x)}$$

uniformly in x as $\tau(x) \to \infty$. Using Corollary 4.1 there is a constant K_9, independent of ε such that

$$E[\textstyle\sum_{i=1}^N |c_{\nu_i} + \tilde{R}_\varepsilon(S_i) - \tilde{R}_\varepsilon(S_{i-1})| \, I\{t_{\nu_i}(S_{i-1}) \geq h_1(\tau(S_{i-1}))\}] \leq o(\sqrt{\tau}) + K_9\sqrt{\varepsilon\tau}$$

as $\tau \to \infty$. Hence there is a constant K_{10} independent of ε such that

$$|R(s, \mathscr{P}) - \hat{R}(s)| \leq Ah_2(\tau(s)) + o(\sqrt{\tau}) + K_{10}\sqrt{\varepsilon\tau}$$

as $\tau \to \infty$. Since this equation holds for any ε, equation (4.5) holds and Theorem 4.4 is established.

5. Sequential design: proofs. To apply the results of Section 4, stopping regions with the correct shape are needed. As the next lemma shows, only procedures which stop according to (2.1) need be considered.

LEMMA 5.1. *Let $\mathscr{P}$ be an arbitrary procedure. There exists a procedure $\mathscr{P}_1$*

satisfying (2.1) *and a constant* K_1 *such that*

$$R(\pi, \alpha, \mathscr{P}_1) \le R(\pi, \alpha, \mathscr{P}) + K_1\alpha, \tag{5.1}$$

for all π *and all* α.

The following technical lemma will be needed in the proof of this lemma.

LEMMA 5.2. *Suppose* $\tau(S_0^{(i)} + \ln\alpha\, \mathbf{1} - \Lambda^{(i)}) = x > 0$, *and* $K_2 > 0$. *Then for some* K_3 *independent of* x,

$$P(\theta \in a^{(i)}) \ge \alpha e^x \pi^{(i)}, \tag{5.2}$$

and

$$\alpha K_2 x \pi^{(i)} \le P(\theta \in a^{(i)}) + K_3\alpha. \tag{5.3}$$

If in addition, $\tau(S_0^{(j)} + \ln\alpha\, \mathbf{1} - \Lambda^{(j)}) = y > 0$ *and* $S_0^{(i)} \ge 0$, *then*

$$\alpha K_2 y \pi^{(j)} \le P(\theta \in a^{(i)}) + K_3\alpha. \tag{5.4}$$

PROOF. The conditions in the lemma give

$$\sup_{n \in \underset{\sim}{a}^{(i)}} \pi^{(n)} = \pi^{(i)}\alpha e^x, \tag{5.5}$$

and

$$\sup_{n \in \underset{\sim}{a}^{(j)}} \pi^{(n)} = \alpha\pi^{(j)}e^y. \tag{5.6}$$

(5.5) implies (5.2) and (5.2) implies (5.3) because

$$\inf_{x>0}(e^x - Kx) > -\infty.$$

Suppose $a^{(i)} = a^{(j)}$. Then (5.5) and (5.6) yield

$$e^y\pi^{(j)} = e^x\pi^{(i)},$$

which implies (5.4) since $\inf_y e^y - K_2 y > -\infty$. If instead $a^{(i)} \ne a^{(j)}$, then $i \in \underset{\sim}{a}^{(j)}$. Since $S_0^{(i)} \ge 0$, $\pi^{(i)} \ge \pi^{(n)}$ for all n and (5.6) gives

$$\pi^{(i)} = \alpha\pi^{(j)}e^y. \tag{5.7}$$

Suppose $y \ge \ln K_2 - 2\ln\alpha$. Then $e^y > K_2 y/\alpha$ and (5.7) gives

$$\alpha K_2 y \pi^{(j)} \le \alpha\pi^{(i)} \le P(\theta \in a^{(i)}).$$

If $y < \ln K_2 - 2\ln\alpha$ then

$$\alpha K_2 y \le 1 + K_3\alpha$$

where $K_3 = \sup_{\alpha\in(0,1)} K_2(\ln K_2 - 2\ln\alpha) - 1/\alpha < \infty$, and again (5.4) holds.

PROOF OF LEMMA 5.1. Let N be the stopping time for $\mathscr{P}$. Let $\mathscr{P}_1$ be the procedure with stopping time N_1 defined by (2.1) which chooses the same experiments as $\mathscr{P}$ until $\mathscr{P}$ stops and chooses e_1 thereafter if $N_1 > N$. Define a random variable I so that $S_N^{(I)} \geq 0$, i.e. I is the index corresponding to the most likely state of nature when $\mathscr{P}$ stops. I_1 will be the corresponding r.v. for $\mathscr{P}_1$. Also let $\mathscr{F}_N$ be the σ-field representing the information when $\mathscr{P}$ stops.

$$\begin{aligned} R(\pi, \alpha, \mathscr{P}_1) - R(\pi, \alpha, \mathscr{P}) & \\ &\leq \ell_{\max} P(\theta \in a^{(I_1)}) + \alpha c_{\max} E(N_1 - N)^+ - \ell_{\min} P(\theta \in a^{(I)}) \\ &\leq p\ell_{\max}\alpha + E(\alpha c_{\max} E((N_1 - N)^+ \mid \mathscr{F}_N) - \ell_{\min} P(\theta \in a^{(I)} \mid \mathscr{F}_N)). \end{aligned}$$

From Corollary 4.1,

$$E((N_1 - N)^+ \mid \mathscr{F}_N, \theta = \theta_i) \leq K_4[\tau(S_N^{(i)} + \ln\alpha\,\mathbf{1} - \Lambda^{(i)}) + 1].$$

Using Lemma (5.2),

$$\begin{aligned} \frac{c_{\max}}{\ell_{\min}} E((N_1 - N)^+ \mid \mathscr{F}_N) &\leq \textstyle\sum_{i=0}^p K_4 \pi^{(i)}(N)(\tau(S_N^{(i)} + \ln\alpha\,\mathbf{1} - \Lambda^{(i)}) + 1) \\ &\leq \alpha^{-1} P(\theta \in a^{(i)} \mid \mathscr{F}_N) + K_5. \end{aligned}$$

Hence

$$R(\pi, \alpha, \mathscr{P}_1) - R(\pi, \alpha, \mathscr{P}) \leq p\ell_{\max}\alpha + K_5\alpha = K_1\alpha$$

proving Lemma 5.1.

LEMMA 5.3. *For any procedure, if* $\tau(S_0^{(i)} + \ln\alpha\,\mathbf{1} - \Lambda^{(i)}) > 0$,

$$\begin{aligned} E[\exp(\tau(S_N^{(i)} + \ln\alpha\,\mathbf{1} - \Lambda^{(i)}))I\{\tau(S_N^{(i)} + \ln\alpha\,\mathbf{1} - \Lambda^{(i)}) > 0\} \mid \theta = \theta_i] & \\ \leq p \exp(\tau(S_0^{(i)} + \ln\alpha\,\mathbf{1} - \Lambda^{(i)})).& \end{aligned}$$

PROOF. Given $\theta = \theta_i$, the sequences $\{\exp(-[S_n^{(i)}]_j)\}_{n\geq 1}$ are positive martingales ($[S_n^{(i)}]_j$ is the jth component of $S_n^{(i)}$). Hence

$$E[\exp(-[S_N^{(i)}]_j) \mid \theta = \theta_i] \leq \exp(-[S_0^{(i)}]_j).$$

Therefore

$$\begin{aligned} E[\exp(\tau(S_N^{(i)} + \ln\alpha\,\mathbf{1} - \Lambda^{(i)}))I\{\tau(S_N^{(i)} + \ln\alpha\,\mathbf{1} - \Lambda^{(i)}) > 0\} \mid \theta = \theta_i] & \\ \leq \textstyle\sum_{j\in a_i} E[\exp(-[S_N^{(i)}]_j - \ln\alpha) \mid \theta = \theta_i] & \\ \leq \textstyle\sum_{j\in a_i} \exp(-[S_0^{(i)}]_j - \ln\alpha) & \\ \leq p \exp(\tau(S_0^{(i)} + \ln\alpha\,\mathbf{1} - \Lambda^{(i)})). & \end{aligned}$$

PROOF OF THEOREM 5.1. Let us begin by showing

$$R(\pi, \alpha) \geq \hat{R}(\pi, \alpha) + o(\alpha\sqrt{-\ln \alpha}) \tag{5.8}$$

uniformly in π as $\alpha \to 0$. If not, then there exist sequences $\{\mathscr{P}_m\}$, $\{\alpha_m\}$ and $\{\pi_m\}$ and an index J such that $\alpha_m \to 0$ and

$$\lim_{m\to\infty} \frac{\pi_m^{(J)}[\alpha_m \hat{R}^{(J)}(S_{Om}^{(J)} + \ln\alpha_m\mathbf{1} - \Lambda^{(J)}) - R^{(J)}(\pi_m, \alpha_m, \mathscr{P}_m)]}{\alpha_m\sqrt{-\ln\alpha_m}} > 0. \tag{5.9}$$

Using Lemma 5.1 we can assume that each $\mathscr{P}_m$ satisfies (2.1). (5.9) clearly implies $\tau_m = \tau(S_{0m}^{(J)} + \ln \alpha_m \mathbf{1} - \Lambda^{(J)}) \to \infty$. Let $\{\mathscr{P}_m'\}$ be a sequence of procedures which have sample sizes given by

$$N_m' = \inf\{n \geq 0: S_{nm}^{(J)} \geq -\ln\alpha\ \mathbf{1} + \Lambda^{(J)}\}$$

and such that $\mathscr{P}_m$ and $\mathscr{P}_m'$ use the same experiments on trials before N_m. Using Theorem 4.4,

$$R^{(J)}(\pi_m, \alpha_m, \mathscr{P}_m') \geq \alpha_m \hat{R}^{(J)}(S_{0m}^{(J)} + \ln\alpha_m \mathbf{1} - \Lambda^{(J)}) + \alpha_m \sqrt{\tau_m}\ o(1)$$

as $m \to \infty$. Now $\pi_m^{(J)} \alpha_m \exp(\tau_m) \leq 1$ (see 5.5) so $\tau_m^{(J)}\ \sqrt{\tau_m}/\sqrt{-\ln\alpha_m}$ is bounded, and since

$$R^{(J)}(\pi_m, \alpha_m, \mathscr{P}_m') \leq R^{(J)}(\pi_m, \alpha_m, \mathscr{P}_m) + \alpha_m c_{\max} E[N_m' - N_m \mid \theta = \theta_J],$$

we can contradict (5.9) by showing that

$$\text{(5.10)} \qquad \lim_{m\to\infty} \frac{\pi_m^{(J)} E[N_m' - N_m \mid \theta = \theta_J]}{\sqrt{-\ln\ \alpha_m}} = 0.$$

Using Corollary (4.1), (5.10) holds provided

$$\text{(5.11)} \qquad \lim_{m\to\infty} \frac{\pi_m^{(J)}}{\sqrt{-\ln\ \alpha_m}} E[\tau(S_{N_m}^{(J)} + \ln\ \alpha_m \mathbf{1} - \Lambda^{(J)}) \cdot I\{\tau(S_{N_m} + \ln\ \alpha_m \mathbf{1} - \Lambda^{(J)}) > 0\} \mid \theta = \theta_J] = 0.$$

Now on $\{\tau(S_{N_m} + \ln\alpha_m \mathbf{1} - \Lambda^{(J)}) > 0\}$, $\tau(S_{N_m} + \ln\alpha_m \mathbf{1} - \Lambda^{(J)}) \geq -2\ \ln\alpha_m$. Using the fact that e^x/x is increasing for $x > 1$, we see that if $\alpha_m < 1/\sqrt{e}$,

$$\tau(S_{N_m}^{(J)} + \ln\alpha_m \mathbf{1} - \Lambda^{(J)}) \leq -2\ \ln\alpha_m \exp(\tau(S_{N_m} + \ln\ \alpha_m \mathbf{1} - \Lambda^{(J)}))\alpha_m^2$$

on $\{\tau(S_{N_m} + \ln\ \alpha_m \mathbf{1} - \Lambda^{(J)}) > 0\}$. Consequently an upper bound for the limit in (5.11) is

$$\limsup_{m\to\infty} \frac{\pi_m^{(J)}}{\sqrt{-\ln\alpha_m}} (-2\alpha_m^2 \ln\alpha_m) E[\exp(\tau(S_{N_m}^{(J)} + \ln\alpha_m \mathbf{1} - \Lambda^{(J)})) \cdot I\{\tau(S_{N_m}^{(J)} + \ln\alpha_m \mathbf{1} - \Lambda^{(J)}) > 0\} \mid \theta = \theta_J]$$

$$\leq \limsup_{m\to\infty} 2\alpha_m^2 \sqrt{-\ln\alpha_m}\ \pi_m^{(J)} p \exp(\tau(S_{0m}^{(J)} + \ln\ \alpha_m \mathbf{1} - \Lambda^{(i)})),$$

where Lemma 5.3 was used for the second line. This last expression equals

$$\limsup_{m\to\infty} \sup_{i \in a_m} 2\alpha_m \sqrt{-\ln\alpha_m}\ p\pi_m^{(i)}(0) = 0.$$

This establishes (5.11) and proves (5.8).

To finish, it is sufficient to show that if $\mathscr{P}$ satisfies (2.5) then

$$R(\pi, \alpha, \mathscr{P}) \leq \hat{R}(\pi, \alpha) + o(\alpha\sqrt{-\ln\alpha}).$$

An immediate consequence of Theorem 4.4 is that for some constant A,

$$R(\pi, \alpha, \mathscr{P}) \leq \hat{R}(\pi, \alpha) + \alpha A \textstyle\sum_{j=0}^{p} \pi^{(j)}(h_2(\alpha) + \tau(S_0^{(j)})) + o(\alpha\sqrt{-\ln\alpha}).$$

Theorem 2.1 now follows as $\sup_\pi \pi^{(j)}\tau(S_0^{(j)}) < \infty$.

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