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A PROOF OF THE CONJECTURE THAT THE TUKEY-KRAMER MULTIPLE COMPARISONS PROCEDURE IS CONSERVATIVE[1]

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In this paper we present a first general proof of Tukey's (1953) conjecture concerning the extension of the Tukey multiple comparisons procedure to the case of unequal sample sizes, thereby proving that the so-called Tukey-Kramer procedure is conservative in all cases. Also a brief history of the conjecture is given and some extensions and further problems concerning the procedure are discussed.

1. Introduction. Consider the usual one-way fixed effects analysis of variance (ANOVA) model

$$X_{ij} = \mu_i + \varepsilon_{ij} \quad (1 \le j \le n_i,\ 1 \le i \le k)$$

where the ε_{ij} are independent $N(0, \sigma^2)$ random variables and μ_i is the mean of the ith treatment $(1 \le i \le k)$. The μ_i and σ^2 are unknown parameters. Let $\bar{X}_i$ be the sample mean of the ith treatment based on n_i observations $(1 \le i \le k)$, and let S^2 be an unbiased estimate of σ^2 which is distributed independently of the $\bar{X}_i$ as a $\sigma^2\chi^2_\nu/\nu$ random variables. Usually the ANOVA mean square error with $\nu = \sum_{i=1}^k n_i - k$ degrees of freedom is used as the estimate S^2.

A commonly occurring inference problem in practice is that of making simultaneous pairwise comparisons between the treatment means μ_i. Tukey (1953) proposed his celebrated T-procedure to do this in the special case when all the n_i are equal to a common sample size n (say). This procedure can be summarized by the following probability statement which gives *exact* $(1 - \alpha)$-level joint confidence intervals for all the differences $\mu_i - \mu_j$:

$$P\left\{\mu_i - \mu_j \in \left[\bar{X}_i - \bar{X}_j \pm q^{(\alpha)}_{k,\nu} \frac{S}{\sqrt{n}}\right];\ 1 \le i, j \le k\right\} = 1 - \alpha \tag{1.1}$$

where $q^{(\alpha)}_{k,\nu}$ is the upper α point of the Studentized range distribution with parameter k and ν degrees of freedom (Miller, 1966, page 38).

When the n_i are unequal, Tukey (1953) suggested that (1.1) be modified by replacing $1/\sqrt{n}$ by $\{(1/n_i + 1/n_j)/2\}^{1/2}$ in the confidence interval for $\mu_i - \mu_j (1 \le i, j \le k)$. He made the conjecture (which we shall refer to as the *Tukey conjecture*) that this procedure "... is apparently in the conservative direction ..." (Tukey,

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1953, page 39), i.e.

$$(1.2)\quad P\left\{\mu_i - \mu_j \in \left[\bar{X}_i - \bar{X}_j \pm q_{k,\nu}^{(\alpha)}\, S \sqrt{\frac{1}{2}\left(\frac{1}{n_i} + \frac{1}{n_j}\right)}\right];\ 1 \le i, j \le k\right\} \ge 1 - \alpha$$

for all values of the n_i. If the n_i are equal then, of course, we have equality in (1.2) because of (1.1). Kramer (1956) independently proposed the same modification, though in the slightly different context of a multiple range test procedure, and so (1.2) is referred to as the *Tukey-Kramer* (TK)*-procedure.*

Over the past thirty years many statisticians have attempted to prove or disprove the Tukey conjecture but its general resolution has remained an unsolved problem. Nevertheless, despite the uncertainty associated with the Tukey conjecture, the TK-procedure is widely preferred to many other procedures because of its intuitive appeal and because it provides shorter intervals. In this paper we offer a first general proof of the validity of the TK-procedure.

We now give a brief history of the problem. In a doctoral dissertation under Tukey's supervision, Kurtz (1956) proved the inequality (1.2) when $k = 3$ and for nearly equal n_i's when $k = 4$. (The case $k = 2$ trivially gives equality in (1.2).) He also studied certain limiting cases (involving highly unequal n_i's) for arbitrary k, and found the conjecture to be true in these cases. Based on these results he expressed "a strong feeling" for the truth of the conjecture for all k. Later, Miller (1966, page 87) advised against the use of the TK-procedure, describing it as "inexact" and as having "no mathematical proof or numerical substantiation." Instead Miller suggested using Scheffé's (1953) S-procedure or the classical Bonferroni procedure, neither of which requires the assumption of equal n_i's. However, these procedures are rather conservative—especially the former if interest is confined to pairwise comparisons of the means. This situation prompted many authors to develop other procedures to deal with the case of unequal sample sizes, e.g. Spjøtvoll and Stoline (1973), Dunn (1974) and Hochberg (1974a, 1975). However, the TK-procedure provides shorter intervals than these other procedures and so it became crucial to settle the validity of (1.2). For this purpose Dunnett (1980) carried out an extensive simulation study which provided a quite strong indication of the truth of the conjecture for all k. Inspired by these simulation results, Brown (1979) succeeded in proving the conjecture for the cases $k = 3$, 4 and 5. Based on all this evidence, Stoline (1981) concluded that "For all practical purposes, the TK method is conservative." The proof presented in this paper finally removes any uncertainty concerning the use of the TK-procedure.

The proof of the Tukey conjecture is given in Section 2 of this paper and a technical lemma required in this proof is given in the Appendix. Some extensions and further problems relating to the TK-procedure are discussed in Section 3.

2. The main result. By conditioning on S it is apparent that the inequality (1.2) follows from the theorem stated below.

THEOREM. *Let $X_i \sim N(0, \sigma_i^2)$, $1 \le i \le k$, be independent where $0 < \sigma_i < \infty$,* *and let* $\xi_{ij} = q(\sigma_i^2 + \sigma_j^2)^{1/2}$ *for some fixed* $q > 0$ $(1 \le i, j \le k)$. *Then the function*

$$F = F(\sigma_1, \cdots, \sigma_k) = P\{|X_i - X_j| \le \xi_{ij};\ 1 \le i, j \le k\}$$

is strictly minimized when the σ_i *are equal.*

REMARK. Note that when the σ_i are equal, F is independent of their value. Also, since F is *strictly* minimized, there will be strict inequality in (1.2) when the n_i are not equal.

The proof of the theorem is rather long and so it is broken into several lemmas. We first explain the main idea and the steps involved in the proof so that it will be easier to follow. The proof depends upon the fact that if we take the partial derivative of F with respect to one of the σ's, σ_k say, then this may be written as the sum of $k - 1$ terms, G_i, $1 \le i \le k - 1$ (Lemma 1) where the sign of each G_i depends only on the sign of $\sigma_i - \sigma_k$ (Lemma 4). Lemma 2 shows that the G_i, as defined in Lemma 1, can be written as the integral over $\mathbb{R}$ of an odd function multiplied by another function $H_r(r)$ (see (2.6) in Lemma 4). The functions $H_i(r)$ involve integrals over intervals R_{ij} which are investigated in Lemma 3. In particular, Lemma 3 shows that the sign of the midpoints of these intervals depend only on the sign of $\sigma_i - \sigma_k$, and using this result, Lemma 5 in the Appendix shows that the sign of $H_i(r) - H_i(-r)$ is the same for all $r > 0$ and depends only on the sign of $\sigma_i - \sigma_k$. Lemma 4 then follows from this last result and Lemma 2.

PROOF OF THE THEOREM. Let

$$\psi_k(x_1, \cdots, x_k) = \left(\prod_{i=1}^{k} f_{\sigma_i}(x_i)\right) I_{\{|x_i - x_j| \le \xi_{ij};\, 1 \le i,j \le k\}}$$

where $f_{\sigma_i}(\cdot)$ is the density function of a $N(0, \sigma_i^2)$ random variable and $I_{\{A\}}$ is the indicator function of the set A. The subscript k on ψ refers to the dimension of its domain. Then

$$F = \int_{-\infty \le x_1, \cdots, x_k \le \infty} \cdots \int \psi_k(x_1, \cdots, x_k)\, dx_1 \cdots dx_k$$

$$= \int_{-\infty \le x_1, \cdots, x_k \le \infty} \cdots \int \psi_{k-1}(x_1, \cdots, x_{k-1}) f_{\sigma_k}(x_k) \times \left(\prod_{i=1}^{k-1} I_{\{|x_k - x_i| \le \xi_{ik}\}}\right) dx_1 \cdots dx_k.$$

Substituting $y_k = x_k/\sigma_k$ gives

$$F = \int_{y_k=-\infty}^{\infty} \int_{|x_i - y_k\sigma_k| \le \xi_{ik},\, 1 \le i \le k-1} \cdots \int (2\pi)^{-1/2} \exp\left\{\frac{-y_k^2}{2}\right\} \times \psi_{k-1}(x_1, \cdots, x_{k-1})\, dx_1 \cdots dx_{k-1}\, dy_k. \tag{2.1}$$

LEMMA 1. *For* $1 \le i \le k - 1$, *define*

$$x_i^* = y_k\sigma_k + \xi_{ik}, \quad N_i = \{j\colon 1 \le j \le k - 1, j \ne i\},$$

and the set $V_i(y_k) \subseteq \mathbb{R}^{k-2}$ *as*

$$V_i(y_k) = \{(x_j\colon j \in N_i)\colon |x_j - y_k\sigma_k| \le \xi_{jk},\ \forall j \in N_i\}.$$

Then

$$\frac{\partial F}{\partial \sigma_k} = \left(\frac{2}{\pi}\right)^{1/2} \sum_{i=1}^{k-1} G_i$$

where

$$G_i = \int_{y_k=-\infty}^{\infty} \int_{V_i(y_k)} \cdots \int \exp\left\{\frac{-y_k^2}{2}\right\} \psi_{k-1}(x_1, \cdots, x_i^*, \cdots, x_{k-1}) \times \left[y_k + \frac{\partial \xi_{ik}}{\partial \sigma_k}\right] \left(\prod_{j \in N_i} dx_j\right) dy_k.$$

PROOF OF LEMMA 1. It follows from (2.1) that

$$\frac{\partial F}{\partial \sigma_k} = (2\pi)^{-1/2} \int_{y_k=-\infty}^{\infty} \exp\left\{\frac{-y_k^2}{2}\right\} D\, dy_k \tag{2.2}$$

where

$$D = \frac{\partial}{\partial \sigma_k} \int_{x_1=y_k\sigma_k-\xi_{1k}}^{y_k\sigma_k+\xi_{1k}} \cdots \int_{x_{k-1}=y_k\sigma_k-\xi_{k-1,k}}^{y_k\sigma_k+\xi_{k-1,k}} \psi_{k-1}(x_1, \cdots, x_{k-1})\, dx_1 \cdots dx_{k-1}.$$

Then if we let $x_i^{**} = y_k\sigma_k - \xi_{ik}$ we obtain

$$\begin{aligned} D = \sum_{i=1}^{k-1} \Bigg\{ &\int_{V_i(y_k)} \cdots \int \psi_{k-1}(x_1, \cdots, x_i^*, \cdots, x_{k-1}) \left[y_k + \frac{\partial \xi_{ik}}{\partial \sigma_k}\right] \left(\prod_{j \in N_i} dx_j\right) \\ &- \int_{V_i(y_k)} \cdots \int \psi_{k-1}(x_1, \cdots, x_i^{**}, \cdots, x_{k-1}) \left[y_k - \frac{\partial \xi_{ik}}{\partial \sigma_k}\right] \left(\prod_{j \in N_i} dx_j\right) \Bigg\} \\ = \sum_{i=1}^{k-1} \{ A_i - B_i\} \quad &\text{(say)}. \end{aligned}$$

If in B_i we make the substitutions $y_k' = -y_k$ and $x_j' = -x_j$, $j \in N_i$; and since $\psi_{k-1}(x_1, \cdots, x_{k-1}) = \psi_{k-1}(-x_1, \cdots, -x_{k-1})$, we see that $B_i = -A_i$, $1 \le i \le k-1$. Therefore

$$\begin{aligned} D &= 2 \sum_{i=1}^{k-1} A_i \\ &= 2 \sum_{i=1}^{k-1} \int_{V_i(y_k)} \cdots \int \psi_{k-1}(x_1, \cdots, x_i^*, \cdots, x_{k-1}) \left[y_k + \frac{\partial \xi_{ik}}{\partial \sigma_k}\right] \left(\prod_{j \in N_i} dx_j\right). \end{aligned}$$

Lemma 1 now follows by putting this expression for D in (2.2) and exchanging the order of the summation over i and the integration over y_k. □

LEMMA 2. *For* $1 \le i \le k-1$, *the quantity* G_i *defined in Lemma* 1 *can be* *expressed as*

$$G_i = a_i \int_{r=-\infty}^{\infty} r \exp\left\{-\frac{1}{2}\left(\frac{1}{\sigma_i^2} + \frac{1}{\sigma_k^2}\right)r^2\right\} \int \cdots \int_{x_j - r \in R_{i,j}, j \in N_i} \psi_{k-2}(x_j\colon j \in N_i)\left(\prod_{j \in N_i} dx_j\right) dr$$

where $a_i > 0$ *is some constant and the set* $R_{ij} \subset \mathbb{R}$ *is defined by*

$$\left\{x_j\colon \left| x_j - r + \frac{\sigma_k^2 \xi_{ik}}{\sigma_i^2 + \sigma_k^2} \right| \le \xi_{jk}\right\} \cap \left\{x_j\colon \left| x_j - r - \frac{\sigma_i^2 \xi_{ik}}{\sigma_i^2 + \sigma_k^2} \right| \le \xi_{ij}\right\}$$
$$\equiv \{x_j\colon x_j - r \in R_{ij}\}.$$

PROOF OF LEMMA 2. Notice that:

$$\psi_{k-1}(x_1, \cdots, x_i^*, \cdots, x_{k-1})$$

(i)
$$= \psi_{k-2}(x_j\colon j \in N_i) \times (2\pi\sigma_i^2)^{-1/2} \times \exp\left\{-\frac{1}{2\sigma_i^2}(y_k\sigma_k + \xi_{ik})^2\right\}\left(\prod_{j \in N_i} I_{\{|x_j - y_k\sigma_k - \xi_{ik}| \le \xi_{ij}\}}\right);$$

(ii)
$$\exp\left\{-\frac{1}{2}y_k^2\right\} \times \exp\left\{-\frac{1}{2\sigma_i^2}(y_k\sigma_k + \xi_{ik})^2\right\} = \exp\left\{-\frac{1}{2}\left(1 + \frac{\sigma_k^2}{\sigma_i^2}\right)\left(y_k + \frac{\sigma_k \xi_{ik}}{\sigma_i^2 + \sigma_k^2}\right)^2\right\} \times \exp\left\{-\frac{1}{2}q^2\right\};$$

and

(iii)
$$\frac{\partial \xi_{ik}}{\partial \sigma_k} = \frac{\partial}{\partial \sigma_k}\{q(\sigma_i^2 + \sigma_k^2)^{1/2}\} = \frac{\sigma_k \xi_{ik}}{\sigma_i^2 + \sigma_k^2}.$$

Then, writing

$$z = y_k + \frac{\sigma_k \xi_{ik}}{\sigma_i^2 + \sigma_k^2}$$

we use (i), (ii), (iii), and the definition of G_i in Lemma 1 to obtain, for $1 \le i \le k - 1$

$$G_i = \exp\left\{-\frac{q^2}{2}\right\} \int_{y_k=-\infty}^{\infty} \int \cdots \int_{V_i(y_k)} (2\pi\, \sigma_i^2)^{-1/2} z \exp\left\{-\frac{1}{2}\left(1 + \frac{\sigma_k^2}{\sigma_i^2}\right)z^2\right\} \times \psi_{k-2}(x_j\colon j \in N_i)\left(\prod_{j \in N_i} I_{\{|x_j - y_k\sigma_k - \xi_{ik}| \le \xi_{ij}\}}\right)\left(\prod_{j \in N_i} dx_j\right) dy_k. \tag{2.3}$$

If we make the further substitution $r = \sigma_k z$, we have:

$$\left(1 + \frac{\sigma_k^2}{\sigma_i^2}\right)z^2 = r^2\left(\frac{1}{\sigma_i^2} + \frac{1}{\sigma_k^2}\right), \quad z\, dy_k = \frac{r\, dr}{\sigma_k^2}$$

$$x_j - y_k\sigma_k = x_j - r + \frac{\sigma_k^2 \xi_{ik}}{\sigma_i^2 + \sigma_k^2},$$

and

$$x_j - y_k\sigma_k - \xi_{ik} = x_j - r - \frac{\sigma_i^2 \xi_{ik}}{\sigma_i^2 + \sigma_k^2}.$$

Substituting these expressions in (2.3) we obtain

$$G_i = (2\pi)^{-1/2}(\sigma_k^2\sigma_i)^{-1}\exp\left\{-\frac{q^2}{2}\right\}$$
$$\times \int_{r=-\infty}^{\infty} r\exp\left\{-\frac{1}{2}\left(\frac{1}{\sigma_i^2}+\frac{1}{\sigma_k^2}\right)r^2\right\} \int \cdots \int_{-\infty\le x_j\le\infty, j\in N_i} \psi_{k-2}(x_j\colon j\in N_i)$$
$$\times \left(\prod_{j\in N_i} I_{[\{|x_j - r + (\sigma_k^2\xi_{ik}/\sigma_i^2+\sigma_k^2)|\le \xi_{jk}\}\cap\{|x_j - r - (\sigma_i^2\xi_{ik}/\sigma_i^2+\sigma_k^2)|\le\xi_{ij}\}]}\right) \times \left(\prod_{j\in N_i} dx_j\right) dr.$$

Then for $a_i = (2\pi)^{-1/2}(\sigma_i\sigma_k^2)^{-1}\exp\{-(q^2/2)\} > 0$, the expression for G_i given in this Lemma follows by the definition of R_{ij}. □

LEMMA 3. *For $1 \le i \le k-1$, $j \in N_i$, the set R_{ij} defined in Lemma 2 is the interval*

$$R_{ij} = \left[\frac{\sigma_i^2\xi_{ik}}{\sigma_i^2+\sigma_k^2} - \xi_{ij}, \frac{-\sigma_k^2\xi_{ik}}{\sigma_i^2+\sigma_k^2} + \xi_{jk}\right]$$

which has length > 0. Also, if the midpoint of R_{ij} is denoted by m_{ij}, then $\forall\, j \in N_i$, $1 \le i \le k-1$,

$$\sigma_i \gtreqless \sigma_k \Leftrightarrow m_{ij} \gtreqless 0.$$

PROOF OF LEMMA 3. Let

$$I_1 = \left[\frac{\sigma_i^2\xi_{ik}}{\sigma_i^2+\sigma_k^2} - \xi_{ij}, \frac{\sigma_i^2\xi_{ik}}{\sigma_i^2+\sigma_k^2} + \xi_{ij}\right]$$

and

$$I_2 = \left[\frac{-\sigma_k^2\xi_{ik}}{\sigma_i^2+\sigma_k^2} - \xi_{jk}, \frac{-\sigma_k^2\xi_{ik}}{\sigma_i^2+\sigma_k^2} + \xi_{jk}\right].$$

Then $R_{ij} = I_1 \cap I_2$. Observe that:

(i) $(\sigma_i^2+\sigma_k^2)^{1/2} > |\sigma_i - \sigma_k| > |(\sigma_i^2+\sigma_j^2)^{1/2} - (\sigma_k^2+\sigma_j^2)^{1/2}|$

and so

$$\xi_{ik} > |\xi_{ij} - \xi_{jk}|. \tag{2.4}$$

Suppose that $I_1 \supseteq I_2$, then

$$\frac{\sigma_i^2\xi_{ik}}{\sigma_i^2+\sigma_k^2} - \xi_{ij} \le \frac{-\sigma_k^2\xi_{ik}}{\sigma_i^2+\sigma_k^2} - \xi_{jk} \Rightarrow \xi_{ik} \le \xi_{ij} - \xi_{jk}$$

which contradicts (2.4). Similarly, if $I_2 \supseteq I_1$, then

$$\frac{-\sigma_k^2\xi_{ik}}{\sigma_i^2+\sigma_k^2}+\xi_{jk} \geq \frac{\sigma_i^2\xi_{ik}}{\sigma_i^2+\sigma_k^2}+\xi_{ij}, \quad \xi_{jk}-\xi_{ij} \geq \xi_{ik}$$

which again contradicts (2.4). So neither of I_1 or I_2 is contained in the other. Also,

$$\text{(ii)} \qquad \begin{aligned}&(\sigma_i^2+\sigma_j^2)^{1/2}+(\sigma_k^2+\sigma_j^2)^{1/2} > \sigma_i+\sigma_k > (\sigma_i^2+\sigma_k^2)^{1/2}\\ &\qquad \Rightarrow \xi_{ij}+\xi_{jk} > \xi_{ik} \Rightarrow \frac{\sigma_i^2\xi_{ik}}{\sigma_i^2+\sigma_k^2}-\xi_{ij} < \frac{-\sigma_k^2\xi_{ik}}{\sigma_i^2+\sigma_k^2}+\xi_{jk}\end{aligned}$$

and so I_1 and I_2 are not disjoint.

Together (i) and (ii) imply that R_{ij} is as stated in this Lemma, and the strict inequality in (ii) means that R_{ij} has length > 0.

Turning now to the second half of the Lemma, we have

$$2m_{ij} = \left(\frac{\sigma_i^2-\sigma_k^2}{\sigma_i^2+\sigma_k^2}\right)\xi_{ik}-\xi_{ij}+\xi_{jk}$$

so that

$$\frac{2m_{ij}}{q} = \frac{\sigma_i^2-\sigma_k^2}{(\sigma_i^2+\sigma_k^2)^{1/2}} - [(\sigma_i^2+\sigma_j^2)^{1/2}-(\sigma_k^2+\sigma_j^2)^{1/2}]. \tag{2.5}$$

Immediately we see that if $\sigma_i = \sigma_k$, then $m_{ij} = 0\ \forall j \in N_i$.

If $\sigma_i \neq \sigma_k$, then a small amount of algebra leads to the following inequality:

$$\left|\frac{\sigma_i^2-\sigma_k^2}{(\sigma_i^2+\sigma_k^2)^{1/2}}\right| > |(\sigma_i^2+\sigma_j^2)^{1/2}-(\sigma_k^2+\sigma_j^2)^{1/2}|.$$

Therefore it follows from (2.5) that m_{ij} has the same sign as $(\sigma_i^2-\sigma_k^2)$, i.e. $m_{ij} \gtrless 0$ for $\sigma_i \gtrless \sigma_k$, $\forall j \in N_i$. □

LEMMA 4. *For the quantity G_i in Lemma* 2, $1 \leq i \leq k-1$, *we have*

$$\sigma_k \gtreqless \sigma_i \Leftrightarrow G_i \gtreqless 0.$$

PROOF OF LEMMA 4. From Lemma 2 we have that

$$G_i = a_i \int_{r=-\infty}^{\infty} r \exp\left\{-\frac{1}{2}\left(\frac{1}{\sigma_i^2}+\frac{1}{\sigma_k^2}\right)r^2\right\} H_i(r)\, dr \tag{2.6}$$

where

$$\begin{aligned}H_i(r) &= \int_{x_j-r\in R_{ij}, j\in N_i}\cdots\int \psi_{k-2}(x_j\colon j\in N_i)\left(\textstyle\prod_{j\in N_i} dx_j\right)\\ &= \int_{x_j-r\in R_{ij}, j\in N_i}\cdots\int \left(\textstyle\prod_{j\in N_i} f_{\sigma_j}(x_j)\right) I_{\{|x_l-x_m|\leq\xi_{lm};\forall l,m\in N_i\}}\left(\textstyle\prod_{j\in N_i} dx_j\right).\end{aligned}$$

By transforming from x_j to $x_j + r$ we can rewrite this last equation as

$$(2.7)\qquad H_i(r) = \int_{x_j \in R_{ij}, j \in N_i} \cdots \int (\textstyle\prod_{j \in N_i} f_{\sigma_j}(r + x_j)) I_{\{|x_l - x_m| \le \xi_{lm}; \forall l, m \in N_i\}} (\prod_{j \in N_i} dx_j).$$

Also by transforming from x_j to $-x_j$ in (2.7), and since $f_{\sigma_j}(x) = f_{\sigma_j}(-x)$, it follows that

$$(2.8)\qquad H_i(-r) = \int_{x_j \in -R_{ij}, j \in N_i} \cdots \int (\textstyle\prod_{j \in N_i} f_{\sigma_j}(r + x_j)) I_{\{|x_l - x_m| \le \xi_{lm}; \forall l, m \in N_i\}} (\prod_{j \in N_i} dx_j).$$

Note that $H_i(r) \ge 0$.

To prove this lemma we now use the results of Lemma 3 above, and Lemma 5 and its Corollary which are contained in the Appendix. In the case $\sigma_i = \sigma_k$ we have by Lemma 3 that $m_{ij} = 0\ \forall j \in N_i$, so $R_{ij} = -R_{ij}\ \forall j \in N_i$, and hence we see from (2.7) and (2.8) that $H_i(r) = H_i(-r)$. It then follows from (2.6) that $G_i = 0$.

In the case $\sigma_i > \sigma_k$ we have by Lemma 3 that $m_{ij} > 0\ \forall j \in N_i$. Then a direct application of Lemma 5 to equations (2.7) and (2.8) gives $H_i(r) \le H_i(-r)$ for $r > 0$. In fact we have strict inequality, $H_i(r) < H_i(-r)$, because the conditions of the Corollary to Lemma 5 are satisfied, namely:

(a) $m_{ij} > 0$ and R_{ij} has length > 0, $\forall j \in N_i$ (see Lemma 3),

and

(b) for $\forall l, m \in N_i$ we have

$$\left| \left(\frac{-\sigma_k^2 \xi_{ik}}{\sigma_i^2 + \sigma_k^2} + \xi_{lk} \right) - \left(\frac{-\sigma_k^2 \xi_{ik}}{\sigma_i^2 + \sigma_k^2} + \xi_{mk} \right) \right| = |\xi_{lk} - \xi_{mk}| < \xi_{lm} \quad \text{(see (2.4))}.$$

Then since $0 \le H_i(r) < H_i(-r)$ for $r > 0$, and $a_i > 0$, it follows from (2.6) that $G_i < 0$.

When $\sigma_i < \sigma_k$ we have by Lemma 3 that $m_{ij} < 0\ \forall j \in N_i$, and so in a similar way it follows from Lemma 5 that $0 \le H_i(-r) < H_i(r)$ for $r > 0$, and hence from (2.6) it follows that $G_i > 0$. □

We now complete the proof of the theorem. It follows from Lemmas 1 and 4 that $\partial F/\partial\sigma_k$ can be expressed as

$$\frac{\partial F}{\partial \sigma_k} = \textstyle\sum_{i=1}^{k-1} b_{ki}$$

for some b_{ki} which satisfy

$$b_{ki} \gtreqless 0 \Leftrightarrow \sigma_k \gtreqless \sigma_i.$$

But σ_k was arbitrarily chosen from among the σ's, so more generally, for $1 \le i \le k$, $\partial F/\partial\sigma_i$ can be expressed as

$$\frac{\partial F}{\partial \sigma_i} = \textstyle\sum_{j=1, j \ne i}^{k} b_{ij}$$

for some b_{ij} which satisfy

$$b_{ij} \gtreqqless 0 \Leftrightarrow \sigma_i \gtreqqless \sigma_j.$$

This leads to the required result that F has a strict minimum when all the σ_i are equal, as we now show.

Let $\sigma_{[1]} \leq \sigma_{[2]} \leq \cdots \leq \sigma_{[k]}$ denote the ordered σ_i's. If we

(1) increase $\sigma_{[1]}$ to $\sigma_{[2]}$,

(2) increase $\sigma_{[1]}$ and $\sigma_{[2]}$ to $\sigma_{[3]}$ (keeping $\sigma_{[1]} = \sigma_{[2]}$),

$\vdots$

$(k-1)$ increase $\sigma_{[1]}, \cdots, \sigma_{[k-1]}$ to $\sigma_{[k]}$ (keeping $\sigma_{[1]} = \cdots = \sigma_{[k-1]}$),

then F will be strictly decreased at each step where an increase is necessary (and there will be such a step unless the σ_i are all equal). This completes the proof of the theorem. □

3. Extensions of the Tukey-Kramer procedure. It is sometimes of interest to have a procedure which gives joint confidence intervals for all contrasts of the treatment means, i.e. for all parametric functions $\sum_{i=1}^k c_i\mu_i$ where $\sum_{i=1}^k c_i = 0$. This is useful as it provides confidence intervals for any group of contrasts which are selected for consideration after looking at the data ("data-snooping"). We can extend the TK-procedure to do this by using the following result of Tukey (1953).

Let C^k be the set of all k-dimensional contrasts,

$$C^k = \{\mathbf{c} = (c_1, \cdots, c_k): \textstyle\sum_{i=1}^k c_i = 0\} \subseteq \mathbb{R}^k.$$

Then for any real vector $(y_1, \cdots, y_k)$ and nonnegative numbers ξ_{ij} satisfying $\xi_{ij} = \xi_{ji}$, $1 \leq i, j \leq k$,

$$\begin{aligned} &|y_i - y_j| \leq \xi_{ij}, \quad 1 \leq i, \quad j \leq k \\ (3.1) \qquad &\Leftrightarrow |\textstyle\sum_{i=1}^k c_i y_i| \leq \dfrac{\sum_{i=1}^k \sum_{j=1}^k c_i^+ c_j^- \xi_{ij}}{(1/2) \sum_{i=1}^k |c_i|} \quad \forall\, \mathbf{c} \in C^k \end{aligned}$$

where $c_i^+ = \max(c_i, 0)$ and $c_j^- = -\min(c_j, 0)$. Therefore if we let

$$\xi_{ij} = q_{k,\nu}^{(\alpha)}\, S\left[\frac{1}{2}\left(\frac{1}{n_i} + \frac{1}{n_j}\right)\right]^{1/2}$$

and

$$y_i = \bar{X}_i - \mu_i$$

it follows from (1.2) that

$$P\left\{\sum_{i=1}^{k} c_i\mu_i \in \right.$$

$$\text{(3.2)}\quad \left.\left[\sum_{i=1}^{k} c_i\bar{X}_i \pm q_{k,\nu}^{(\alpha)}\, S\, \frac{\sum_{i=1}^{k}\sum_{j=1}^{k} c_i^{+}c_j^{-}(\tfrac{1}{2})(1/n_i + 1/n_j))^{1/2}}{(\tfrac{1}{2})\sum_{i=1}^{k} |c_i|}\right];\ \forall \mathbf{c} \in C^k\right\}$$

$$\geq 1-\alpha.$$

However, while it is true that when only pairwise differences of the treatment means are considered, the TK-procedure is preferred to other procedures because it provides shorter intervals, this may no longer be true when we consider all contrasts of the treatment means. In particular, Scheffé's (1953) S-procedure (Miller, 1966, page 49) which provides exact $(1-\alpha)$-level joint confidence intervals for all contrasts may be preferred in this case.

A further extension of the TK-procedure is to designs other than the one-way layout, when estimates of the μ_i, $\hat{\mu}_i$ say, are not independent (e.g. the one-way analysis of covariance model). Suppose that the vector $\hat{\boldsymbol{\mu}} = (\hat{\mu}_1, \cdots, \hat{\mu}_k)$ has a multivariate normal distribution with mean vector $\boldsymbol{\mu} = (\mu_1, \cdots, \mu_k)$ and covariance matrix $\sigma^2\mathbf{V}$, where σ^2 is unknown and $\mathbf{V} = (v_{ij})$ is a known $(k \times k)$ positive-definite, symmetric matrix. It was proposed independently by Tukey (1953) and Kramer (1957) that in this case, $1/\sqrt{n}$ in (1.1) be replaced by $[(v_{ii} + v_{jj} - 2v_{ij})/2]^{1/2}$ in the confidence interval for $\mu_i - \mu_j$. Tukey conjectured that this procedure is also conservative, i.e.

$$\text{(3.3)}\quad P\{\mu_i - \mu_j \in [\hat{\mu}_i - \hat{\mu}_j \pm q_{k,\nu}^{(\alpha)} S[\tfrac{1}{2}\,(v_{ii} + v_{jj} - 2v_{ij})]^{1/2}];\ 1 \leq \mathrm{i}, \mathrm{j} \leq \mathrm{k}\} \geq 1-\alpha.$$

Notice that if $\mathbf{V}$ is diagonal and if $v_{ii} = 1/n_i$ then (3.3) reduces to (1.2). Also, using (3.1), (3.3) can be extended to the case of all contrasts of the μ_i.

In the case $k = 3$, (3.3) has been proved by Brown (1982). Also Hochberg (1974b) has shown that whenever $v_{ii} + v_{jj} - 2v_{ij} = v$, $\forall i, j$, for some constant v (i.e. the variance of $\hat{\mu}_i - \hat{\mu}_j$ is the same for all i, j), then there is equality in (3.3) for all k. The question of the general validity of (3.3) remains an unsolved problem.

APPENDIX

In Lemma 5 and its proof we use the following notation. For any set A, $-A = \{x\colon -x \in A\}$; and if $B \subseteq A$ then $A - B = \{x\colon x \in A, x \notin B\}$. Also if A is a finite set then $|A|$ denotes the number of elements in A. Finally $\mathbb{N}$ is the set of natural numbers $\{1, 2, 3, \cdots\}$.

LEMMA 5. *For a finite set $J \subset \mathbb{N}$, and for some fixed $r > 0$, $\delta_{ij} \geq 0$ and $\tau_i > 0$, define*

$$g_{|J|}(x_i\colon i \in J) = \left(\prod_{i\in J} f_{\tau_i}(x_i + r)\right) I_{\{|x_i - x_j| \leq \delta_{ij};\forall i,j\in J\}}$$

where $f_{\tau_i}(x)$ is the density of a $N(0, \tau_i^2)$ random variable. Also for some fixed $d_i \geq$

0 and $m_i \in \mathbb{R}$, *define for* $n \in \mathbb{N}$,

$$A_n = \{(x_1, \cdots, x_n): m_i - d_i \le x_i \le m_i + d_i, 1 \le i \le n\} \subseteq \mathbb{R}^n$$

so that

$$-A_n = \{(x_1, \cdots, x_n): -m_i - d_i \le x_i \le -m_i + d_i, 1 \le i \le n\} \subseteq \mathbb{R}^n.$$

Then if $m_i \ge 0$ *for* $1 \le i \le n$ *we have*

$$\int \cdots \int_{A_n} g_n(x_1, \cdots, x_n)\, dx_1 \cdots dx_n \le \int \cdots \int_{-A_n} g_n(x_1, \cdots, x_n)\, dx_1 \cdots dx_n \tag{A1}$$

for all $n \in \mathbb{N}$.

PROOF OF LEMMA 5. We shall prove (A1) by induction on n. For $m_1, d_1 \ge 0$,

$$\int_{m_1-d_1}^{m_1+d_1} f_{\tau_1}(x_1 + r)\, dx_1 \le \int_{-m_1-d_1}^{-m_1+d_1} f_{\tau_1}(x_1 + r)\, dx_1,$$

so (A1) is true for $n = 1$.

Now assume that (A1) is true for $n = 1, 2, \cdots, k - 1$ and we will show that this implies that (A1) is true for $n = k$. The proof is divided into four parts:

(i) Let $B_k = A_k - (A_k \cap -A_k)$, so $-B_k = -A_k - (A_k \cap -A_k)$. Then a necessary and sufficient condition for (A1) in the case $n = k$ is that

$$\int \cdots \int_{B_k} g_k(x_1, \cdots, x_k)\, dx_1 \cdots dx_k \le \int \cdots \int_{-B_k} g_k(x_1, \cdots, x_k)\, dx_1 \cdots dx_k. \tag{A2}$$

(Note: If $m_i - d_i > 0$ for any i, $1 \le i \le k$, then $A_k \cap -A_k = \varnothing$, the empty set, and $B_k = A_k$. Also if $m_i = 0$ for each i, $1 \le i \le k$, then $A_k = -A_k$ and $B_k = \varnothing$. Throughout this proof, for convenience, we define an integral over an empty set to be 0.)

(ii) Define $D_k = \{(x_1, \cdots, x_k) \in B_k: x_i > -m_i + d_i, 1 \le i \le k\}$ and define $C_k = B_k - D_k$ (so $-C_k = -B_k - (-D_k)$).

First note that $(x_1, \cdots, x_k) \in B_k \Rightarrow (x_1, \cdots, x_k) \in A_k \Rightarrow x_i \le m_i + d_i$, $1 \le i \le k$. Therefore $D_k = \varnothing$ if any $m_i = 0$.

If $D_k \ne \varnothing$ then $(x_1, \cdots, x_k) \in D_k \Rightarrow x_i > 0$, $1 \le i \le k$, since

$$x_i > -m_i + d_i \quad \text{by the definition of} \quad D_k$$

and

$$x_i \geq m_i - d_i \quad \text{by the definition of} \quad A_k \quad (\text{and } D_k \subseteq A_k).$$

Therefore, because $f_{\tau_i}(x_i + r) < f_{\tau_i}(-x_i + r)$ for $x_i > 0$, we have

(A3) $$g_k(x_1, \cdots, x_k) \leq g_k(-x_1, \cdots, -x_k) \quad \text{for} \quad (x_1, \cdots, x_k) \in D_k.$$

Hence either $D_k = \varnothing$ or

$$\int_{D_k} \cdots \int g_k(x_1, \cdots, x_k)\, dx_1 \cdots dx_k \leq \int_{-D_k} \cdots \int g_k(x_1, \cdots, x_k)\, dx_1 \cdots dx_k.$$

So to show (A2) it is sufficient to show that

(A4) $$\int_{C_k} \cdots \int g_k(x_1, \cdots, x_k)\, dx_1 \cdots dx_k \leq \int_{-C_k} \cdots \int g_k(x_1, \cdots, x_k)\, dx_1 \cdots dx_k.$$

(iii) Let $\mathscr{I}$ be the set of all nonempty, proper subsets of $\{1, 2, \cdots, k\}$. For each $I \in \mathscr{I}$ let $I^* = \{1, 2, \cdots, k\} - I$ and define

$$M(I) = \{(x_i \colon i \in I) \colon m_i - d_i \leq x_i \leq m_i + d_i,\ x_i > -m_i + d_i;\ \forall i \in I\} \subseteq \mathbb{R}^{|I|}$$

and

$$L((x_i \colon i \in I)) = \{(x_j \colon j \in I^*) \colon m_j - d_j \leq x_j \leq m_j + d_j,\ x_j \leq -m_j + d_j,$$
$$|x_i - x_j| \leq \delta_{ij};\ \forall i \in I, j \in I^*\} \subseteq \mathbb{R}^{|I^*|}.$$

For each $(x_1, \cdots, x_k) \in C_k$, $\exists i$, $1 \leq i \leq k$, such that $x_i > -m_i + d_i$ (otherwise $(x_1, \cdots, x_k) \in A_k \cap -A_k$), and $\exists j$, $1 \leq j \leq k$, such that $x_j \leq -m_j + d_j$ (otherwise $(x_1, \cdots, x_k) \in D_k$). Therefore it follows from the definitions of M, L and $\mathscr{I}$ that

(A5) $$\int_{C_k} \cdots \int g_k(x_1, \cdots, x_k)\, dx_1 \cdots dx_k = \sum_{I \in \mathscr{I}} \int_{M(I)} \cdots \int g_{|I|}(x_i \colon i \in I) \int_{L((x_i : i \in I))} \cdots \int g_{|I^*|}(x_j \colon j \in I^*)\, dx_1 \cdots dx_k$$

and

(A6) $$\int_{-C_k} \cdots \int g_k(x_1, \cdots, x_k)\, dx_1 \cdots dx_k = \sum_{I \in \mathscr{I}} \int_{-M(I)} \cdots \int g_{|I|}(x_i \colon i \in I) \int_{-L(-(x_i : i \in I))} \cdots \int g_{|I^*|}(x_j \colon j \in I^*)\, dx_1 \cdots dx_k.$$

Suppose $(x_i: i \in I) \in M(I)$ for some $I \in \mathscr{J}$. Then using similar arguments to those in part (ii) of this proof, we have $x_i > 0$ for each $i \in I$, and hence

$$g_{|I|}(x_i: i \in I) \leq g_{|I|}(-x_i: i \in I). \tag{A7}$$

It follows from (A5), (A6) and (A7) that to show (A4) it is sufficient to show that for each $(x_i: i \in I) \in M(I)$ for any $I \in \mathscr{J}$, either $L((x_i: i \in I)) = \varnothing$ or

$$\int \cdots \int_{L((x_i: i \in I))} g_{|I^*|}(x_j: j \in I^*)(\textstyle\prod_{j \in I^*} dx_j) \leq \int \cdots \int_{-L((x_i: i \in I))} g_{|I^*|}(x_j: j \in I^*)\ (\textstyle\prod_{j \in I^*} dx_j). \tag{A8}$$

(iv) For $(x_j: j \in I^*)$ to be in the set $L((x_i: i \in I))$ it is necessary, by the definition of L, that $m_j - d_j \leq x_j$ and $x_j \leq -m_j + d_j$ for each $j \in I^*$. So if $m_j - d_j > 0$ for any $j \in I^*$, then $L((x_i: i \in I)) = \varnothing$. Otherwise

$$(x_j: j \in I^*) \in L((x_i: i \in I)) \Leftrightarrow x_j \in E_j\ \forall j \in I^*$$

where

$$E_j = \textstyle\bigcap_{i \in I} [x_i - \delta_{ij}, x_i + \delta_{ij}] \cap [m_j - d_j, -m_j + d_j].$$

Now E_j is the intersection of intervals each with midpoint ≥ 0, so either it is empty, or it is an interval with midpoint ≥ 0. If $E_j = \varnothing$ for any $j \in I^*$ then $L((x_i: i \in I)) = \varnothing$, and if E_j is an interval with midpoint ≥ 0 for each $j \in I^*$, then (A8) is true by the induction assumption since $1 \leq |I^*| \leq k - 1$.

So in all cases either $L((x_i: i \in I)) = \varnothing$ or (A8) is true, which proves (A1) is true at the kth stage. That (A1) is true for all $n \in \mathbb{N}$ follows by induction. □

COROLLARY TO LEMMA 5. *Suppose that*

(a) $d_i, m_i > 0\ \forall i$, and

(b) $|(m_i + d_i) - (m_j + d_j)| < \delta_{ij\ \forall i, j}$.

Then

$$\int \cdots \int_{A_n} g_n(x_1, \cdots, x_n)\, dx_1 \cdots dx_n < \int \cdots \int_{-A_n} g_n(x_1, \cdots, x_n)\, dx_1 \cdots dx_n \tag{A9}$$

for all $n \in \mathbb{N}$.

PROOF OF COROLLARY. If $m_1, d_1 > 0$ then

$$\int_{m_1 - d_1}^{m_1 + d_1} f_{\tau_1}(x_1 + r)\, dx_1 < \int_{-m_1 - d_1}^{-m_1 + d_1} f_{\tau_1}(x_1 + r)\, dx_1,$$

so (A9) is true for $n = 1$.

For $n = k \geq 2$, conditions (a) and (b) $\Rightarrow \exists\ \delta x_i > 0$, $1 \leq i \leq k$, such that

(i) $F_k = \{(x_1, \cdots, x_k): m_i + d_i - \delta x_i \leq x_i \leq m_i + d_i, 1 \leq i \leq k\} \subseteq D_k$ and
(ii) $(x_1, \cdots, x_k) \in F_k \Rightarrow |x_i - x_j| \leq \delta_{ij}, \quad 1 \leq i, j \leq k.$

To see this, notice that (i) is satisfied if $m_i + d_i - \delta x_i > -m_i + d_i$ and $m_i + d_i - \delta x_i \geq m_i - d_i$, i.e. $\delta x_i < 2m_i$ and $\delta x_i \leq 2d_i$ for each i (which we can satisfy by condition (a)); and (ii) is satisfied if

$$\delta x_i < \tfrac{1}{2} \min_{1 \leq i,j \leq k}\{\delta_{ij} - |(m_i + d_i) - (m_j + d_j)|\} \quad \text{for each} \quad i,$$

(and the right hand side of this inequality is > 0 by condition (b)).

Now (i) $\Rightarrow (m_1 + d_1 - \delta x_1, \cdots, m_k + d_k - \delta x_k) \in D_k$

$\Rightarrow m_i + d_i - \delta x_i > 0, \quad 1 \leq i \leq k$

(see part (ii) of the proof of Lemma 5)

$$\Rightarrow \int_{m_i+d_i-\delta x_i}^{m_i+d_i} f_{T_i}(x_i + r)\,dx_i < \int_{-(m_i+d_i)}^{-(m_i+d_i-\delta x_i)} f_{T_i}(x_i + r)\,dx_i, \quad 1 \leq i \leq k.$$

Therefore, because the indicator function term of g_k is 1 everywhere in F_k by condition (ii) above, we have

$$\int \cdots \int_{F_k} g_k(x_1, \cdots, x_k)\,dx_1 \cdots dx_k$$

$$\text{(A10)} \qquad = \prod_{i=1}^{k} \int_{m_i+d_i-\delta x_i}^{m_i+d_i} f_{T_i}(x_i + r)\,dx_i < \prod_{i=1}^{k} \int_{-(m_i+d_i)}^{-(m_i+d_i-\delta x_i)} f_{T_i}(x_i + r)\,dx_i$$

$$= \int \cdots \int_{-F_k} g_k(x_1, \cdots, x_k)\,dx_1 \cdots dx_k.$$

Define $G_k = D_k - F_k$. Since $G_k \subseteq D_k$ it follows from (A3) that

$$\text{(A11)} \qquad \int \cdots \int_{G_k} g_k(x_1, \cdots, x_k)\,dx_1 \cdots dx_k \leq \int \cdots \int_{-G_k} g_k(x_1, \cdots, x_k)\,dx_1 \cdots dx_k.$$

Then (A10) and (A11) imply that

$$\int \cdots \int_{D_k} g_k(x_1, \cdots, x_k)\,dx_1 \cdots dx_k < \int \cdots \int_{-D_k} g_k(x_1, \cdots, x_k)\,dx_1 \cdots dx_k.$$

Therefore the inequality (A4), which was verified as part of the proof of Lemma

5, is sufficient to show that

$$\int_{A_k} \cdots \int g_k(x_1, \cdots, x_k)\, dx_1 \cdots dx_k < \int_{-A_k} \cdots \int g_k(x_1, \cdots, x_k)\, dx_1 \cdots dx_k.$$

This completes the proof of the Corollary. □

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